The Chiral de Rham Complex of Tori and Orbifolds

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Introduction

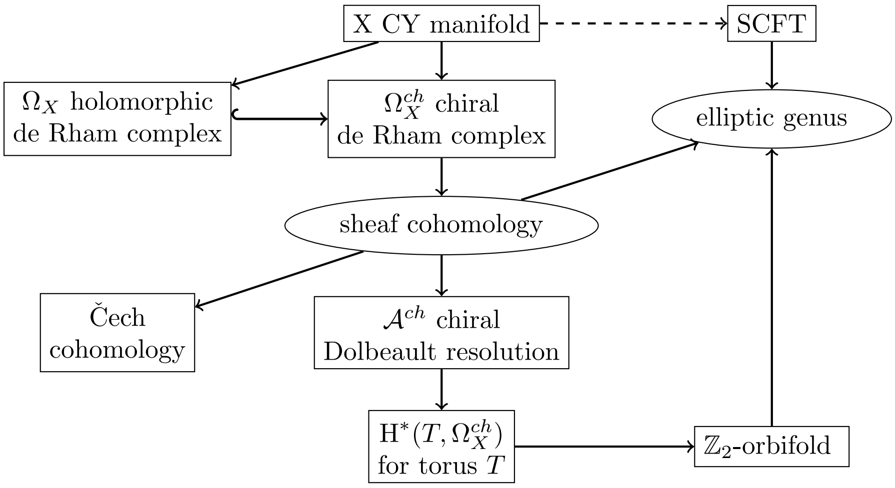
In string theory conformal field theories (CFTs), or rather superconformal field theories (SCFTs), describe the dynamics of a string moving through a Calabi-Yau manifold. The physical theory, a non-linear sigma model, allows physicists to associate a SCFT to such a Calabi-Yau manifold. Due to the notorious path integral formalism needed for this correspondence the respective SCFT is only well defined for tori and their orbifolds. A mathematically precise way of defining a SCFT for any Calabi-Yau manifold is out of reach at the moment. However, part of the SCFT structure, the vertex algebra, behaves mathematically much nicer and it is thus a better candidate for a precise definition. Vertex algebras on their own are well studied and proved their importance in the theory of Monstrous Moonshine, which ultimately led to a Fields medal for Borcherds.

In [49] Malikov, Schechtman and Vaintrob defined a sheaf of vertex algebras, the chiral de Rham complex, for any complex manifold. Adding the Calabi-Yau condition ensures the existence of the structure of a topological $N = 2$ superconformal algebra in the space of global sections. The corresponding sheaf cohomology allows, in a precise way, to thus associate one vertex algebra to every Calabi-Yau manifold. However, all this is very abstract. It is difficult to compute and there are very few examples. Not even the simplest case of the torus is computed explicitly.

Borisov explicitly gave the vertex algebra structure for Čech cohomology in [6] and together with Libgober in [7] proved that a certain operator trace over the cohomology space yields the elliptic genus, a topological invarant also appearing in SCFT and at the topological level. The natural question is if this is compatible with orbifolding: In [8] they showed the compatibility of the elliptic genus with the orbifold construction. Since the starting point of this chain of constructions is the chiral de Rham complex of [49] this is evidence that the chiral de Rham cohomology is, in fact, compatible with orbifolding and related to the SCFT.

While the orbifold elliptic genus gives relations only on the level of operator traces, Frenkel and Szczesny in [25] described a general orbifold procedure for the chiral de Rham complex cohomology, proposing an orbifold space of states. They conjecture that, similar to the CFT/SCFT setting, the orbifold procedure yields the vertex algebra of the chiral de Rham complex of the Calabi-Yau resolution of the orbifold.

In this thesis we want to investigate the chiral de Rham complex and its relation to SCFT in the simplest examples of Calabi-Yau manifolds, tori and K3 surfaces. Special K3 surfaces, Kummer surfaces, can be constructed via a $\mathbb{Z}_2$-orbifold of the torus. This is the reason for the importance of the orbifold construction in this context. Our approach is summarized in the following diagram:



where the dashed arrow from a Calabi-Yau manifold to SCFT represents the ill-defindness of this physical construction in general. In the cases we use it, tori and their orbifolds, this is well defined though. Our goal is to check it for the Kummer construction.

There are two main open problems with the proposal of the chiral de Rham orbifold construction in [25]. First, the verification of their conjecture. Secondly, the prediction of the OPEs of all elements of the proposed state of states, including in particular the 16 twist fields and their 16 conjugated partners. In our view these two are interrelated: Under the assumption that the conjecture is true, we show how to derive the OPEs of the 16 twist fields of $T^4/\mathbb{Z}_2$ from the chiral de Rham complex of the Kummer surface (Proposition 5.4.4). We then show how to obtain their 16 conjugated partners and investigate their OPEs with the 16 twist fields (Proposition 5.4.5 and Proposition 5.4.6). This is our contribution to the problem. It remains open to show that this is, in fact, the chiral de Rham complex of the resolution. So far, we have been able to show that some global sections predicted by the orbifold cannot be easily found in chiral de Rham complex of a Kummer surface (Proposition 5.4.2).

Our main result establishes the existence of a chiral Dolbeault resolution for the chiral de Rham complex on any complex manifold (Theorem 5.1.7). We also compute the sheaf cohomology of the holomorphic chiral de Rham complex of a complex torus both using this chiral Dolbeault resolution (Proposition 5.2.1) and Čech cohomology. Interestingly, and somewhat unexpectedly, this is independent of the choice of complex structure on the torus (Corollary 5.2.2). We then use the orbifold procedure on the torus chiral de Rham cohomology in order to compare it with its Kummer surfaces counterpart (Proposition 5.4.4). For both the torus and the orbifold we compare the field content with that of the SCFT.

