Billiards, cohomological field theories, and de Jonquières divisors

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Billiards in convex polygons

- Ideal billiard ball
- Mass concentrated at one point
- No friction, no spin
- Optical rule
Billiards in convex polygons
Billiards in rational polygons

Rational polygon: all angles are rational multiples of $\pi$

- Many tools available
- Connections with algebraic geometry, Teichmüller theory, ...

Motivation: the group generated by the reflections of a rational polygon is finite
Unfolding rational polygons
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Unfolding rational polygons
Surfaces from polygons

Glue identified edges of polygon $\Leftrightarrow$ surface with flat metric away from some (conical) singularities

Singularities arise from corners of polygon

Angle around singularities is integer multiple of $2\pi$
Surfaces from polygons

\[ \alpha = \frac{\pi}{8} \text{ and } \beta = \frac{3\pi}{8} \]
Surfaces from polygons

\[ \gamma = 2 \times \frac{3\pi}{8} = \frac{3\pi}{4} \]

\[ 8 \times \frac{3\pi}{4} = 6\pi = 3 \times 2\pi \]
Surfaces from polygons

Gauss-Bonnet type theorem:
- $n$ singularities with angle $(a_i + 1)2\pi$
- $\sum_{i=1}^{n} a_i = 2g - 2$

Take complex coordinate $z$ on surface

$$\omega := p(z)dz$$

$\omega$ vanishes at conical singularities with order $a_i$
Surfaces from polygons

\[ \gamma = 2 \times \frac{3\pi}{8} = \frac{3\pi}{4} \]

\[ 8 \times \frac{3\pi}{4} = 6\pi = (2 + 1)2\pi \]

\[ 2g - 2 = 2 \Rightarrow g = 2 \]
Strata of holomorphic differentials

\[ \mathcal{H}_g(\mu) \]

- Riemann surface \( C \) of genus \( g \)
- \( n \) conical singularities \( p_i \) with angles \( (a_i + 1)2\pi \)
- (or with differential \( \omega \) vanishing at \( p_i \) at order \( a_i \))
- \( \mu = (a_1, \ldots, a_n) \) partition of \( 2g - 2 \)
Strata of holomorphic differentials

Kontsevich and Zorich: $\mathcal{H}_g(\mu)$ has at most three connected components

Lelièvre, Monteil, Weiss: there are at most finitely many points $y$ on the polygon not reachable by a billiard trajectory from an arbitrary point $x$
To summarise

- Billiards in polygons
- Riemann surface of genus $g$ with $n$ conical singularities
- Riemann surface of genus $g$ with differential vanishing at $n$ points with prescribed order $a_i$ such that
  \[ \sum a_i = 2g - 2 \]
- Strata of differentials $\mathcal{H}_g(\mu)$
Studying $\overline{\mathcal{M}}_{g,n}$

$\mathcal{M}_{g,n} =$ moduli space of Riemann surfaces of genus $g$ with $n$ marked points

$$(C; p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$$

- $\dim \mathcal{M}_{g,n} = 3g - 3 + n$
- compactification $\overline{\mathcal{M}}_{g,n}$
Studying $\overline{M}_{g,n}$

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**Question**

What is the cohomology of $\overline{\mathcal{M}}_{g,n}$?
Studying $\overline{\mathcal{M}}_{g,n}$

$\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$

Take closure: $\overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$

**Question**
What is the fundamental class $[\overline{\mathcal{H}}_g(\mu)]$?

**Answer**
(potentially) Cohomological field theory!
\( \mathcal{M}_{g,n} \) and 2-dimensional CFT

CFT

- 2-dimensional QFT invariant under conformal transformations
- defined over compact Riemann surfaces

Stick to holomorphic side

CFT = 2-dimensional QFT covariant w.r.t. holomorphic coordinate changes
$\overline{M}_{g,n}$ and 2-dimensional CFT

Infinitesimal change of holomorphic coordinate

$$z \mapsto z + \epsilon f(z)$$

Local holomorphic vector field

$$f(z) \frac{d}{dz}$$
\( M_{g,n} \) and 2-dimensional CFT

Infinitesimal change of holomorphic coordinate

\[ z \mapsto z + \epsilon f(z) \]

