From de Jonquières’ Counts to Cohomological Field Theories

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Women at the Intersection of Mathematics and High Energy Physics
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What is Enumerative Geometry?

How many geometric structures of a given type satisfy a given collection of geometric conditions?
What is Enumerative Geometry?

Appolonius’ problem (approx. 200 BC)

8 circles tangent to 3 other circles
What is Enumerative Geometry?

- **3264** conics tangent to 5 given conics (1864 Chasles)
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- **609,250** conics on a quintic threefold (1985 Katz)
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- **27** lines on cubic surface (1849 Cayley-Salmon)
- **2875** lines on a quintic threefold (1886 Schubert)
- **609.250** conics on a quintic threefold (1985 Katz)
- **317.206.375** cubics on a quintic threefold (1991 Ellingsrud-Strømme)
What is Enumerative Geometry?

Question

Number of rational curves of any degree on quintic threefold?
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**Answer**
Mirror symmetry!
1991 Candelas, de la Ossa, Green, Parkes
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- 1995 Kontsevich
- 1996 Givental
- Clemens conjecture?
What is Enumerative Geometry?

Question
How many points in the plane lie at the intersection of two given lines?
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How many points in the plane lie at the intersection of two given lines?

Answer
It depends!
- Lines in general position $\Rightarrow$ exactly one
- Parallel lines $\Rightarrow$ none
- Lines coincide $\Rightarrow$ an infinite number of points
What is Enumerative Geometry?

- Parameter (moduli) space
- Compactify!
- Do excess intersection theory
Ernest de Jonquières
de Jonquières’ multitangency conditions

$C$ smooth, genus $g$
$f : C \to \mathbb{P}^r$ non-degenerate
Degree of $f = \# \{ f(C) \cap H \} =: d$
de Jonquières’ multitangency conditions

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\[ d = 3 \]
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Degree of $f = \# \{ f(C) \cap H \} =: d$

\[
\begin{align*}
  f^{-1}\{ f(C) \cap H \} &= p_1 + 2p_2
\end{align*}
\]
de Jonquières’ multitangency conditions

de Jonquières counts the number of pairs \((p_1, p_2)\) such that there exists a hyperplane \(H \subset \mathbb{P}^r\) with

\[
f^{-1}\{f(C) \cap H\} = p_1 + 2p_2
\]
de Jonquières’ multitangency conditions

de Jonquières (and Mattuck, Macdonald) count the $n$-tuples

$$(p_1, \ldots, p_n)$$

such that there exists a hyperplane $H \subset \mathbb{P}^r$ with

$$f^{-1}\{f(C) \cap H\} = a_1 p_1 + \ldots + a_n p_n$$

where

$$a_1 + \ldots + a_n = d$$
de Jonquières’ multitangency conditions

The (virtual) de Jonquières numbers are the coefficients of

\[ t_1 \cdot \ldots \cdot t_n \]

in

\[ (1 + a_1^2 t_1 + \ldots + a_n^2 t_n)^g (1 + a_1 t_1 + \ldots + a_n t_n)^{d-r-g} \]
Constructing the moduli space

An embedding $f : C \to \mathbb{P}^r$ of degree $d$ is given by

A pair $(L, V)$
- a line bundle $L$ of degree $d$ on $C$
- an $(r + 1)$-dimensional vector space $V$ of sections of $L$
An embedding $f : C \rightarrow \mathbb{P}^r$ of degree $d$ is given by

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Choose $(\sigma_0, \ldots, \sigma_r)$ basis of $V$

$\downarrow$

$f : C \rightarrow \mathbb{P}^r$

$p \mapsto [\sigma_0(p) : \ldots : \sigma_r(p)]$
Constructing the moduli space

Space of all divisors of degree $d$ on $C$

$$C_d = \underbrace{C \times \ldots \times C}_{d \text{ times}} / S_d$$

For example

$$p_1 + 2p_2 \in C_3$$
Constructing the moduli space

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For example

\[ p_1 + 2p_2 \in C_3 \]