The organization of this thesis is the following:

In the first three chapters we review the ingredients of CFT and SCFT and the orbifold construction we need from the literature. In Chapter 4 we review the chiral de Rham complex and in Chapter 5 give our results concerning the chiral Dolbeault resolution, tori and their $\mathbb{Z}_2$-orbifolds. In more detail the layout is the following:

In Chapter 1 we give a brief account of a conformal field theory (CFT) definition. We quickly restrict to the special CFTs we are most interested in, toroidal CFT. We then turn to vertex algebras and review the shape of $\mathcal{W}$-algebras of toroidal CFTs depending on their geometric interpretation (Theorem 1.3.6).

Chapter 2 extends the previous chapter by incorporating odd elements, i.e. fermions. We encounter the Ising model, the first of the minimal series with application in statistical mechanics. It is also an example of coupled spin structures and the higher dimensional analogs are the models needed to define toroidal SCFTs. We can then define the elliptic genus in the SCFT context.

In Chapter 3 we review orbifold techniques for CFTs and SCFTs. We concentrate on the toroidal examples, which we want to mimic in the context of the chiral de Rham complex.

In Chapter 4 we review the holomorphic chiral de Rham complex construction [49] of Malikov, Schechtman and Vaintrob as well as Borisov's and Libgober's work ([6], [7]) concerning cohomology and toric geometry. Here we re-encounter the elliptic genus in the chiral de Rham setting.

Chapter 5 contains the construction of [25] as well as the results of this thesis.

Chapter 1

Conformal Field Theory

Conformal field theories (CFTs) have been a well studied topic since Belavin, Polyakov and Zamolodchikov in their seminal paper [4] combined representation theory of the Virasoro algebra with the idea of local operators to discover the minimal models. This is a series of theories related to second order phase transitions in statistical mechanics, the first of which is the Ising model. Another application of CFT is string theory, where the CFT describes the dynamics of strings compactified on a Calabi-Yau manifold.

In the first section of this Chapter we briefly review the definition of a conformal field theory (CFT). In Section 2 we review the first example and object of study, toroidal CFTs, describing strings compactified on a torus. Section 3 treats the second important structure of this thesis, vertex algebras, and how they arise in CFT as well as the concept of rationality and how this depends on the modulus of toroidal CFTs. Our discussion is by no means complete and meant to be a reminder and collection of terminology.

The main references for this chapter are [31], [57] and [59][1].

1.1. Definition

This is based on [59] and Section 2.2 of [58], also see Chapter 2 of [57]. We discuss two-dimensional euclidean unitary conformal field theories (CFT) here. Such a conformal field theory is an object with a lot of structure made up from many parts. Roughly, we have an infinite dimensional vector space that represents states of our system and we have a way of associating functions to any tuple of states. Conformality features as a representation of the Virasoro algebra.

The *Virasoro algebra at central charge* $c \in \mathfrak{c} \cong \mathbb{C}$ is an infinite dimensional Lie algebra which appears as the central extension of the holomorphic Witt algebra of holomorphic infinitesimal conformal transformations of $\mathbb{C}^*$. Central extensions arise in the world of quantum physics, where symmetries of a classical system are expected to lift to a central extension acting on the quantization of the classical system. The Virasoro algebra has $\mathbb{C}$-vector space basis $\{\, c, L_n \mid n \in \mathbb{Z} \,\}$, where c is central and the generators L_n have Lie bracket

$$[L_n, L_m] = (m-n)L_{m+n} + \delta_{n+m,0}\frac{c}{12}m\left(m^2-1\right). \tag{1.1}$$

[1] Since [59] is a book manuscript with still changing numbering we will only refer to the respective chapters

The central element c is called *central charge*. Any two distinct central charges $c \neq c'$ give inequivalent Virasoro algebras. The universal enveloping algebra of the Virasoro algebra will be denoted by Vir_c.

The first ingredient of a CFT is the space of states $\mathbb{H}$, an infinite dimensional $\mathbb{C}$-vector space with a real structure, i.e. an anti-$\mathbb{C}$-linear involution $\mathbb{H} \to \mathbb{H}, \varphi \mapsto \varphi^*$, and a compatible positive definite Hermitian scalar product $\langle \cdot, \cdot \rangle$, i.e. $\forall \varphi, \xi \in \mathbb{H} : \langle \varphi, \chi \rangle = \overline{\langle \chi, \varphi \rangle} = \overline{\langle \varphi^*, \chi^* \rangle}$. Furthermore $\mathbb{H}$ is the representation space of a representation $\rho : \mathrm{Vir}_c \otimes \overline{\mathrm{Vir}}_{\bar{c}} \to \mathrm{End}_{\mathbb{C}}(\mathbb{H})$ of two commuting copies $\mathrm{Vir}_c \otimes \overline{\mathrm{Vir}}_{\bar{c}}$ of the Virasoro algebra. We will usually refer to $\mathbb{H}$ as the representation when it is clear what is meant by ρ. The generators of Vir_c and $\overline{\mathrm{Vir}}_{\bar{c}}$ are denoted by L_n and $\bar{L}_n$ for $n \in \mathbb{Z}$. For both copies of the Virasoro algebra the representation is

1. *real*:
$$\forall n \in \mathbb{Z}, \quad \forall \varphi \in \mathbb{H} : \quad (L_n \varphi)^* = L_n \varphi^* \tag{1.2}$$
and analogously for $\bar{L}_n$.

2. *unitary*:
$$\forall n \in \mathbb{Z}, \quad \forall \varphi, \chi \in \mathbb{H} : \quad \langle L_n \varphi, \chi \rangle = \langle \varphi, L_{-n} \chi \rangle \tag{1.3}$$
and analogously for $\bar{L}_n$. We write $(L_n)^\dagger = L_{-n}$.