Local meromorphic vector field

\[ f(z) \frac{d}{dz} \]
\(M_{g,n}\) and 2-dimensional CFT

Infinitesimal change of holomorphic coordinate

\[ z \mapsto z + \epsilon f(z) \]

Local meromorphic vector field

\[ f(z) \frac{d}{dz} \]

\[ \Downarrow \]

Virasoro algebra:

\[ L_n = -z^{n+1} \frac{d}{dz} \Rightarrow [L_n, L_m] = (m - n)L_{m+n}, n \in \mathbb{Z} \]

dec...
$\overline{M}_{g,n}$ and 2-dimensional CFT

Local meromorphic vector field

$$f(z) \frac{d}{dz}$$

\updownarrow

Infinitesimal deformation of complex structure

\updownarrow

Infinitesimal deformation of an algebraic curve
$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ representation labels

$V_{\vec{\lambda}}(C; p_1, \ldots, p_n)$ space of conformal blocks
$\overline{M}_{g,n}$ and 2-dimensional CFT

$(C; p_1, \ldots, p_n) \in \overline{M}_{g,n}$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ representation labels

$V_{\vec{\lambda}}(C; p_1, \ldots, p_n)$ space of conformal blocks
$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

$(\tilde{C}; p_1, \ldots, p_n, q_+, q_-) \in \overline{\mathcal{M}}_{g,n+2}$

$\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n, \lambda, \lambda^\dagger)$ representation labels

$V_{\tilde{\lambda}}(\tilde{C}; p_1, \ldots, p_n, q_+, q_-)$ space of conformal blocks
\( \overline{\mathcal{M}}_{g,n} \) and 2-dimensional CFT

Verlinde bundle

\[ \mathcal{V}_{\vec{\lambda}} \to \overline{\mathcal{M}}_{g,n} \]

Each fibre is given by space of conformal blocks

\[ V_{\vec{\lambda}}(C; p_1, \ldots, p_n) \to (C; p_1, \ldots, p_n) \]
The characters $\text{ch}(\mathcal{V}_\chi)$ define a CohFT on $\overline{\mathcal{M}}_{g,n}$!
The characters $ch(V_{\chi})$ define a CohFT on $\overline{M}_{g,n}$!

A CohFT

- a vector space of fields $U$
- a non-degenerate pairing $\eta$
- a distinguished vector $1 \in U$
- a family of correlators

$$\Omega_{g,n} \in H^*(\overline{M}_{g,n}, \mathbb{Q}) \otimes (U^*) \otimes n$$

satisfying gluing...
Quantum multiplication \(*\) on \(U\)

\[
\eta(v_1 \ast v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) \in \mathbb{Q}
\]

\((U, \ast)\) Frobenius algebra of the CohFT

Teleman: classification of all CohFT with semisimple Frobenius algebra
$\overline{M}_{g,n}$ and CohFT

$\mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n) \text{ such that differential vanishes at } p_i \text{ to order } a_i\}$

$[\overline{\mathcal{H}}_g(\mu)] = ?$

Maybe $[\overline{\mathcal{H}}_g(\mu)]$ is one of the $\Omega_{g,n}$
\( \overline{\mathcal{H}}_g(\mu) \) and CohFT

\[ [\overline{\mathcal{H}}_g(\mu)] \] is not a CohFT class!

Conjecture (Pandharipande, Pixton, Zvonkine): it is related to one Witten \( R \)-spin class

\[ W^R_{g,\mu} \in H^{2g-2}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \]
de Jonquières’ divisors

\[ \mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n) \text{ such that differential vanishes at } p_i \text{ to order } a_i \} \]

\[ = \{(C; p_1, \ldots, p_n) \text{ such that } \Omega^1_C \text{ has section that vanishes at } p_i \text{ to order } a_i \} \]

What if \( L \neq \Omega^1_C \)?
Ernest de Jonquières
Disclaimer
de Jonquières’ multitangency conditions

$C$ smooth, genus $g$

$f : C \rightarrow \mathbb{P}^r$ non-degenerate

Degree of $f = \#\{f(C) \cap H\} =: d$

\[ d = 3 \]
de Jonquières’ multitangency conditions

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de Jonquières’ multitangency conditions

