Mixed Hodge Modules for Generic Vanishing

1 Why MHM for generic vanishing?

Recall Hacon's proof for the generic vanishing in the canonical case. Let X be a smooth complex projective variety and $A := Alb(X)$ its Albanese variety. Denote by

$$a : X \to A$$

the corresponding Albanese mapping and by $\hat{A} = \mathrm{Pic}^0(A)$ the dual abelian variety. Using Kollár's result on the splitting of the direct image

$$Ra_*\omega_X$$

in $D^b_{coh}(\mathcal{O}_A)$ and some other technical results, one concludes that it is enough to prove that

$$\operatorname{codim} V^l(R^i a_* \omega_X) \geq l \tag{1}$$

for all $i = 0, 1, \dots, k = \dim X - \dim a(X)$. Expressed in terms of the Fourier-Mukai transform

$$R\phi_p : D^b_{coh}(\mathcal{O}_A) \to D^b_{coh}(\mathcal{O}_{\hat{A}}),$$

the inequality (1) is equivalent to

$$\operatorname{codim} Supp R^l \phi_p(R^i a_* \omega_X) \geq l, \tag{2}$$

for all $i = 0, 1, \dots, k$. Using the fact that the sheaves $R^i a_* \omega_X$ satisfy a Kodaira-type vanishing theorem and that $\hat{A}$ is an abelian variety, we have that $R^i a_* \omega_X$ is the dual of a sheaf $\mathcal{F}_i$ on $\hat{A}$, in other words

$$R\phi_p(R^i a_* \omega_X) \simeq R\mathcal{H}om(\mathcal{F}_i, \mathcal{O}_{\hat{A}}).$$

Therefore

$$\operatorname{codim} Supp R^l \phi_p(R^i a_* \omega_X) = Ext^l(\mathcal{F}_i, \mathcal{O}_{\hat{A}}) \geq l.$$

the inequality is now a consequence of a theorem on regular local rings. Thus we have generic vanishing for topologically trivial line bundles.

The idea is now to get generic vanishing for general objects of Hodge-theoretic origin. The strategy is as follows:

- use Saito's decomposition theorem instead of Kollár's
- use a Kodaira-type vanishing theorem for MHM (also thanks to Saito), which becomes useful on abelian varieties. This allows one to generalise the second part of the proof to any coherent sheaf of Hodge-theoretic origin on an abelian variety.

2 Hodge structures

The basic idea of Hodge theory is that the cohomology of an algebraic variety has more structure than one sees when viewing the same object as a bare topological space.

Recall that a *(polarised) pure Hodge structure of weight k* consists of

- a finite dimensional $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ satisfying the decomposition

$$H := H_{\mathbb{Q}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

and

$$H^{p,q} = \overline{H^{q,p}},$$

where the conjugation is relative to the real structure on H.

- a polarisation, i.e. a quadratic form Q on $H_{\mathbb{Q}}$ satisfying the following Hodge-Riemann relations

$$\begin{aligned} &Q(u,v) = (-1)^m Q(v,u) \\ &Q(H^{p,q}, H^{r,s}) = 0 \text{ unless } p = s, q = r \\ &\sqrt{-1}^{p-q} Q(u, \bar{u}) > 0 \text{ for } u \in H^{p,q}, u \neq 0. \end{aligned}$$

Example 1. The cohomology of an n-dimensional projective algebraic manifold X satisfies the Hodge decomposition, and its polarised part is given by

$$H_{prim} := \ker(H^n(X) \to H^n(Y)),$$

where Y is a smooth hyperplane section.

An equivalent notion to the Hodge decomposition is the *Hodge filtration*

$$F^p = \bigoplus_{i \geq p} H^{i,k-i}$$

which satisfies, for a weight-k structure, the relation

$$H = F^p \oplus \overline{F^{k-p+1}}.$$

Note that the Hodge filtration is not just book keeping! It turns out that for a family of projective algebraic manifolds the Hodge filtration varies holomorphically, while the Hodge decomposition does not.