3. The central charges c and $\bar{c}$ are constant on $\mathbb{H}$.

4. The elements L_0 and $\bar{L}_0$ act as positive semidefinite diagonalizable operators on $\mathbb{H}$ such that $\mathbb{H}$ decomposes into a direct sum of countable many simultaneous non-zero eigenspaces
$$\mathbb{H} = \bigoplus_{(h,\bar{h}) \in R} \mathbb{H}_{(h,\bar{h})}, \quad \mathbb{H}_{h,\bar{h}} = \left\{ \varphi \in \mathbb{H} \;\middle|\; L_0 \varphi = h \varphi, \quad \bar{L}_0 \varphi = \bar{h} \varphi \right\}. \tag{1.4}$$
Here $R \subset \mathbb{R}^2_{\geq 0}$ is countable without accumulation points and for all $(h, \bar{h}) \in R$ we have $\dim \mathbb{H}_{h,\bar{h}} < \infty$. Note that R induces a bigrading on $\mathbb{H}$ and the tupels $(h, \bar{h})$ will be called *conformal weight*.

5. The conformal weight zero space is $\mathbb{H}_{0,0} \cong \mathbb{C}$ with a *unique vacuum* Ω, a real unit vector generating $\mathbb{H}_{0,0}$.

Properties 3 and 4 of such a representation are called *perfect* in [59]. There are two important subspaces of $\mathbb{H}$:

Definition 1.1.1. The subspaces
$$\mathcal{W} := \bigoplus_{h} \mathbb{H}_{(h,0)}, \quad \overline{\mathcal{W}} := \bigoplus_{\bar{h}} \mathbb{H}_{(0,\bar{h})} \tag{1.5}$$
are called *holomorphic and antiholomorphic $\mathcal{W}$-algebras.*

These subspaces have a particular nice structure which we will discuss in Section 1.3.

The second ingredient of CFT is the notorious *operator product expansion (OPE)*: Denote by $\mathbb{H}\{\{z,\bar{z}\}\}$ the space of functions $f = f(z,\bar{z}) : \mathbb{C} \to \overline{\mathbb{H}}$, that are real analytic on $\mathbb{C}^*$ and have the following behavior around $z = 0$:

$$f(z,\bar{z}) = \sum_{s\in S, n\in\mathbb{Z}} a_{sn} z^{s+n} \bar{z}^{s}. \tag{1.6}$$

Here $S \subset \mathbb{R}$ is again countable and $a_{sn} \in \mathbb{H}$ with only finitely many nonzero coefficients with $s + n < 0$ or $s < 0$. We call an expression of the form (1.6) a *field*[2].

Definition 1.1.2 ([59], Chapter 4)**.** An *operator product expansion (OPE)* is a $\mathbb{C}$ vector space homeomorphism $OPE : \mathbb{H} \otimes \mathbb{H} \to \mathbb{H}\{\{z,\bar{z}\}\}$ which is compatible with the bigrading of $\mathbb{H}$, i.e. such that

$$\forall \varphi_i \in \mathbb{H}_{h_i,\bar{h}_i} : \quad OPE(\varphi_1 \otimes \varphi_2) = \sum_{\substack{(r,\bar{r})\in \\ R_{\varphi_1,\varphi_2}}} C_{r,\bar{r}} z^r \bar{z}^{\bar{r}} \quad \text{with} \quad C_{r,\bar{r}} \in \mathbb{H}_{h_1+h_2+r,\bar{h}_1+\bar{h}_2+\bar{r}}. \tag{1.7}$$

Dependency of antiholomorphic coordinates is explicitly stated in order to distinguish between holomorphic and real-analytic functions.

On the holomorphic $\mathcal{W}$-algebra only inserting one element of $\mathcal{W}$ into the OPE induces a map $\mathcal{W} \to \mathrm{End}(\mathcal{W})[[z^{\pm}]]$, associating a Laurent series $\varphi(z,\bar{z})$ to every element φ of $\mathcal{W}$. This series has only finitely many singular terms when applied to any element of $\mathcal{W}$. Such a Laurent series is a field in the vertex algebra sense (cf. Section 1.3) and the map $\mathcal{W} \to \mathrm{End}(W)[[z^{\pm}]]$ is called the *state-field-correspondence*. The representation of the Virasoro algebra allows to define the Virasoro field $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{n-2}$. Its OPE with itself

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{1.8}$$

encodes all Lie brackets of the Virasoro algebra! From this point of view the OPE can be considered to be a convenient way of encoding relations. Analogously we can define $\bar{T}(\bar{z})$ on the antiholomorphic side. Furthermore the field associated to the vacuum Ω is the identity operator $\Omega(z) = \mathbb{1}$.

The OPE is not quite enough to capture the structure of the functions associated to states of a CFT. We also need *n-point functions*. For this let $\mathcal{F}^{(n)}$ denote the space of maps $\mathbb{C}^n \to \mathbb{C}$ that are real-analytic outside partial diagonals.

Definition 1.1.3 ([59], Chapter 4)**.** A system of *n-point functions on* $\mathbb{H}$ is a vector space homeomorphism $\mathbb{H}^{\otimes n} \to \mathcal{F}^{(n)}, \varphi_1 \otimes \cdots \otimes \varphi_n \mapsto \langle \varphi_1(z_1,\bar{z}_1) \cdots \varphi_n(z_n,\bar{z}_n)\rangle$ such that

1. φ_i are pairwise local: for every permutation $\sigma \in S_n$

$$\langle \varphi_1(z_1,\bar{z}_1) \cdots \varphi_n(z_n,\bar{z}_n)\rangle = \langle \varphi_{\sigma(1)}(z_{\sigma(1)},\bar{z}_{\sigma(1)}) \cdots \varphi_{\sigma(n)}(z_{\sigma(n)},\bar{z}_{\sigma(n)})\rangle. \tag{1.9}$$

[2]Note that this is the field definition of WENDLAND ([59]) which is closer to the physics definition of a field but in the holomorphic setting agrees with the vertex algebra definition of KAC ([40]).

