Torelli's Theorem for the Cubic Fourfold

Disclaimer: we work exclusively over $\mathbb{C}$.

Torelli's Theorem for a given family of projective varieties is a result that allows us to distinguish members of the family by their Hodge structure. The answer is yes for

- Curves - can recover the curve C from its associated ppav $(J(C), \Theta)$ (Andreotti)
- Abelian varieties, K3 surfaces (Piateski-Shapiro & Safarevich)
- Cubic threefolds (Clemens & Griffiths)
- Most projective hypersurfaces (Donagi)

and no for certain families of surfaces of general type, such as

- surfaces with $q = p_g = 0$ - here the Hodge structure is always trivial, so we cannot use it to differentiate them
- surfaces with $p_g = 1$ and $K.K = 1$ - here the period map has degree at least two.

Actually, the cubic hypersurfaces in $\mathbb{P}^5$ were one of the possible exceptions in Donagi's paper and this is the case that we shall tackle. The statement we aim to prove is:

Theorem 1. *Let X and X' be two smooth cubic fourfolds and let*

$$\phi : H^4(X, \mathbb{Z}) \to H^4(X', \mathbb{Z})$$

be an isomorphism of polarised Hodge structures preserving the class h^2 of a linear section. Then there exists a projective isomorphism $f : X' \to X$ such that $\phi = f^*$.

There are a few proofs of this result:

- Voisin '86 - analyses the geometry of cubic fourfolds containing a plane;
- Looijenga '06 - analyses the geometry of some specific singular cubic fourfolds;
- Charles '12 - uses Verbitsky's global Torelli theorem for irreducible holomorphic symplectic varieties (the Fano variety of lines of a cubic fourfold is such a variety).

We focus on Voisin's proof. The idea is to reformulate the statement of Theorem 1in terms of families. More precisely, we shall describe classifying spaces for cubic fourfolds and for their Hodge structures and prescribe a map between them, associating to a cubic fourfold a Hodge structure. Theorem 1 will then be equivalent to the injectivity of this map.

1 The classifying space for cubic fourfolds

The projective space of all cubic fourfolds in $\mathbb{P}^5$ is $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3))) = \mathbb{P}^{55}$ in which the smooth cubic fourfolds form a Zariski open subset U. Two cubic fourfolds are isomorphic if and only if they are congruent under the action of $PGL(5)$. It turns out (see Mumford's et al GIT) that the smooth cubic fourfolds are properly stable and thus the quotient $\mathcal{C} := U//PGL(5)$ exists as a quasi-projective variety and is a coarse moduli space for smooth cubic fourfolds. By a parameter count, $\mathcal{C}$ has dimension 20.

2 The classifying space for Hodge structures

We make a (very short) summary of Griffith's formalism of Hodge theory, as presented in "Recent developments in Hodge theory".

Definition 1 (Hodge Structure)**.** Let H_R be a finite dimensional vector space, containing a lattice H_Z and let $H = H_R \otimes \mathbb{C}$ be its complexification. A *Hodge structure* of weight k consists of a direct sum decomposition

$$H = \bigoplus_{p+q=k} H^{p,q} \text{ such that } H^{q,p} = \overline{H^{p,q}}.$$

Definition 2 (Morphism of Hodge structures)**.** A linear map $\phi : H \to H'$ between vector spaces with Hodge structures is a *morphism of Hodge structures* if it is defined over $\mathbb{Q}$, relative to the lattices H_Z and H'_Z, and if $\phi(H^{p,q}) \subset H'^{p,q}$. More generally, ϕ is a morphism of type (l, l) if $\phi(H^{p,q}) \subset H'^{p+l,q+l}$.

Consider now a nondegenerate bilinear form $(,)$ on H (symmetric if k is even, skew if odd).

Definition 3 (Polarisations of Hodge structures)**.** The Hodge structure is *polarised by* $(,)$ if

1. $(H^{p,q}, H^{p',q'}) = 0$ for $p \neq q'$, $q \neq p'$,
2. $(\sqrt{-1})^{p-q}(v, \bar{v}) > 0$ for a nonzero $v \in H^{p,q}$.