Example 2. In the case of a Riemann surface (i.e. an algebraic curve) we have

$$F^0 = H^1, F^1 = H^{1,0}$$

and moreover $H^1 = H^{1,0} + H^{0,1}$, or equivalently.

$$H = F^1 + \overline{F^1} = F^1 + H^{0,1}.$$

Deligne extended this theory to the case of singular varieties. The key notion here is that of a *mixed Hodge structure*. This consists of a triple $(H, F^{\cdot}, W_{\cdot})$ such that

- H is a finite dimensional $\mathbb{Q}$-vector space
- $F^{\cdot}$ is a finite decreasing filtration
- $W_{\cdot}$ is a finite increasing filtration

and $\left(Gr_k^W H := W_k/W_{k-1}, F^{\cdot}\right)$ is a pure Hodge structure.

Example 3. We illustrate the concept with a nodal curve S_0 of genus 1. In its homology, we have an obvious choice of 1-cycles

- one cycle going through the node
- two cycles going through the genus.

These cycles generate the whole homology H^1 and give a basis for it. Take the dual basis. Note that cannot carry a Hodge structure of weight 1 because such a structure would have to have an even dimension, as follows from the Hodge decomposition

$$H^1 = H^{0,1} + H^{1,0}.$$

However, it still carries a lot of structure. Consider the normalisation map

$$p : \tilde{S}_0 \to S_0.$$

It induces a surjective map on cohomology

$$p^* : H^1(S_0) \to H^1(\tilde{S}_0),$$

whose kernel K is a $\mathbb{Z}$-module generated by γ_1. Therefore

$$H^1(S_0)/K \simeq H^1(\tilde{S}_0).$$

But $H^1(\tilde{S}_0)$ now has a pure Hodge structure determined by the filtration

$$F^1 = H^{1,0} \subset H^1(\tilde{S}_0)(\text{ or } H^1(S_0)) \text{ and } F^0 = H^1.$$

3 Variations of Hodge Structures

Griffiths first considered Hodge structures in order to study the following geometric situation: let X and S be two connected complex manifolds and $f : X \to S$ a surjective, proper, holomorphic map with connected fibres everywhere of maximal rank. Then the fibre $X_t = f^{-1}(t) \subset X$ is a compact complex manifold. Suppose moreover that each fibre is a polarised abelian variety so that each fibre comes with a cohomology class $\eta_n \in H^2(X_t, \mathbb{Z})$ of a projective embedding. Therefore the collection $\{\eta_n\}$ is a section of $R^2 f_*(\mathbb{Z})$ and so we have a family of polarised algebraic manifolds.

Viewing $f : X \to S$ as a C^∞-fibre bundle, we also have a flat complex vector bundle $H^k_{\mathbb{C}} \to S$ with fibres $H^k(X_t, \mathbb{C})$. We analogously define the subbundles

$$H^k_{\mathbb{Z}} \subset H^k_{\mathbb{R}} \subset H^k_{\mathbb{C}}.$$

Grauert then tells us that there exist subbundles $H^{p,q} \subset H^k_{\mathbb{C}}$ with fibres $H^{p,q}(X_t)$ over $t \in S$. The Hodge filtration gives yet another subbundle

$$F^p = \bigoplus_{i \geq p} H^{i,k-i} \subseteq H^k_{\mathbb{C}}.$$

Griffiths showed that in fact F^p are holomorphic vector bundles and moreover they satisfy the following *transversality* condition

$$\forall p, \nabla \mathcal{O}(F^p) \subset \mathcal{O}(F^{p-1} \otimes \Omega^1_S),$$

where $\mathcal{O}(-)$ denotes the sheaf of germs of holomorphic sections. It turns out that these properties make it so that the bundle F^p are the right way of formulating variations of Hodge structure. We now give the proper definition of the variations.