2. The n-point functions are *Poincaré covariant*: for all isometries and all dilations $f \in \mathrm{PSL}_2(\mathbb{C})$ and for all $\varphi_i \in \mathbb{H}_{h_i,\bar{h}_i}$:

$$\begin{aligned} &\langle \varphi_1(f(z_1,\bar{z}_1))\cdots\varphi_n(f(z_n,\bar{z}_n))\rangle = \\ \prod_{i=1}^{n}\Big[(f'(z_i,\bar{z}_i))^{-h_i}&(\overline{f'(z_i,\bar{z}_i)})^{-\bar{h}_i}\Big]\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_n(z_n,\bar{z}_n)\rangle. \end{aligned} \tag{1.10}$$

3. On the n-point functions translations are represented by $\alpha\partial_z + \bar{\alpha}\partial_{\bar{z}}$, $\alpha \in \mathbb{C}$: for $\varphi_i \in \mathbb{H}$:

$$\begin{aligned} &\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_{n-1}(z_{n-1},\bar{z}_{n-1})(L_1\varphi_n)(z_n,\bar{z}_n)\rangle = \\ &\frac{\partial}{\partial z_n}\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_{n-1}(z_{n-1},\bar{z}_{n-1})\varphi_n(z_n,\bar{z}_n)\rangle, \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_{n-1}(z_{n-1},\bar{z}_{n-1})(\bar{L}_1\varphi_n)(z_n,\bar{z}_n)\rangle = \\ &\frac{\partial}{\partial \bar{z}_n}\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_{n-1}(z_{n-1},\bar{z}_{n-1})\varphi_n(z_n,\bar{z}_n)\rangle. \end{aligned} \tag{1.12}$$

4. The maps $\mathbb{H}^{\otimes n} \to \mathcal{F}^{(n)}$ are compatible with the real structures on $\mathbb{H}$ and on $\mathcal{F}^{(n)}$:

$$\overline{\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_n(z_n,\bar{z}_n)\rangle} = \langle \varphi_1^*(\bar{z}_1,z_1)\cdots\varphi_n^*(\bar{z}_n,z_n)\rangle. \tag{1.13}$$

One may hope for Poincaré covariance to hold for all elements $f \in \mathrm{PSL}_2(\mathbb{C})$ and $\varphi_i \in \mathbb{H}$. While this cannot be achieved in general, there is a subset of states on which it does hold, called *quasi-primary*. The space of quasi-primaries is

$$\mathbb{H}^{QP} \coloneqq \left\{\varphi \in \mathbb{H} \,\middle|\, L_{-1}\varphi = 0 = \bar{L}_{-1}\varphi\right\}. \tag{1.14}$$

Definition 1.1.4 ([59], Chapter 4)**.** A system of n-point functions on $\mathbb{H}$ is *conformally covariant* if

1. For all $f \in \mathrm{PSL}_2(\mathbb{C})$ and for all $\varphi_i \in \mathbb{H}^{QP}_{(h_i,\bar{h}_i)}$:

$$\begin{aligned} &\langle \varphi_1(f(z_1,\bar{z}_1))\cdots\varphi_n(f(z_n,\bar{z}_n))\rangle = \\ \prod_{i=1}^{n}\Big[(f'(z_i,\bar{z}_i))^{-h_i}&(\overline{f'(z_i,\bar{z}_i)})^{-\bar{h}_i}\Big]\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_n(z_n,\bar{z}_n)\rangle. \end{aligned} \tag{1.15}$$

2. The n-point functions involving the state $T \coloneqq L_2\Omega$, created by the Virasoro field from the vacuum, for z in some neighborhood of z_n obey:

$$\begin{aligned} \forall m \in \mathbb{Z}, \forall \varphi_i \in \mathbb{H}: \quad &\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_{n-1}(z_{n-1},\bar{z}_{n-1})T(z)\varphi_n(z_n,\bar{z}_n)\rangle = \\ \sum_{m\in\mathbb{Z}}(z-z_n)^{m-2}&\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_{n-1}(z_{n-1},\bar{z}_{n-1})(L_m\varphi_n)(z_n,\bar{z}_n)\rangle. \end{aligned} \tag{1.16}$$

The n-point functions and the OPE need to be compatible, this is ensured by:

Definition 1.1.5 ([59], Chapter 4)**.** A system of n-point functions $\langle\cdots\rangle$ on $\mathbb{H}$ is a *representation of an OPE* if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{H}^{\otimes 2}\otimes\mathbb{H}^{\otimes(n-2)} & \longrightarrow & \mathcal{F}^{(n)} \\ \downarrow & & \downarrow \\ \mathbb{H}^{\otimes(n-1)}\{z,\bar{z}\} & \longrightarrow & \mathcal{F}^{n-1}\{z,\bar{z}\} \end{array}. \tag{1.17}$$

All these parts accumulate in the definition of a CFT:

Definition 1.1.6 ([59], Chapter 5)**.** A *(two-dimensional euclidean) unitary conformal field theory (CFT)* $\mathcal{C}$ at central charges $c, \bar{c}$ is a perfect unitary real representation $\mathbb{H}$ of $\mathrm{Vir}_c \otimes \overline{\mathrm{Vir}}_{\bar{c}}$ together with a conformally covariant system of n-point functions $\langle \cdots \rangle$ such that:

1. All conformal spins $h - \bar{h}$ for $(h, \bar{h}) \in R$ are integral.
2. The system of n-point functions provides a representation of an OPE. Moreover,
$$\forall \psi \in \mathbb{H}^{QP}_{h,\bar{h}}, \quad \forall \varphi \in \mathbb{H}^{QP} : \quad \langle \psi, \varphi \rangle = \lim_{x,w \to 0} \bar{x}^{-2h} x^{-2\bar{h}} \langle \psi^* \left(\bar{x}^{-1} \right) \varphi(w) \rangle. \tag{1.18}$$
3. $\mathbb{H}$ possesses a modular invariant partition function
$$Z(\tau) := \mathrm{tr}_{\mathbb{H}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right), \tag{1.19}$$
with $q := \exp(2\pi i \tau)$, $\bar{q} = \exp(-2\pi i \bar{\tau})$ and $\tau \in \mathcal{H}$, the upper half plane.
4. *Universality* holds: If $\mathbb{H}'$ is a pre-Hilbert space with a real structure and n-point functions $\langle \cdots \rangle'$, such that the above hold with $\mathbb{H} \subset \mathbb{H}'$, where $\mathbb{H}$ is a subrepresentation of $\mathrm{Vir}_c \otimes \overline{\mathrm{Vir}}_{\bar{c}}$ and $\langle \cdots \rangle$ is the restriction of $\langle \cdots \rangle'$ to $\mathbb{H}$, then $\mathbb{H} = \mathbb{H}'$.