The prototypical example is the decomposition according to Hodge type of the k-th complex cohomology group of a compact Kähler manifold X.

Suppose the Kähler metric of X is induced from the Fubini-Study metric by a projective embedding, we have that the Lefschetz operator L is defined over $\mathbb{Q}$ on the level of cohomology. Now, the Kähler form ω is of type $(1, 1)$, so L is a morphism of Hodge structures of type $(1, 1)$. Thus the Hodge structure of $H^k(X, \mathbb{C})$ restricts to a Hodge structure on the primitive cohomology $H^k(X, \mathbb{C})^0$. Introduce also the Hodge bilinear form:

$$\begin{aligned} (,) : H^k(X, \mathbb{C})^0 \times H^k(X, \mathbb{C})^0 &\to \mathbb{C} \\ ([\alpha], [\beta]) &= (-1)^{\frac{k(k-1)}{2}} \int \omega^{n-k} \wedge \alpha \wedge \beta. \end{aligned}$$

The Hodge-Riemann bilinear relations immediately imply that $(,)$ polarises the Hodge structure on the primitive part of the cohomology groups.

We are interested of course in the case when X is a cubic fourfold. Its Hodge diamond has the form:

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 0 && 0 &&& \\ && 0 && 1 && 0 && \\ & 0 && 0 && 0 && 0 & \\ 0 && 1 && 21 && 1 && 0 \end{array}$$

We focus on the middle cohomology i.e. $H^4(X, \mathbb{Z})$ since all the non-trivial Hodge information is contained there. It is not too hard to see that

$$H^4(X, \mathbb{Z}) \simeq (+1)^{\oplus 21} \oplus (-1)^{\oplus 2},$$

i.e. $(,)$ is diagonalisable over $\mathbb{Z}$ with entries $\pm$ along the diagonal. To see why, note that Poincaré duality implies that the bilinear form $(,)$ is unimodular. The Riemann bilinear relations tell us that the signature of $H^4(X,\mathbb{Z})$ is $(21,2)$. So we can view $H^4(X,\mathbb{Z})$ as a unimodular lattice of signature $(21,1)$. Moreover, if $h \in H^{1,1}(X)$ is the hyperplane class, then the induced class $h^2 \in H^4(X,\mathbb{Z})$ has self-intersection $(h^2,h^2)=3$, so the lattice is odd. The decomposition then follows from the theory of indefinite quadratic forms.

One can check also that $H^4(X,\mathbb{Z})^0$ is the orthogonal complement of h^2. Furthermore, $H^4(X,\mathbb{Z})^0$ is generated by elements with self-intersection 2, so this lattice is even.

Now take any odd unimodular lattice Λ of signature $(21,2)$, and $\eta \in \Lambda$ such that $(\eta,\eta)=3$ and with even orthogonal complement Λ^0.

There exists therefore an isometry

$$\varphi : H^4(X,\mathbb{Z}) \to \Lambda$$

that sends h^2 to η. Such an isometry is called a *marking*. Given such a marking, we can identify $H^4(X,\mathbb{Z})^0$ with $\Lambda^0_{\mathbb{C}} = \Lambda^0 \otimes_{\mathbb{Z}} \mathbb{C}$.

From Hodge theory, $H^{3,1}(X,\mathbb{C})$ is a distinguished subspace of $\Lambda^0_{\mathbb{C}}$, in the following sense:

1. $H^{3,1}(X,\mathbb{C})$ is isotropic, i.e. it is spanned by a form α such that $(\alpha,\alpha)=0$;
2. The hermitian form $-(\alpha,\overline{\beta})$ is positive on $H^{3,1}(X,\mathbb{C})$.

