The Geometry of Divisors with Multiplicity on Projective Curves

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Third Meeting of Young Women in Mathematics
14 February 2017

A tour of Brill-Noether theory

Group theory

19th century: group $\leftrightarrow$ subset of GL_n

20th century: abstract groups

$$\begin{cases} \text{structure and classification of abstract groups} \\ \text{representation theory} \end{cases}$$

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Algebraic geometry
Algebraic curves

19th century: irreducible polynomial in two variables

Curve in a higher dimensional projective space $\leftrightarrow$ subset of projective space defined by polynomial equations

Classification of algebraic curves $\leftrightarrow$ classification of all such subsets of projective space

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Classification of algebraic curves $\leftrightarrow$ describing all components of the Hilbert scheme whose general point corresponds to an integral curve

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20th century

Abstract curve

Classification of algebraic curves $\leftrightarrow$ study of moduli spaces $\mathcal{M}_g$

$$\begin{cases} \text{study of set of all abstract curves} \\ \text{study the ways in which a curve can be mapped to } \mathbb{P}^r \end{cases}$$

Brill-Noether theory $=$ representation theory for curves

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Moduli spaces

$\mathcal{M}_g$ = moduli space of curves

$G^r_d(C)$ = set of all nondegenerate maps $C \to \mathbb{P}^r$ of degree d

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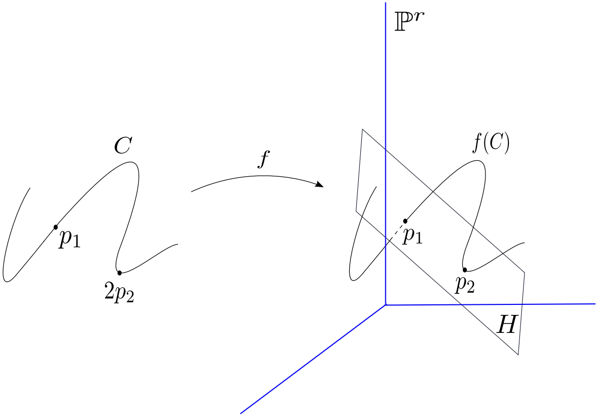
C smooth, genus g
$f : C \to \mathbb{P}^r$ non-degenerate
Degree of f = degree of $f^*H =: d$



$$f^*H = p_1 + 2p_2$$

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A $g^r_d = (L, V)$

- a line bundle L of degree d on C
- an $(r + 1)$-dimensional vector space $V \subset H^0(L)$

$$f : C \to \mathbb{P}^r$$
$$p \mapsto [\sigma_0(p) : \ldots : \sigma_r(p)]$$

$G^r_d(C) =$ space of all g^r_d-s on C

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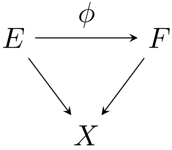
Estimate for dimension of $G^r_d(C)$
Describe $G^r_d(C)$ as a determinantal variety over $Pic^d(C)$



$$X_k(\phi) = \{p \in X \mid rk(\phi_p) \leq k\}$$

$$\dim X_k(\phi) \geq \dim X - (rk(E) - k)(rk(F) - k)$$

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$$\dim G^r_d(C) \geq g - (r+1)(g-d+r)$$

Brill-Noether number

$$\rho(g,r,d) = g - (r+1)(g-d+r)$$

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Existence and non-existence results

- $\rho \geq 0 \Rightarrow G^r_d(C) \neq \emptyset$ for any C
- $\rho < 0 \Rightarrow G^r_d(C) = \emptyset$ for a general C

Results about the geometry of $G^r_d(C)$ for general C

- $\dim G^r_d(C) = \rho$
- $G^r_d(C)$ is smooth
- $\rho = 0 \Rightarrow C$ has a finite number of g^r_d-s

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Results about the geometry of the g^r_d-s and their corresponding maps $f : C \to \mathbb{P}^r$
(both C and the g^r_d-s are general)

- if $r \geq 3$, then f is an embedding
- if $r = 2$, then f maps C birationally to a curve with at most nodes as singularities
- if $r = 1$, then f expresses C as a simply branched cover over $\mathbb{P}^1$

Multitangency conditions

Mémoire sur les contacts multiples d'ordre quelconque des courbes de degré r, qui satisfont à des conditions données, avec une courbe fixe du degré m; suivi de quelques réflexions sur la solution d'un grand nombre de questions concernant les propriétés projectives des courbes et des surfaces algébriques.

(Par M. *E. de Jonquières* à Paris.)