Modularity of the partition function proves to be an essential ingredient in the orbifold construction. Summing up, a CFT $\mathcal{C}$ depends on the representation of two commuting Virasoro algebras $\mathbb{H}$ together with its real structure, scalar product and choice of vacuum Ω as well as on the n-point functions and the OPE.

1.2. Toroidal CFT

We will now turn to our first example of CFTs as well as main object of study of this thesis, toroidal CFT. These theories arise in string theory and describes the dynamics of a string moving through a torus subject to an action functional and thus are special cases of non-linear sigma model with a torus as target space ([11], [52]). In these cases the non-linear sigma model from physics can be made mathematically precise. For the vertex algebra side of the toroidal theories see [40], [42], [26].

We will first study the model for a torus of smallest dimension, a circle, before turning to the general case. We will see the dependency on the moduli of these theories. While we denoted general CFTs by $\mathcal{C}$, toroidal CFTs will be denoted by $\mathcal{T}$. This is based on Chapter 3 of [59], also see Section 8.1 of [31].

1.2.1. The free boson compatified on the circle

The non-linear sigma model of the free boson compactified on the circle describes the motion of a closed string in a circle subject to an action. Let $r > 0$ be the radius of the circle through which the string travels. The parameter space of the string is a complex torus $\mathbb{C}/\Lambda_\tau$, called the *world-sheet*, with complex parameter $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. The string itself is a smooth map from $\mathbb{C}/\Lambda_\tau$ to the circle $\mathbb{R}/2\pi r\mathbb{Z}$. The action functional is given by its area:

$$\begin{array}{rcl} S_q\colon\ \mathcal{C}^\infty(\mathbb{C}/\Lambda_\tau, \mathbb{R}/2\pi r\mathbb{Z}) & \longrightarrow & \mathbb{R} \\ \varphi & \longmapsto & \frac{1}{(2\pi)^2}\int_{\mathbb{C}/\Lambda_\tau} \partial_\xi\varphi(\xi)\partial_{\bar{\xi}}(\xi)2i d\xi\wedge d\bar{\xi}, \end{array} \tag{1.20}$$

where $q := \mathrm{e}^{2\pi i\tau}$. Each string φ can be represented by a function $\varphi : \mathbb{C} \to \mathbb{R}$ satisfying

$$\varphi(\xi + 1) = \varphi(\xi) + 2\pi r n, \tag{1.21}$$
$$\varphi(\xi + \tau) = \varphi(\xi) + 2\pi r n' \tag{1.22}$$

for all $\xi \in \mathbb{C}$ and for some $n, n' \in \mathbb{N}$. Such a string φ is said to belong to the sector ${}_n\square_{n'}$, where $\square$ represents the fundamental domain of the torus.

To define a corresponding physical theory, let $\mathbb{H} := \mathcal{C}^\infty(\mathbb{C}/\Lambda_\tau, \mathbb{R}/2\pi r\mathbb{Z})$ be the space of states of the free boson compactified on the circle. We can define a partition function that weights all possible paths by the exponential of the negative of the action functional

$$Z(\tau) := \int_{\mathbb{H}} D\varphi\, \mathrm{e}^{-2\pi S_q(\varphi)}. \tag{1.23}$$

While in general such a path-integral approach leads to convergence issues one can make sense of this for tori. Using Zeta value regularization and Poisson resummation one finds that $Z(\tau)$ is convergent. Let $\tau = \tau_1 + i\tau_2$ and $\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^\infty(1 - q^n)$ be the Dedekind eta-function, then

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2}\sum_n \exp\left(-2\pi\tau_2 n^2 r^2\right)\sum_{n'}\left(-\frac{2\pi r^2}{\tau_2}(n' - n\tau_1)^2\right) \tag{1.24}$$
$$= \frac{1}{|\eta(\tau)|^2}\sum_{n,m} q^{\frac{1}{2}p_l^2}\bar{q}^{\frac{1}{2}p_r^2}. \tag{1.25}$$

Here p_l and p_r are the left and right momenta, $p_l := \frac{m}{2r} + nr$ and $p_r := \frac{m}{2r} - nr$. The states contributing to the sum $\sum_{n,m}$ come from static embeddings of the string into the circle, while $\frac{1}{|\eta(\tau)|^2}$ gives the vibration modes.

This model can also be realized using Fock spaces. Let b be the Lie algebra with $\mathbb{C}$-vector space basis $\{1; a_n \,|\, n \in \mathbb{Z}\}$ with central element 1 and Lie bracket

$$[a_n, a_m] = m\delta_{m+n,0}. \tag{1.26}$$

Let $\mathcal{B}$ be the universal enveloping algebra of b, called *free boson algebra*. Choose $Q \in \mathbb{R}$ and consider the left-handed ideal $\mathcal{I}$ in $\mathcal{B}$ generated by

$$a_{-n} \text{ for } n > 0, \quad 1_{b^{\otimes 1}} - 1_{b^{\otimes 0}}, \quad a_0 - Q. \tag{1.27}$$

The quotient $B^Q := \mathcal{B}/\mathcal{I}$ is called *Fock space representation of the free boson algebra* $\mathcal{B}$. This is a lowest weight representation of the free boson algebra $\mathcal{B}$ with lowest weight vector $|h\rangle \in B^Q$ with $a_0|h\rangle = Q|h\rangle$.