Consider the quadric hypersurface $Q \subset \mathbb{P}\Lambda^0_{\mathbb{C}}$ defined by vanishing of $(,)$, i.e. $(\alpha,\alpha)=0$. Let U be the topologically open subset of Q where $-(\alpha,\beta)>0$. The real Lie group $SO(\Lambda^0_{\mathbb{R}}) = SO(20,2)$ acts transitively on U. Moreover, this group has two components, one of which acts by exchanging $H^{3,1}(X,\mathbb{C})$ and $H^{1,3}(X,\mathbb{C})$. Thus the two connected components of U parametrise the subspaces $H^{3,1}(X,\mathbb{C})$ and $H^{1,3}(X,\mathbb{C})$. We denote the first one by $\tilde{\mathcal{D}}$. It is a 20-dimensional open complex manifold, called the *local period domain* for cubic fourfolds with markings. So the marking associates to X an element of $\tilde{\mathcal{D}}$ (its period point). This is the classifying space for polarised Hodge structures arising from cubic fourfolds that we were looking for.

Let Γ be the automorphism group of $H^4(X,\mathbb{Z})$ which preserves $(,)$ and the class h^2. In particular, take $\Gamma^+ \subset \Gamma$ to be the subgroup that stabilises $\tilde{\mathcal{D}}$.

Definition 4. The orbit space $\mathcal{C} := \Gamma^+ \setminus \tilde{\mathcal{D}}$ is called the *global period domain.*

Indeed one even has

Proposition 2. *The global period domain for cubic fourfolds is a 20-dimensional quasi-projective variety.*

Since each cubic fourfold determines a point in $\mathcal{D}$, we have

Definition 5. The map

$$\mathcal{P} : \mathcal{C} \to \mathcal{D}$$

associating to each cubic fourfold its corresponding point in the global period domain is called the *period map.*

By Hodge theory, this is a holomorphic map of analytic spaces, etc (in fact, it is even algebraic). Moreover, Hodge theory also shows that it is a local isomorphism. In this context, Theorem 1 is equivalent the following

Theorem 3. *The period map for cubic fourfolds is injective.*

Remark. This version of the theorem is called *global Torelli Theorem.* Other variants exists, which look for example at whether the period map is an immersion (local Torelli) or generically injective (generic Torelli).

One way to prove a result of this type for other kinds of projective hypersurfaces is to consider the 'differential' of the period map, show that it determines the Jacobian ideal of X and finally prove that this ideal determines X up to projective automorphism. Unfortunately due to the local nature of the period map for the cubic fourfold, this method does not work in this case. So one needs to work directly with the Hodge structure. The main difficulty of this approach is the fact that one does not have much of a grasp on the integral cohomology classes of type $(2,2)$. So we need some input from somewhere else, namely from the geometry of the cubic fourfold.

So let us try to gain more insight into the middle cohomology of X. We take a somewhat indirect approach, by comparing the cohomology of X to that of its variety of lines.

Definition 6. Let F be the *Fano variety of lines* of X, i.e. the subvariety of the Grassmannian $\mathbb{G}(1,5)$ parametrising lines contained in X.

Beauville and Donagi proved that

Proposition 4. *F is a simply connected, irreducible symplectic variety of dimension 4.*

Its Hodge diamond is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 1 & & 21 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 21 & & 232 & & 21 & & 1 \end{array}$$

Moreover, in the same paper they also proved the following useful result about the cohomology of F:

Proposition 5. *The natural map*

$$Sym^2 H^2(F,\mathbb{Q}) \to H^4(F,\mathbb{Q})$$

is an isomorphism of Hodge structures.

Consequently, the cohomology of F is practically determined by $H^2(F,\mathbb{Z})$. The next step, therefore, is to connect this space to the cohomology of X. This is what the Abel-Jacobi map is for.

