What is Symplectic Geometry?

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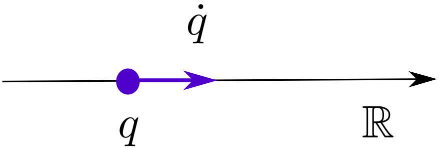
What is symplectic geometry?

Even dimensional geometry

What is a symplectic structure?

Classical mechanical systems

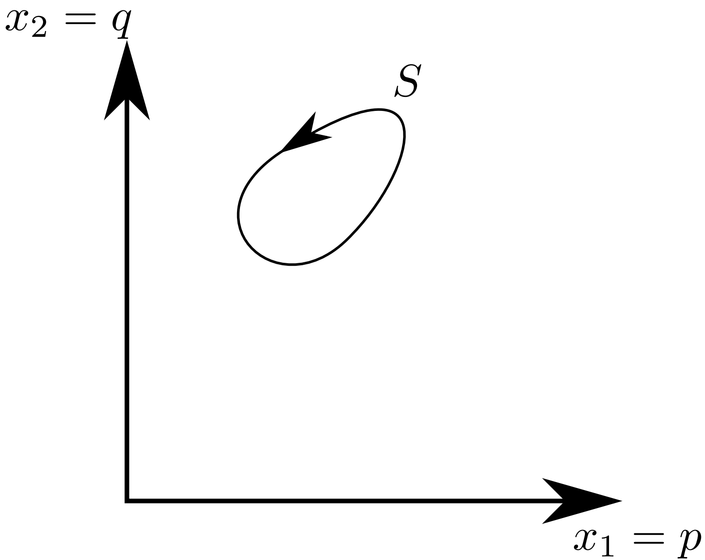


$$p := \dot{q}$$

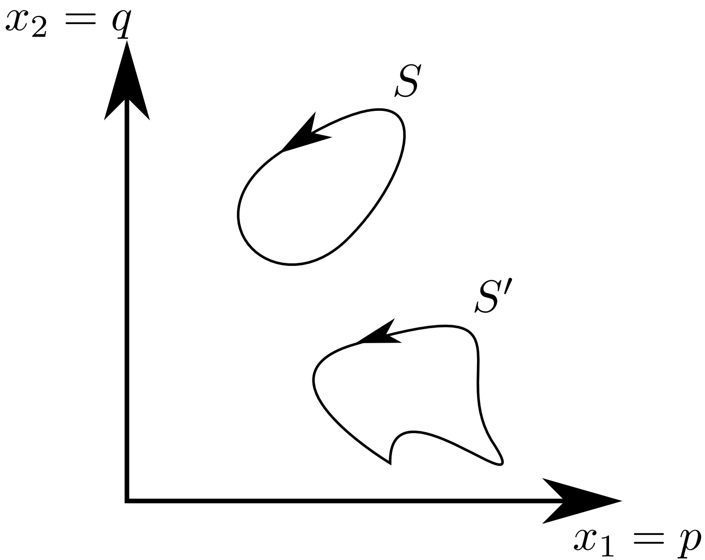
$$(x_1, x_2) := (p, q) \in \mathbb{R}^2$$

Symplectic structure: area form $\omega := dp \wedge dq$

What is a symplectic structure?



What is a symplectic structure?



What is a symplectic structure?

Particle in the plane

Position coordinates: q_1, q_2
Momentum coordinates: $p_1 := \dot{q}_1, p_2 := \dot{q}_2$

$$(x_1, x_2, x_3, x_4) = (p_1, q_1, p_2, q_2) \in \mathbb{R}^4$$

Symplectic form

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

What is a symplectic structure?

Particle in $\mathbb{R}^n$

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n}$$

What is a symplectic structure?

Even dimensional smooth manifold M

Definition (Symplectic structure)

Closed, non-degenerate 2-form ω on M

Flabbiness

- Darboux Theorem: locally all symplectic forms look like

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n}$$

- Moser Stability Theorem:

$$\omega_t, t \in [0,1]$$
$$\int_S \omega_t = \int_S \omega_0 \text{ for any closed surface } S \subset M$$

The ω_t are indistinguishable up to smooth bijections on M

What is the shape of a symplectic ball?

Definition (Symplectomorphisms)

Diffeomorphisms $\phi : M \to M$ that preserve ω, i.e.

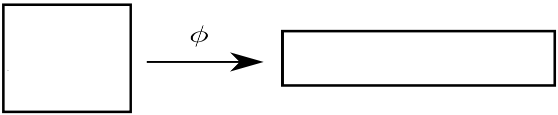
$$\int_S \omega = \int_{\phi(S)} \omega \text{ for all surfaces } S \subset M$$

They necessarily preserve volume!

What is the shape of a symplectic ball?

$\mathbb{R}^2$

$$\phi(x_1, x_2) = (2x_1, \frac{1}{2}x_2)$$



Moser: any area preserving transformation

What is the shape of a symplectic ball?

$\mathbb{R}^4$

$B(r) := \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq r\}$

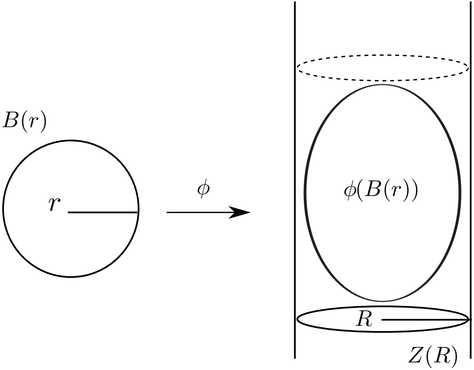
$$\phi(x_1, x_2, x_3, x_4) = (2x_1, \frac{1}{2}x_2, 2x_3, \frac{1}{2}x_4)$$

$$\psi(x_1, x_2, x_3, x_4) = (2x_1, 2x_2, \frac{1}{2}x_3, \frac{1}{2}x_4)$$

What is the shape of a symplectic ball?

Theorem (Gromov Non-Squeezing Theorem)

There is no symplectomorphism ϕ that embeds $B(r)$ into a cylinder $Z(R)$ of radius R smaller than r.



What is a symplectic capacity?

Definition (Cylindrical capacity)

For any non-empty subset $U \subset M$,

$$c_Z := \inf\{\pi r^2 \mid \exists \phi \ s.t. \ \phi(U) \subset Z(r)\}$$

Definition (Symplectic radius)

For any non-empty subset $U \subset M$,

$$c_B := \sup\{\pi r^2 \mid \exists \phi \ s.t. \ \phi(B(r)) \subset U\}$$

What is a symplectic capacity?

Definition (Symplectic capacity)

A map c from the set of all symplectic manifolds to $\mathbb{R}_{\geq 0}$ satisfying

1. Monotonicity: if $(M, \omega) \subset (M', \omega')$, then $c(M, \omega) < c(M', \omega')$
2. Conformality: $c(M, \alpha\omega) = |\alpha| c(M, \omega)$ for $\alpha \in \mathbb{R}$
3. Normalisation: $c(B(r)) = c(Z(r)) = \pi r^2$