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de Jonquières’ multitangency conditions

\[ C \text{ smooth, genus } g \]
\[ f : C \to \mathbb{P}^r \text{ non-degenerate} \]
\[ \text{Degree of } f = \# \{ f(C) \cap H \} =: d \]

\[ f^{-1} \{ f(C) \cap H \} = p_1 + 2p_2 \]
de Jonquières’ multitangency conditions

dech Jonquières counts the number of pairs \((p_1, p_2)\) such that there exists a hyperplane \(H \subset \mathbb{P}^r\) with

\[
f^{-1}\{f(C) \cap H\} = p_1 + 2p_2
\]
de Jonquières’ multitangency conditions

de Jonquières (and Mattuck, Macdonald) count the $n$-tuples $(p_1, \ldots, p_n)$ such that there exists a hyperplane $H \subset \mathbb{P}^r$ with

$$f^{-1}\{f(C) \cap H\} = a_1p_1 + \ldots + a_np_n$$

where

$$a_1 + \ldots + a_n = d$$
The (virtual) de Jonquières numbers are the coefficients of

\[ t_1 \cdot \ldots \cdot t_n \]

in

\[ (1 + a_1^2 t_1 + \ldots + a_n^2 t_n)^g (1 + a_1 t_1 + \ldots + a_n t_n)^{d-r-g} \]
Constructing the moduli space

An embedding \( f : C \rightarrow \mathbb{P}^r \) of degree \( d \) is given by

A pair \((L, V)\)

- a line bundle \( L \) of degree \( d \) on \( C \)
- an \((r + 1)\)-dimensional vector space \( V \) of sections of \( L \)
Constructing the moduli space

An embedding $f : C \to \mathbb{P}^r$ of degree $d$ is given by

A pair $(L, V)$

- a line bundle $L$ of degree $d$ on $C$
- an $(r + 1)$-dimensional vector space $V$ of sections of $L$

Choose $(\sigma_0, \ldots, \sigma_r)$ basis of $V$

$$f : C \to \mathbb{P}^r$$

$$p \mapsto [\sigma_0(p) : \ldots : \sigma_r(p)]$$
Constructing the moduli space

Space of all divisors of degree $d$ on $C$

$$C_d = \underbrace{C \times \ldots \times C}_{d \text{ times}} / S_d$$

For example

$$p_1 + 2p_2 \in C_3$$
Constructing the moduli space

Space of all divisors of degree \( d \) on \( C \)

\[ C_d = C \times \ldots \times C / S_d \]

For example

\[ p_1 + 2p_2 \in C_3 \]

We define de Jonquières divisors

\[ p_1 + \ldots + p_n \in C_n \]

such that

\[ f^{-1}\{ f(C) \cap H \} = a_1p_1 + \ldots + a_n p_n \]
Constructing the moduli space

\[ D = p_1 + \ldots + p_n \] is de Jonquières divisor

\[ \uparrow \]

there exists a section \( \sigma \) whose zeros are

\[ a_1 p_1 + \ldots + a_n p_n \]
To summarise...

- Fix curve $C$ of genus $g$
- Fix embedding given by $(L, V)$
- $C_n :=$ space of divisors of degree $n$
- Defined de Jonquières divisors via multitangency conditions

$$DJ_n = \{ D \in C_n \mid D \text{ de Jonquières divisor for } L \}$$
Analysing the moduli space

\[ DJ_n = \{ D \in C_n \mid D \text{ de Jonquières divisor for } L \} \]

- determinantal subvariety of \( C_n \) (degeneracy locus)
- if \( DJ_n \neq \emptyset \), then \( \dim DJ_n \geq n - d + r \)
Analysing the moduli space

Relevant questions

- \( n - d + r < 0 \) \( \Rightarrow \) non-existence of de Jonquières divisors
- \( n - d + r \geq 0 \) \( \Rightarrow \) existence of de Jonquières divisors
- \( n - d + r = 0 \) \( \Rightarrow \) finite number of de Jonquières divisors
- \( \dim DJ_n = n - d + r \)
Taking a variational perspective

- Allow $C$ to vary in $\mathcal{M}_{g,n}$
- Vary the de Jonquières structure with it
To summarise...

- Billiards in rational polygons
- Obtained Riemann surface with differential vanishing at marked points
- Looked at strata $\mathcal{H}_g(\mu)$ using CohFT
- Generalised to de Jonquières divisors