Consider the following incidence correspondence

$$Z = \{(l,x) \in F \times X \mid x \in l\} \subset F \times X.$$

Setting p and q to be the corresponding projections,

$$\begin{array}{ccc} Z & \xrightarrow{p} & F \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

we have

Definition 7. The *Abel-Jacobi map* Φ is defined as the map of cohomology groups

$$\Phi = p_* q^* : H^4(X,\mathbb{Z}) \to H^2(F,\mathbb{Z}).$$

A priori it is not at all clear how the Abel-Jacobi map behaves with respect to the polarisation on the primitive part of the cohomology, or what this polarisation should even be on $H^2(F,\mathbb{Z})^0$.

To answer these questions, let g be the class of a hyperplane on F (using the Plücker embedding in the Grassmannian). Since $\Phi(h^2)$ corresponds to the lines meeting a codimension-2 subspace of $\mathbb{P}^5$, then $\Phi(h^2) = g$. Let us now define a symmetric bilinear form $\langle , \rangle$ on $H^2(X,\mathbb{Z})^0$. Following Beauville and Donagi, assume that g and $H^2(X,\mathbb{Z})^0$ are orthogonal with respect to this form, and set $\langle g, g\rangle = 6$ and $\langle u, v\rangle = \frac{1}{6}g^2uv$ for $u, v \in H^2(F,\mathbb{Z})^0$. Extending by linearity, we obtain an integral form on the whole of $H^2(F,\mathbb{Z})$. This is actually the canonical polarisation arising from the symplectic structure of F. Beauville and Donagi prove that the Abel-Jacobi map indeed preserves the bilinear forms on primitive cohomology:

Proposition 6. *The Abel-Jacobi map induces an isomorphism of polarised Hodge structures between* $H^4(X,\mathbb{Z})^0$ *and* $H^2(F,\mathbb{Z})^0$. *Moreover,*

$$\langle \Phi(\alpha), \Phi(\beta)\rangle = -(\alpha, \beta).$$

So we now have access to geometric information from both F and X to solve our problem. To take advantage of this, Voisin focuses on a certain subvariety of F, constructed as follows.

We restrict our attention to the cubic fourfolds X that contain a plane P. In fact this setup defines another plane P' which parametrises the 3-planes that contain P:

$$P' := \{\text{3-planes such that } P \subset \mathbb{P}^3 \subset \mathbb{P}^5\}.$$

Such a $\mathbb{P}^3$ intersects X in a singular cubic surface $P \cup Q$, where Q is a quadric surface. Hence we have a 2-dimensional (in fact the plane $\mathbb{P}^2$) family of quadric surfaces. In fact, the projection

$$Bl_P X \to P'$$

is a quadric surface fibration with generic fibre $\mathbb{P}^1 \times \mathbb{P}^1$ and discriminant locus given by a plane sextic curve C. Now, recall that a smooth quadric surface admits two rulings, while a singular one is a cone so it admits only one (the lines passing through the singular point). Thus we have the double-cover

$$r : S := \{\text{rulings of the fibres}\} \to P'$$

ramified along C. Actually S is a K3 surface: from the theory of double covers we get that

$$h^1(\mathcal{O}_S) = h^1(\mathcal{O}_{\mathbb{P}^2}) = 0$$

and finally

$$K_S = r^*(K_{\mathbb{P}^2} + \frac{1}{2}C) = r^*(-3H + \frac{1}{2}(6H)) \equiv 0.$$

As mentioned in the introduction, the Torelli theorem is known for K3 surfaces. In order to exploit this fact, we need to be able to relate the Hodge structures of X and S and to reconstruct X from the data of these constructions.