Definition 1. A *variation of Hodge structure of weight k* on a smooth complex variety X is a collection of the following data:

1. a flat vector bundle (V, ∇),
2. a local system $V_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces with the isomorphism of flat vector bundles
$$V \simeq V_{\mathbb{Q}} \otimes_{\mathbb{Q}}$$
given by the de Rham functor.
3. a decreasing Hodge filtration $F^\bullet$ on V satisfying Griffiths transversality
$$\nabla(F^p V) \subset F^{p-1} V \otimes \Omega^1_S).$$

Remark. We note here that the interpretation of Griffiths transversality is that it shows that the filtration is not flat with respect to ∇ and therefore it does not descend to a filtration of $V_{\mathbb{Q}}$.

Going back to the example at the beginning of this section, we see that in that case we have
$$V_{\mathbb{Q}} = R^k f_*(\mathbb{Q})$$
$$V = R^k f_*(\mathbb{Q}) \otimes \mathcal{O}_S = H^k_{dR}(X/S).$$

Of course, the following question arises: what if the family $X \to S$ has smooth fibres away from a closed set $V \subset S$? Then generically $H^k(X_s)$ (or $R^k f_*(\mathbb{Q})$) underlie a *variation of a mixed Hodge structure*. But what do we mean by that exactly?

In the smooth case we used the Riemann-Hilbert correspondence, i.e. the following equivalence of categories:
$$\{\text{vector bundles with flat connections}\} \leftrightarrow \{\text{local systems}\}$$
given by the de Rham functor.

In the singular case we have instead an equivalence of derived categories
$$\{\text{regular holonomic } D\text{-modules}\} \leftrightarrow \{\text{constructible sheaves}\}$$
given again by the de Rham functor which in this case has the following form
$$DR(M) = [M \to \Omega^1_X \otimes M \to \ldots \to \Omega^n_X \otimes M]$$
in degrees $-n, \ldots, 0$. Let us now unpack the definitions behind this correspondence.

We start with the *constructible sheaves*. These are sheaves that can be built up from local systems. More precisely, let X be a complex manifold. Then a $\mathbb{C}_X$-module M is *constructible* if there exists a stratification
$$\bigsqcup X_i = X$$
such that each stratum X_i is a connected non-empty set, and the restriction M_{X_i} is a local system. In the derived category, a complex $M^\bullet$ is constructible if all its coherent sheaves are constructible. If X were a smooth variety, then a $\mathbb{C}_X$-module M would be constructible if there existed a stratification by algebraic varieties such that M was a constructible $\mathbb{C}_X$-analytic module.

We now turn to the D-module side. Our guiding example is the case of a locally free sheaf of $\mathcal{O}_X$-modules of finite rank M. Then M is a sheaf of sections of a vector bundle on V and flat connections on this vector bundle correspond to D-module structures on M. However, in general, M is not a sheaf of sections, but we still have the correspondence between D-module structures on M and the existence of a $\mathbb{C}$-linear morphisms $\nabla : T_X \to End(M)$ with the usual properties... Let us now define *regular holonomic D-modules* for the case when X is a smooth curve. To begin with, holonomicity refers to the fact that the system of PDEs determined by the D-module is maximally determined, i.e. there is an equivalence of categories

$$\{\text{holonomic } D\text{-modules}\} \leftrightarrow \{\text{systems of PDEs with finite-dimensional space of solutions}\}.$$

A holonomic D_X-module M is called *regular* if there exists an open dense set $X \subseteq X$ such that $M|_U$ is a regular integrable connection. What we mean by this is that the meromorphic connection induced from $M|_U$ is regular at every point $x \in \overline{X} \setminus X$, where $\overline{X}$ denotes the smooth completion of X. Even more precisely, it means that given the open embedding $j : X \to \overline{X}$, the stalk $(j_* M)_x$ can be endowed with the structure of a meromorphic connection at x:

$$\nabla(m) = d\xi \otimes \partial m,$$

where (ξ, ∂) are local coordinated for $D_{\overline{X}}$ and $m \in (j_* M)_x$.