Theorem 1.2.1 ([59], Chapter 3)**.** *Let* $R \in \mathbb{R}^+$ *and*

$$\Gamma_R := \left\{ \frac{1}{\sqrt{2}} \left(nR + \frac{m}{R}, nR - \frac{m}{R} \right) \middle| n, m \in \mathbb{Z} \right\}. \tag{1.28}$$

Then

$$\mathbb{H}_{circ}(R) := \sum_{(Q,\bar{Q}) \in \Gamma_R} B^Q \otimes \bar{B}^{\bar{Q}} \tag{1.29}$$

is the space of states of the free boson compatified on the circle of radius $r = \frac{\sqrt{2}}{2} R$.

The lattice Γ_R is called the *charge lattice*. It gives the connection between the charges and the geometric interpretation, i.e. the radius. We can define a real structure on $\mathbb{H}_{circ}(R)$ via *charge conjugation* $B^Q \to B^{-Q}, \varphi \mapsto \varphi^*$:

$$\left(\sum \alpha_{k_1 \cdots k_p} a_{k_1} \cdots a_{k_p} |h\rangle \right)^* := (-1)^p \sum \overline{\alpha_{k_1 \cdots k_p}} a_{k_1} \cdots a_{k_p} |h\rangle, \quad \alpha_{k_1 \cdots k_p} \in \mathbb{C}. \tag{1.30}$$

For the scalar product let $v := |h\rangle$ and set $\langle v, v \rangle := 1$. For any standard basis vector $w = a_{k_1} \cdots a_{k_p} |h\rangle$ with $p > 0$ set $\langle v, w \rangle := 0$ and extend linearly to all $w \in B^Q$. If $w' = a_{l_1} \cdots a_{l_q} |h\rangle$ is another standard vector, set

$$\langle w', w \rangle := \langle v, a_{-l_q} \cdots a_{-l_1} a_{k_1} \cdots a_{k_p} v \rangle. \tag{1.31}$$

The scalar product on $\mathbb{H}_{circ}(R)$ is then obtained by Hermitian sesquilinear extension.

Now we need to realize the state-field correspondence. For the vibration modes $a_n, \bar{a}_n$ this is done via the so-called $u(1)$-*currents*

$$j(z) := a_1(z) = \sum_{n \in \mathbb{Z}} a_n z^{n-1} \quad \text{resp.} \quad \bar{j}(\bar{z}) := \bar{a}_1(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{a}_n \bar{z}^{n-1} \tag{1.32}$$

with non-trivial OPEs

$$j(z) j(w) \sim \frac{1}{(z-w)^2} \quad \text{resp.} \quad \bar{j}(\bar{z}) \bar{j}(\bar{w}) \sim \frac{1}{(\bar{z}-\bar{w})^2}. \tag{1.33}$$

Other vibration modes follow from derivatives and normal ordered products. The Virasoro field is realized as

$$T(z) := \frac{1}{2} : j(z) j(z) : \tag{1.34}$$

with OPE

$$T(z) T(w) \sim \frac{\frac{1}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \tag{1.35}$$

and analogously on the antiholomorphic side, making each B^Q into a representation of the Virasoro algebra at central charge $c = 1$. The conformal weight of a lowest weight vector $|h, \bar{h}\rangle \in B^Q \otimes \bar{B}^{\bar{Q}}$ is $(\frac{Q^2}{2}, \frac{\bar{Q}^2}{2})$. The fields for the lowest weight vectors of $B^Q \otimes \bar{B}^{\bar{Q}}$

for $(Q,\bar{Q}) \neq (0,0)$ follow from integrating the $u(1)$-currents and exponentiating out the logarithmic contribution. We define X by

$$j(z) = i\partial_z X(z), \tag{1.36}$$
$$\bar{j}(\bar{z}) = i\partial_{\bar{z}}\bar{X}(\bar{z}) \tag{1.37}$$

and use it to define the fields for the lowest weight vectors of $B^Q \otimes \bar{B}^{\bar{Q}}$

$$U_{Q,\bar{Q}}(z,\bar{z}) =: e^{iQX(z)+i\bar{Q}\bar{X}(\bar{z})} : . \tag{1.38}$$

Expanding the exponential series one can calculate the OPE between two such fields. There still is a problem concerning one of the properties of a CFT, for some charges $(Q,\bar{Q}),(Q',\bar{Q}') \in \Gamma_R$ we have

$$U_{Q,\bar{Q}}(z,\bar{z})U_{Q',\bar{Q}'}(w,\bar{w}) \sim -U_{Q',\bar{Q}'}(w,\bar{w})U_{Q,\bar{Q}}(z,\bar{z}), \tag{1.39}$$

which contradicts locality because of the sign. In order to implement locality these fields have to be augmented using phases η_γ, cocycle factors c_γ and two-cocycles $\varepsilon : \Gamma_R \times \Gamma_R \to \{\pm 1\}$ for $\gamma := (Q,\bar{Q}) \in \Gamma_R$ ([59], Chapter 5). The two-cocycles are given by

$$\begin{array}{cccc} \varepsilon\colon & \Gamma_R \times \Gamma_R & \longrightarrow & \{\pm 1\} \\ & ((Q,\bar{Q}),(Q',\bar{Q}')) & \longmapsto & (-1)^{\frac{1}{2}(Q+\bar{Q})(Q'-\bar{Q}')}. \end{array} \tag{1.40}$$

Theorem 1.2.2 ([59], Chapter 5)**.** *There is a choice of phases $\eta : \Gamma_R \to \{\pm 1\}$ such that for the two-cocycle*

$$\begin{array}{cccc} \tilde{\varepsilon}\colon & \Gamma_R \times \Gamma_R & \longrightarrow & \{\pm 1\} \\ & (\alpha,\beta) & \longmapsto & \varepsilon(\alpha,\beta)\eta_\alpha\eta_\beta\eta_{\alpha,\beta} \end{array} \tag{1.41}$$

there are well-defined cocycle factors c_γ.