Let $D \subset F$ be the subvariety of lines that meet P. We can describe it as follows:

$$D = \{(l, s) \in F \times S \mid l \text{ belongs to the ruling of the quadric } r(s), \text{ parametrised by } s\}.$$

The projection on the first factor embeds D in F. Moreover, D is a fibration over S where the fibres are the smooth rational curves that are the base curves of the rulings. We denote the corresponding projection by $s : D \to S$. We now have the following nice consequence on the cohomology

Proposition 7. *Let $Z_D \subset D \times X$ be the restriction of the incidence correspondence variety to D. Then the restriction to D of the Abel-Jacobi map is an isomorphism of Hodge structures*

$$\Phi : H^4(X, \mathbb{Q}) \to H^2(D, \mathbb{Q}).$$

Proof. We look instead at the dual spaces and the corresponding map

$$\Phi^t = q_* p^* : H_2(D, \mathbb{Q}) \to H_4(X, \mathbb{Q}).$$

The projection $q : Z_D \to X$ has degree 2: if $x \in X$ is not contained in the plane P, and if the quadric surface determined by x is smooth, then exactly two lines of this quadric will pass through x. Thus, since the degree of q is finite, the map $q_* : H_4(Z_D, \mathbb{Q}) \to H_4(X, \mathbb{Q})$ is surjective.

Moreover, $H_4(Z_D, \mathbb{Q}) = p^*(H_2(D, \mathbb{Q}) \oplus H_4(D, \mathbb{Q}))$, and we also have that the map $H_4(D, \mathbb{Q}) \to H_4(X, \mathbb{Q})$ factors through $H_4(Y, \mathbb{Q})$, where Y is a hyperplane section of X. Since the image of $H_4(Y, \mathbb{Q})$ is contained in the image of $H_2(D, \mathbb{Q})$, we have surjectivity. The result then follows from a dimension count. □

Now, since D is a fibration over S with fibres rational curves, we have that the Hodge structure of D is inherited from the one of S, and so the period map on the space of cubics containing a plane can be identified with the period map on F_2 (since S has degree 2).

We can make this a bit more precise: abusing notation, let P and Q denote the cohomology classes in $H^4(X, \mathbb{Z})$ of the plane P and of a quadric, respectively, such that $P + Q = h^2$. Denote by L the image in $H^2(D, \mathbb{Z})$ under the restricted Abel-Jacobi map of the orthogonal complement of the subspace spanned by P and Q.

Proposition 8. *The subspace L is contained in $s^*(H^2(S, \mathbb{Z})^0)$ and for any α, β orthogonal to P and Q, we have that*

$$(\alpha, \beta)_X = -\langle \Phi\alpha, \Phi\beta \rangle_S.$$

As for the Hodge structures, first note that $H^2(S, \mathbb{Z})$ has a distinguished class k with the property that

$$L = \{\alpha \in H^2(S, \mathbb{Z})^0 \mid \langle \alpha, k \rangle_S = 0 \mod 2\}.$$

We can view k as a class in $H^2(S, \mathbb{Z}/2\mathbb{Z})$ inducing morphisms in $Hom(H^2(S, \mathbb{Z})^0, \mathbb{Z}/2\mathbb{Z})$. So different classes k will induce different such homomorphisms. So k can be obtained by strictly algebraic considerations.

One computes the following products:

$$(P, P)_X = 3, \ (Q, Q)_X = 4, \ (P, Q) = -2.$$

Hence the lattice $\langle P, Q \rangle$ can be seen as a lattice of rank 2 with bilinear form over $\mathbb{Z}$ given by the matrix

$$\begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}.$$

Then the idea is that the lattice $H^4(X, \mathbb{Z})$ is obtained from the orthogonal sum

$$\langle P, Q \rangle \oplus L.$$

For this to work we also need that the ring L satisfies $L^*/L \simeq \mathbb{Z}/8\mathbb{Z}$ and that we can extend the module $\langle P, Q \rangle \oplus L$ into a unimodular lattice with integer bilinear form.

The only thing left to consider is what happens given two lattices L and L' determine isomorphic Hodge structures on the cubic, then the we get an automorphism of $H^2(S)$ sending one to the other, while preserving the Hodge structure and the polarisation. Therefore we proved

Proposition 9. *The Hodge structure on X equipped with the lattice $\langle P, Q\rangle$ determine S. Conversely, the polarised Hodge structure of S together with the distinguished class k determine the Hodge structure of X.*

So we have essentially constructed a morphism from the space of cubics containing a plane to F_2.