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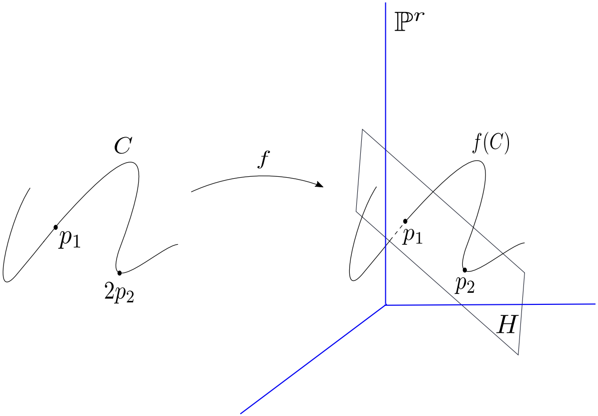
1. Le théorème suivant résout, d'une manière générale, toutes les questions de contacts d'ordre quelconque des courbes algébriques avec une courbe fixe. Afin d'en faciliter et d'en abréger l'énoncé, nous adopterons les notations suivantes:

Notations. t désigne le nombre des contacts proposés, dont les ordres sont n_1, n_2, n_3, ... n_{t-1}, n_t;

Multitangency conditions



$$f^*H = p_1 + 2p_2$$

Multitangency conditions and de Jonquières divisors

de Jonquières counts the number of pairs (p_1, p_2) with

$$f^*H = p_1 + 2p_2$$

for some hyperplane $H \subset \mathbb{P}^r$

Multitangency conditions and de Jonquières divisors

de Jonquières (and Mattuck, Macdonald) count the n-tuples

$$(p_1, \ldots, p_n)$$

with

$$f^* H = a_1 p_1 + \ldots + a_n p_n$$

where

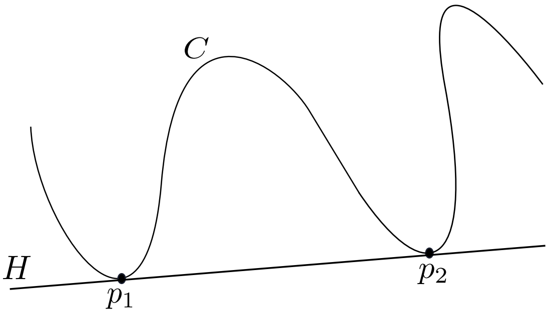
$$a_1 + \ldots + a_n = d$$

for some hyperplane $H \subset \mathbb{P}^r$

Multitangency conditions and de Jonquières divisors

$\mathbb{P}^2$

$a_1 = a_2 = 2 \Rightarrow$ counting bitangent lines



$a_1 = \ldots = a_m = 2 \Rightarrow$ counting m-tangent lines

$a_1 = 3 \Rightarrow$ counting flex points

$a_1 = 4 \Rightarrow$ counting hyperflexes

Multitangency conditions and de Jonquières divisors

The (virtual) de Jonquières numbers are the coefficients of

$$t_1 \cdot \ldots \cdot t_n$$

in

$$(1 + a_1^2 t_1 + \ldots + a_n^2 t_n)^g (1 + a_1 t_1 + \ldots + a_n t_n)^{d-r-g}$$

Multitangency conditions and de Jonquières divisors

Space of all divisors of degree d on C

$$C_d = \underbrace{C \times \ldots \times C}_{d \text{ times}} / S_d$$

For example

$$p_1 + 2p_2 \in C_3$$

We define de Jonquières divisors

$$p_1 + \ldots + p_n \in C_n$$

such that

$$f^*H = a_1 p_1 + \ldots + a_n p_n \in C_d$$

Multitangency conditions and de Jonquières divisors

$g^r_d = (L, V)$ with $V \subset H^0(L)$ and $\dim V = r + 1$

$D = p_1 + \ldots + p_n$ is de Jonquières divisor

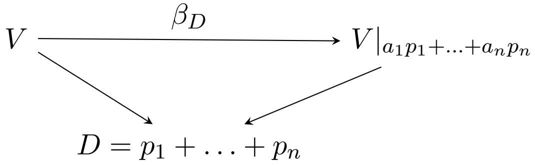
$$\Leftrightarrow$$

the map $V \xrightarrow{\beta_D} V|_{a_1 p_1 + \ldots + a_n p_n}$ has kernel

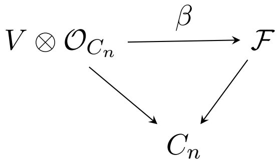
$$\Leftrightarrow$$

$$rk(\beta_D) \leq \dim V - 1 = r$$

Multitangency conditions and de Jonquières divisors



Multitangency conditions and de Jonquières divisors



De Jonquières divisors:

$$DJ_n = \{D \in C_n \mid rk(\beta_D) \leq r\}$$

Multitangency conditions and de Jonquières divisors

$$\dim DJ_n \geq n - d + r$$

Relevant questions

- $n - d + r < 0 \Rightarrow$ non-existence of de Jonquières divisors
- $n - d + r \geq 0 \Rightarrow$ existence of de Jonquières divisors
- $n - d + r = 0 \Rightarrow$ finite number of de Jonquières divisors
- $\dim DJ_n = n - d + r$

Why do we care?

$L = K_C$

Diaz ('84), Polishchuk ('03), Farkas-Pandharipande ('15), Bainbridge-Chen-Gendron-Grushevsky-Möller ('16), ...

Fix partition $\mu = (a_1, \ldots, a_n)$ of $d = 2g - 2$

$\mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n) \mid$
K_C admits the de Jonquières divisor $a_1 p_1 + \ldots a_n p_n\}$

Why do we care?

$\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$

- $\mathcal{H}_g(\mu)$ has expected dimension
- compactification $\widetilde{\mathcal{H}}_g(\mu)$ has expected codimension in $\overline{\mathcal{M}}_{g,n}$
- fundamental class of $\widetilde{\mathcal{H}}_g(\mu)$ related to Pixton tautological class

What if $L \neq K_C$?