Denote by $D^b_{rh}(D_X)$ the derived category of complexes with regular holonomic cohomology and by $Mod_{rh}(D_X)$ its full subcategory of regular holonomic D-modules. In fact $Mod_{rh}(D_X)$ is the heart of $D^b_{rh}(D_X)$. Since the de Rham functor is t-exact, it makes sense to consider its image $DR(Mod_{rh})$. We define *perverse sheaves* to be exactly this image of $Mod_{rh}(D_X)$ under the Riemann-Hilbert correspondence.

4 Hodge Modules

Hodge modules give a generalisation of the variations of Hodge structures. We can think of them as of a sheaf of Hodge structures on a manifold. In fact one constructs them inductively, using as base case a variation of Hodge structure on a dense open subset U of X. One also needs some gluing data on the complements $Z = X \setminus U$, together with compatibility conditions. Due to technical reasons, this extra data comes in the form of certain filtrations on the nearby and vanishing cycle complexes of M whose k-th graded parts are themselves weight k Hodge modules supported on the singular loci. As a word of caution, we note here that this induction actually constructs the category of Hodge modules, and Hodge modules are therefore objects of this category.

Essentially, a Hodge module is a filtered regular holonomic D-module M with an underlying rational structure (a perverse sheaf). Let us first define the notion of *rational filtered D-modules*. This is a category closed under Verdier duality and it consists of the following

- a regular holonomic D-module
- a perverse sheaf of $\mathbb{Q}$-vector spaces $M_{\mathbb{Q}}$ together with the isomorphisms

$$DR(M) \simeq M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

- a good filtration F_0 on M.

We will not define exactly what a good filtration is, suffice to say that it is a generalisation of the Griffiths transversality condition that we saw in the definition of variations of Hodge structures. Recall that Griffiths transversality of a filtration meant that

$$\nabla(F^p V) \subset F^{p-1} V \otimes \Omega^1.$$

In particular, for a vector field v, we have that

$$\nabla_v(F^p V) \subset F^{p-1} V.$$

Then $F_{-p} := F^p V$ is an increasing filtration compatible with the sheaf of differential operators D by degree. This is a prototype for a good filtration.

Example 4. The main example of filtered regular holonomic D-modules is, of course, the variation of Hodge structure.

In order to perform the induction, we need the notion of *strict support*:

Definition 2. Let $Z \subset X$ be an irreducible subvariety. A D-module M has strict support Z if the support of every nonzero subobject or quotient object of M is equal to Z.

We now inductively define our object of interest:

Definition 3. The category $HM(x, n)$ of *(pure) Hodge modules of weight n* is the largest full subcategory of the category of rational filtered D-modules (plus some other technical things) such that

1. Hodge modules of weight n supported on points are pure (rational) Hodge structures of weight n.
2. For all Zariski open $U \subset X$, $f : U \to \mathbb{C}$ a non-constant holomorphic function, and M having support not contained in $f^{-1}(0)$, if $M \in HM(X, n)$, then the weight filtrations of the D-modules of nearby and vanishing cycles are also Hodge modules of the appropriate weight supported on $f^{-1}(0)$.

With this definition, $HM(X, n)$ is an abelian category, and moreover

$$HM(X, n) = \bigoplus_{Z \subset X} HM_Z(X, n),$$

where $HM_Z(X, n)$ is the category of Hodge modules with strict support on Z.

We mentioned earlier that Hodge modules are generalisations of variations of Hodge structures. More precisely, Saito proves that any weight m variation of Hodge structures over an open subset of a closed subset

$$U \xrightarrow{j} Z \xrightarrow{i} X$$

can be extended to a Hodge module in $HM_Z(X, n) \subset HM(X, n)$, where $n = \dim Z + m$. The underlying perverse sheaf of the extension is the associated intersection cohomology complex

$$i_* j_{*!} L[\dim U],$$

where L is the perverse sheaf of the variation.