We define de Jonquières divisors

\[ p_1 + \ldots + p_n \in C_n \]

such that

\[ f^{-1}\{f(C) \cap H\} = a_1 p_1 + \ldots + a_n p_n \]
Constructing the moduli space

\[ D = p_1 + \ldots + p_n \] is de Jonquières divisor

\[ \uparrow \]

there exists a section \( \sigma \) whose zeros are

\[ a_1 p_1 + \ldots + a_n p_n \]
Constructing the moduli space

\[ D = p_1 + \ldots + p_n \] is de Jonquières divisor

\[ \iff \]

there exists a section \( \sigma \) whose zeros are

\[ a_1 p_1 + \ldots + a_n p_n \]

\[ \iff \]

the map

\[ \beta_D : V \rightarrow V|_{a_1 p_1 + \ldots + a_n p_n} \]

\[ \sigma \mapsto \sigma|_{a_1 p_1 + \ldots + a_n p_n} \]

has nonzero kernel
Constructing the moduli space

the map

$$\beta_D : V \to V|_{a_1p_1 + \ldots + a_np_n}$$
$$\sigma \mapsto \sigma|_{a_1p_1 + \ldots + a_np_n}$$

has nonzero kernel

$$\uparrow$$

$$\text{rank}(\beta_D) \leq \dim V - 1 = r$$
Constructing the moduli space

\[ V \xrightarrow{\beta_D} V|_{a_1p_1 + \ldots + a_np_n} \]

\[ D = p_1 + \ldots + p_n \in C_n \]
Constructing the moduli space

De Jonquières divisors:

\[ DJ_n = \{ D \in C_n \mid \text{rank}(\beta_D) \leq r \} \]

determinantal variety
To summarise...

- Fix curve $C$ of genus $g$
- Fix embedding given by $(L, V)$
- $C_n :=$ space of divisors of degree $n$
- Defined de Jonquières divisors via multitangency conditions
- Described space $DJ_n$ of de Jonquières divisors as determinantal variety over $C_n$
Analysing the moduli space

\[ \dim DJ_n \geq n - d + r \]
Analysing the moduli space

$$\dim DJ_n \geq n - d + r$$

Relevant questions

- $n - d + r < 0 \Rightarrow$ non-existence of de Jonquières divisors
- $n - d + r \geq 0 \Rightarrow$ existence of de Jonquières divisors
- $n - d + r = 0 \Rightarrow$ finite number of de Jonquières divisors
- $\dim DJ_n = n - d + r$
Taking a variational perspective

- Allow $C$ to vary in $\mathcal{M}_{g,n}$
- Vary the de Jonquières structure with it
Taking a variational perspective

\( \mathcal{M}_{g,n} \) = moduli space of smooth curves of genus \( g \) with \( n \) marked points

\[(C; p_1, \ldots, p_n) \in \mathcal{M}_{g,n}\]

- \( \dim \mathcal{M}_{g,n} = 3g - 3 + n \)
- compactification \( \overline{\mathcal{M}}_{g,n} \)

What is the cohomology of \( \mathcal{M}_{g,n} \)?
Taking a variational perspective

\[ \mathcal{M}_{g,n} = \text{moduli space of smooth curves of genus } g \text{ with } n \text{ marked points} \]

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- compactification \( \overline{\mathcal{M}}_{g,n} \)

Question
What is the cohomology of \( \overline{\mathcal{M}}_{g,n} \)?
Taking a variational perspective

\[ L = K_C = \text{bundle of differential forms on } C \]
\[(L, V) = (K_C, \Gamma(C, K_C))\]
Now \( d = 2g - 2 \) and \( r = g - 1 \)
Taking a variational perspective

\[ L = K_C = \text{bundle of differential forms on } C \]
\[(L, V) = (K_C, \Gamma(C, K_C))\]
Now \(d = 2g - 2\) and \(r = g - 1\)

Fix partition \(\mu = (a_1, \ldots, a_n)\) of \(2g - 2\)

\[ \mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n) \text{ such that } K_C \text{ admits the de Jonquières divisor } a_1p_1 + \ldots + a_np_n\} \]
Taking a variational perspective

\[ \mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n} \text{ determinantal subvariety} \]

- flat surfaces, dynamical systems, Teichmüller theory: Masur, Eskin, Zorich, Kontsevich,... Bainbridge-Chen-Gendron-Grushevsky-Möller ('16), ...
- algebraic geometry: Diaz ('84), Polishchuk ('03), Farkas-Pandharipande ('15)
Taking a variational perspective

\[ \mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n} \]
Take closure: \( \overline{\mathcal{H}_g(\mu)} \subset \overline{\mathcal{M}}_{g,n} \)

**Question**
What is the fundamental class \([\overline{\mathcal{H}_g(\mu)}]\)?

**Answer**
(potentially) Cohomological field theory!
$\mathcal{M}_{g,n}$ and 2-dimensional CFT

CFT
- 2-dimensional QFT invariant under conformal transformations
- defined over compact Riemann surfaces

