What is Symplectic Geometry?

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What is symplectic geometry?

Even dimensional geometry
What is a symplectic structure?

Classical mechanical systems

\[ p := \dot{q} \]

\( (x_1, x_2) := (p, q) \in \mathbb{R}^2 \)

Symplectic structure: area form \( \omega := dp \wedge dq \)
What is a symplectic structure?

\[ x_2 = q \]

\[ x_1 = p \]
What is a symplectic structure?
What is a symplectic structure?

Particle in the plane

Position coordinates: \( q_1, q_2 \)
Momentum coordinates: \( p_1 := \dot{q}_1, p_2 := \dot{q}_2 \)

\[(x_1, x_2, x_3, x_4) = (p_1, q_1, p_2, q_2) \in \mathbb{R}^4\]

Symplectic form

\[\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4\]
What is a symplectic structure?

Particle in $\mathbb{R}^n$

$$\omega = dx_1 \land dx_2 + dx_3 \land dx_4 + \ldots + dx_{2n-1} \land dx_{2n}$$
What is a symplectic structure?

Even dimensional smooth manifold $M$

Definition (Symplectic structure)
Closed, non-degenerate 2-form $\omega$ on $M$
Flabbiness

- **Darboux Theorem**: locally all symplectic forms look like

\[ \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n} \]

- **Moser Stability Theorem**:

\[ \int_S \omega_t = \int_S \omega_0 \text{ for any closed surface } S \subset M \]

The \( \omega_t \) are indistinguishable up to smooth bijections on \( M \)
What is the shape of a symplectic ball?

**Definition (Symplectomorphisms)**

Diffeomorphisms $\phi : M \rightarrow M$ that preserve $\omega$, i.e.

$$\int_S \omega = \int_{\phi(S)} \omega \text{ for all surfaces } S \subset M$$

They necessarily preserve volume!
What is the shape of a symplectic ball?

\[ \mathbb{R}^2 \]

\[ \phi(x_1, x_2) = (2x_1, \frac{1}{2}x_2) \]

Moser: any area preserving transformation
What is the shape of a symplectic ball?

\[ \mathbb{R}^4 \]

\[ B(r) := \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq r\} \]

\[ \phi(x_1, x_2, x_3, x_4) = (2x_1, \frac{1}{2}x_2, 2x_3, \frac{1}{2}x_4) \]

\[ \psi(x_1, x_2, x_3, x_4) = (2x_1, 2x_2, \frac{1}{2}x_3, \frac{1}{2}x_4) \]
What is the shape of a symplectic ball?

Theorem (Gromov Non-Squeezing Theorem)

There is no symplectomorphism $\phi$ that embeds $B(r)$ into a cylinder $Z(R)$ of radius $R$ smaller than $r$. 

$B(r)$ \hspace{2cm} \phi \hspace{2cm} \phi(B(r))$ 

$r$ \hspace{2cm} $R$ 

$Z(R)$
What is a symplectic capacity?

Definition (Cylindrical capacity)
For any non-empty subset $U \subset M$, 

$$c_Z := \inf \{ \pi r^2 \mid \exists \phi \text{ s.t. } \phi(U) \subset Z(r) \}$$

Definition (Symplectic radius)
For any non-empty subset $U \subset M$, 

$$c_B := \sup \{ \pi r^2 \mid \exists \phi \text{ s.t. } \phi(B(r)) \subset U \}$$
What is a symplectic capacity?

Definition (Symplectic capacity)
A map $c$ from the set of all symplectic manifolds to $\mathbb{R}_{\geq 0}$ satisfying

1. Monotonicity: if $(M, \omega) \subset (M', \omega')$, then $c(M, \omega) < c(M', \omega')$

2. Conformality: $c(M, \alpha \omega) = |\alpha| c(M, \omega)$ for $\alpha \in \mathbb{R}$

3. Normalisation: $c(B(r)) = c(Z(r)) = \pi r^2$