Example 5. For an easy example, consider the trivial Hodge module $Q^H[n]$. The underlying D-module is $\mathcal{O}_X$ and it has the trivial filtration $F_0 \mathcal{O}_X = \mathcal{O}_X$. The perverse sheaf $Q[n]$ is the constant local system in degree $-n$.

5 Mixed Hodge Modules

As in the case of pure Hodge modules, we describe the category of mixed Hodge moduleson X, denoted $MHM(X)$. Instead of constructing it from scratch, we give its most important properties:

1. There exist forgetful functors

$$MHM(X) \to Perv(X, \mathbb{Q})$$
$$MHM(X) \to Mod_{rh}(X)$$

   compatible with the Riemann-Hilbert correspondence.

2. The objects in $MHM(X)$ admit a weight filtration $W_\bullet$ such that $gr_k^W M \in HM(X,k)$.

3. $MHM(pt)$ is the category of (polarisable) mixed Hodge structures.

4. Since the construction is also inductive, we need some properties pertaining to the gluing data. For this, consider first the Tate Hodge structure (of weight 0) $\mathbb{Q}^H \in MHM(pt)$. Then the map $f : X \to \{pt\}$ induces a complex $f_* f^* \mathbb{Q}^H$ of mixed Hodge modules which means that we have a mixed Hodge structure on the cohomology $H^*(X,\mathbb{Q})$.

6 The Decomposition Theorem

We now return to pure Hodge modules. Let $f : X \to Y$ be a projective morphism. Then Saito proves that if M is a Hodge module, then

1. $R^k f_* M$ is a Hodge module of weight $n+k$ on Y.

2. Hard Lefschetz analogue: the Lefschetz operator induces the isomorphism

$$R^{-i} f_* M \to R^i f_* M[i].$$

3. The primitive part of $R^{-i} f_* M$ is a polarised Hodge module.

We recall also Deligne's result which shows that the Hard Lefschetz implies

$$Rf_* M \simeq \bigoplus_i R^i f_* M[-i].$$

The last ingredient needed for the decomposition is a structure theorem for polarisable Hodge modules: again, Saito proves that any polarisable Hodge module of weight n on X is an intermediate extension of a variation of Hodge structures of weight $n - \dim Z$ on a smooth open subset of Z for some closed subvariety $Z \subset X$.

Given a variation of Hodge structure V on an open subset of Z, denote by $IC_Z(V)$ the intermediate extension as a Hodge module on X.

Theorem 1. *Let $f : X \to Y$ be a projective morphism. Then*

$$Rf_* IC_X(V) = \bigoplus_i R^i f_* IC_X(V)[-i]$$
$$R^i f_* IC_X(V) \simeq \bigoplus_{Z \subset X} IC_Z(V_Z^i)$$

for some variation of Hodge structure V_Z^i.

We also have the following vanishing results, also due to Saito. The proof follows after a reduction to perverse sheaves.

Theorem 2. *Let L be an ample line bundle on X a smooth projective variety, and consider the pair $(M,F) \in MHM(X)$. Then*

$$H^i(X, gr_k^F DR(M) \otimes L) = 0, \forall i > 0$$
$$H^i(X, gr_k^F DR(M) \otimes L^{-1}) = 0, \forall i < 0.$$

Example 6. Let us consider the trivial case $M = \mathbb{Q}^H[n]$ on X with $\mathcal{M} = \mathcal{O}_X$. Then we have

$$gr^F_{-n} DR(\mathcal{O}_X) = \omega_X,$$

and the vanishing theorem is simply Kodaira vanishing. On the other hand

$$gr^F_{-p} DR(\mathcal{O}_X) = \Omega^p_X(n-p)$$

and the vanishing theorem reduces to Nakano vanishing.