Stick to holomorphic side

CFT = 2-dimensional QFT covariant w.r.t. holomorphic coordinate changes
\[ M_{g,n} \text{ and 2-dimensional CFT} \]

Infinitesimal change of holomorphic coordinate

\[ z \mapsto z + \epsilon f(z) \]

Local holomorphic vector field

\[ f(z) \frac{d}{dz} \]
\( \mathcal{M}_{g,n} \) and 2-dimensional CFT

Infinitesimal change of holomorphic coordinate

\[
z \mapsto z + \epsilon f(z)
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Local meromorphic vector field

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f(z) \frac{d}{dz}
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Infinitesimal change of holomorphic coordinate

\[ z \mapsto z + \epsilon f(z) \]

Local meromorphic vector field

\[ f(z) \frac{d}{dz} \]

\[ \Downarrow \]

Virasoro algebra:

\[ L_n = -z^{n+1} \frac{d}{dz} \Rightarrow [L_n, L_m] = (m - n)L_{m+n}, n \in \mathbb{Z} \]

etc...
\( \overline{M}_{g,n} \) and 2-dimensional CFT

Local meromorphic vector field

\[ f(z) \frac{d}{dz} \]

\( \uparrow \)

Infinitesimal deformation of complex structure

\( \uparrow \)

Infinitesimal deformation of an algebraic curve
\( \overline{\mathcal{M}}_{g,n} \) and 2-dimensional CFT

\[
(C; p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}
\]

\( \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \) representation labels

\( V_{\vec{\lambda}}(C; p_1, \ldots, p_n) \) space of conformal blocks
$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

$(C; p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$

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$V_{\vec{\lambda}}(C; p_1, \ldots, p_n)$ space of conformal blocks
$\overline{M}_{g,n}$ and 2-dimensional CFT

$(\tilde{C}; p_1, \ldots, p_n, q_+, q_-) \in \overline{M}_{g,n+2}$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n, \lambda, \lambda^\dagger)$ representation labels

$V_{\vec{\lambda}}(\tilde{C}; p_1, \ldots, p_n, q_+, q_-)$ space of conformal blocks
\( \overline{\mathcal{M}}_{g,n} \) and 2-dimensional CFT

Verlinde bundle

\[
\mathcal{V}_{\lambda} \rightarrow \overline{\mathcal{M}}_{g,n}
\]

Each fibre is given by space of conformal blocks

\[
\mathcal{V}_{\lambda}(C; p_1, \ldots, p_n) \rightarrow (C; p_1, \ldots, p_n)
\]
The characters $\text{ch}(\mathcal{V}_\chi)$ define a CohFT on $\overline{\mathcal{M}}_{g,n}$!
The characters $ch(V_{\chi})$ define a CohFT on $\overline{M}_{g,n}$!

A CohFT

- a vector space of fields $U$
- a non-degenerate pairing $\eta$
- a distinguished vector $1 \in U$
- a family of correlators

$$\Omega_{g,n} \in H^*(\overline{M}_{g,n}, \mathbb{Q}) \otimes (U^*)^\otimes n$$

satisfying gluing...
Quantum multiplication $\ast$ on $U$

$$\eta(v_1 \ast v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) \in \mathbb{Q}$$

$(U, \ast)$ Frobenius algebra of the CohFT

Teleman: classification of all CohFT with semisimple Frobenius algebra
$\overline{\mathcal{M}}_{g,n}$ and CohFT

$\mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n) \text{ such that } K_C \text{ admits the de Jonquières divisor } a_1p_1 + \ldots + a_np_n\}$

$[\overline{\mathcal{H}}_g(\mu)] = ?$

Maybe $[\overline{\mathcal{H}}_g(\mu)]$ is one of the $\Omega_{g,n}$
$\overline{\mathcal{H}}_g(\mu)$ and CohFT

$[\overline{\mathcal{H}}_g(\mu)]$ is not a CohFT class!

Conjecture (Pandharipande, Pixton, Zvonkine): it is related to one Witten $R$-spin class

$$W_{g,\mu}^R \in H^{2g-2}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$
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- Tour of enumerative geometry
- Described de Jonquières divisors on fixed curve $C$ with fixed embedding $(L, V)$
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- Obtained subspace of $\overline{M}_{g,n}$ for particular case $L = K_C$

What if $L \neq K_C$?