This theorem allows us to give a well-defined CFT. The augmented fields are

$$W_\gamma(z,\bar{z}) := \eta_\gamma U_\gamma(z,\bar{z})c_\gamma \tag{1.42}$$

with OPEs

$$j(z)W_\gamma(w,\bar{w}) \sim \frac{Q}{z-w}W_\gamma(w,\bar{w}), \tag{1.43}$$
$$T(z)W_\gamma(w,\bar{w}) \sim \frac{\frac{Q^2}{2}}{(z-w)^2}W_\gamma(w,\bar{w}) + \frac{1}{z-w}\partial_w W_\gamma(w,\bar{w}), \tag{1.44}$$
$$W_\gamma(z,\bar{z})W_{\gamma'}(w,\bar{w}) \sim (z-w)^{QQ'}(\bar{z}-\bar{w})^{\bar{Q}\bar{Q}'}\varepsilon(\gamma,\gamma')\{W_{\gamma+\gamma'}(w,\bar{w}) + \dots\}. \tag{1.45}$$

The CFT of the free boson on the circle of radius $r = \frac{\sqrt{2}}{2}R$ now consists of the representation $\mathbb{H}_{circ}(R)$, its real structure (1.30), scalar product (1.31) and the fields just defined and will be denoted by $\mathcal{T}(R) = \mathcal{T}(\Gamma_R)$. The n-point functions follow from the OPEs. We see that this CFT depends on the charge lattice Γ_R and thus a positive real number R.

However, not every such value R gives a distinct theory. There is an interesting duality called *T-duality*: Some values of R lead to the same physics:

$$\mathcal{T}(R) = \mathcal{T}\left(\frac{1}{R}\right). \tag{1.46}$$

This is a baby-version of *mirror symmetry* ([38]). In fact, the charge lattice, on which the theory depends, is the lowest-dimensional example of a *Narain lattice*, an element of $O(1,1;\mathbb{R})$ and the moduli space of $c = 1$ toroidal CFTs is ([57], [39])

$$\mathcal{M}_{1,1} := O(1;\mathbb{R}) \times O(1;\mathbb{R}) \backslash O(1,1;\mathbb{R}) / O'(1,1;\mathbb{Z}), \tag{1.47}$$

where

$$O'(1,1;\mathbb{Z}) := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} O(1,1;\mathbb{Z}) \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{1.48}$$

This is a bit over-complicated considering the moduli space is also $\mathbb{R}_{\geq 1}$, but useful in view of higher dimensional toroidal CFTs.

1.2.2. Toroidal CFT in arbitrary dimension

We turn to the general case following WENDLAND's abstract definition of a toroidal CFT ([11], [52]) in arbitrary dimension:

Definition 1.2.3 ([57], Definition 4.1.1)**.** A unitary conformal field theory with central charge $c = D \in \mathbb{N}$ is called *toroidal*, if its holomorphic and antiholomorphic W-algebras each contain a $u(1)_1^D$ affine Kac-Moody algebra. A choice of Abelian currents generating these algebras will be denoted $j^1, \dots, j^D; \bar{j}^1, \dots, \bar{j}^D$ if it is normalized to

$$j^k(z) j^l(z) \sim \frac{\delta^{kl}}{(z-w)^2} \quad \text{resp.} \quad \bar{j}^k(\bar{z}) \bar{j}^l(\bar{z}) \sim \frac{\delta^{kl}}{(\bar{z}-\bar{w})^2}. \tag{1.49}$$

Charges with respect to $(j;\bar{j}) := (j^1, \dots, j^D; \bar{j}^1, \dots, \bar{j}^D)$ are denoted by $p = (p_l; p_r)$.

The commuting Virasoro fields are given by the Sugawara construction ([33]) as

$$T(z) = \frac{1}{2}\sum_{k=1}^{D} : j^k j^k : (z), \quad \bar{T}(\bar{z}) = \frac{1}{2}\sum_{k=1}^{D} : \bar{j}^k \bar{j}^k : (z). \tag{1.50}$$

Just as in the one-dimensional case, it can also be seen more explicitly as a direct sum of representations of the $u(1)_1^D$-current algebra with primary fields whose charges are given by a charge lattice. For this let 1_D be the $D \times D$ identity matrix and

$$\mathbf{E} := \frac{1}{\sqrt{2}}\begin{pmatrix}1_D & 1_D\\ 1_D & -1_D\end{pmatrix} \quad \text{and} \quad O'(D,D;\mathbb{Z}) := \mathbf{E} O(D,D;\mathbb{Z}) \mathbf{E}. \tag{1.51}$$

Theorem 1.2.4 ([57], Theorem 4.1.2). *A toroidal CFT with central charge $c = D \in \mathbb{N}$ is uniquely determined by its charge lattice Γ, an even, self-dual lattice with signature (D,D). The moduli space of toroidal CFTs agrees with that of even, self-dual lattices with signature (D,D):*

$$\mathcal{M}_{D,D} := O(D;\mathbb{R}) \times O(D;\mathbb{R}) \backslash O(D,D;\mathbb{R}) / O'(D,D;\mathbb{Z}). \tag{1.52}$$

As in the one-dimensional case there is a relation between the charge lattice and a geometric interpretation (target space). The Narain lattice can be shown ([57], 4.2) to depend on $(\Lambda, B) \in O(D;\mathbb{R}) \backslash \mathrm{GL}(D;\mathbb{R}) \times \mathrm{Skew}(D;\mathbb{R})$:

$$\Gamma = \Gamma(\Lambda, B) = \left\{ (p_l(\mu,\lambda); p_r(\mu,\lambda)) := \frac{1}{\sqrt{2}} \left(\mu - \tilde{B}\lambda + \lambda; \mu - \tilde{B}\lambda - \lambda \right) \middle| (\mu,\lambda) \in \Lambda^* \oplus \Lambda \right\} \tag{1.53}$$

with so-called *B-field* $B = \Lambda^T \tilde{B} \Lambda$. Any element of the charge lattice gives rise to a representation of the $u(1)_1^D$ affine Kac-Moody-algebra as in $\mathcal{T}(\Gamma_R)$ and associated fields can be constructed similarly. We denote the corresponding CFT by $\mathcal{T}$ or $\mathcal{T}(\Gamma)$ respectively $\mathcal{T}(\Lambda, B)$, when we want to emphasize the dependence on Γ respectively Λ and B. For the partition function we have

$$Z_{\mathcal{T}}(\tau) = \frac{1}{|\eta(\tau)|^{2D}} Z_\Gamma(\tau) := \frac{1}{|\eta(\tau)|^{2D}} \sum_{(p_l;p_r)\in\Gamma} q^{\frac{1}{2}p_l^2} \bar{q}^{\frac{1}{2}p_r^2}. \tag{1.54}$$

For central charge $c = 2$ there is a neat way to collect the parameters of Λ and B into two complex numbers $\tau = \tau_1 + i\tau_2$ and $\rho = \rho_1 + i\rho_2$:

$$\Lambda = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \rho_1 \\ -\rho_1 & 0 \end{pmatrix}, \tag{1.55}$$

where τ encodes the complex structure of the target torus and ρ, called *complexified Kähler modulus* involves the volume of the torus ρ_2 and the B-field degree of freedom. Here T-duality takes the form of

$$\mathcal{T}(\tau,\rho) = \mathcal{T}(\rho,\tau), \tag{1.56}$$

i.e. exchanging complex structure and complexified Kähler modulus.

1.3. Vertex operator algebra

The $\mathcal{W}$-algebra mentioned in the definition of CFT has an interesting mathematical structure in its own. Functions associated to vectors in the CFT space of states involve both holomorphic and antiholomorphic contributions and combinations thereof. Restricting to the holomophic contributions gives rise to the structure of a vertex operator algebra. For more on vertex algebra see [40] by Kac, [30] by Gannon or [26] by Frenkel and Ben-Zvi. The advantage of this object is that the n-point functions are a biproduct of the OPE. We do not need to define n-point functions for vertex algebras, and thus the definition of the OPE becomes simpler.

Definition 1.3.1 ([40], 1.3)**.** A *vertex operator algebra* is a datum $(V, Y, L, \mathbb{1})$ where

1. V is a $\mathbb{C}$-vector space called *space of states* with Z_2-grading $V = V^{even} \oplus V^{odd}$ and $\mathbb{Z}$-grading $V = \oplus_{m\in\mathbb{Z}} V_m$.
2. $\mathbb{1} \in W_0$ is a special vector called *vacuum*.
3. Y is a linear map
$$Y(\cdot, z) : V^{even} \longrightarrow \mathrm{End}(V)[[z^{\pm}]] \text{ for even elements and} \tag{1.57}$$
$$Y(\cdot, z) : V^{odd} \longrightarrow z^{\frac{1}{2}} \cdot \mathrm{End}(V)[[z^{\pm}]] \text{ for odd elements,} \tag{1.58}$$
called *state-field correspondence.* The image $A(z) := Y(A, z)$ is called *field.*
4. $L : W \to W$ is a special operator called *translation operator.*

The datum satisfies the following axioms:

1. (*Vacuum axiom*) $Y(\mathbb{1}, z) = \mathbb{1}_V$ is the identity operator on V and $Y(A, z)\mathbb{1}\,|_{z=0} = A$ retrieves the original vector.
2. (*Translation axiom*) For all vectors $A \in V: \quad [L, A(z)] = \partial A(z)$.
3. (*Locality axiom*) All fields are local to each other, meaning for all vectors $A, B \in V$
$$(z-w)^N [A(z), B(w)] = 0 \quad \text{for } N >> 0. \tag{1.59}$$

A vertex algebra is called *conformal at central charge c* if in addition there is a special vector $T \in V_2$ called *conformal vector*, whose associated field $T(z) = \sum_n L_n z^{n-2}$ is a Virasoro field at central charge c. The translation operator is $T = L_1$. For all $m \in \mathbb{Z} : L_0\,|_{V_m} = m \cdot \mathbb{1}_V$ and for $A \in V_m$ we have $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{n-m}$ with $A_n \in \mathrm{End}(V)$ of degree n.

We will simply call V the vertex algebra when it is clear what the remaining data is.

The locality axiom is equivalent to
$$[A(z), B(w)] = \sum_{j=0}^{N-1} \frac{C_j(w)}{j!} \partial_w^j \delta(z-w) \tag{1.60}$$
in $\mathrm{End}(V)[[z^{\pm}, w^{\pm}]]$ ([40], Theorem 2.3). Here C_j is in V and δ is the formal delta distribution $\delta(z-w) = \frac{1}{z}\sum_n \left(\frac{w}{z}\right)^n$. In the domain $|z| > |w|$ one can expand the formal delta distribution using the geometric series. In this case we write
$$A(z)B(w) \sim \sum_{j=0}^{N-1} \frac{C_j(w)}{(z-w)^{j+1}} \tag{1.61}$$
and call this the OPE. In the realm of vertex algebra one does not need a definition of n-point functions on its own. They arise naturally as the successive application of the OPE.

Another way to regard a vertex algebra V is as an infinite dimensional vector space together with products $A_n B$ for $n \in \mathbb{Z}$ of any two elements A, B of V satisfying the relations imposed by locality. These relations are known as *Borcherd's identity* ([40], Proposition 4.8 (b)).

Example 1.3.2 (Free boson algebra)**.** We have already encountered an example of a vertex algebra in the definition of toroidal CFT. The free boson algebra representation B^0 is a vertex algebra. With $T(z) := \frac{1}{2} : j(z)j(z) :$ it is even conformal.

Theorem 1.3.3 ([59], Chapter 4)**.** *Let $\mathcal{W}$ denote holomorphic $\mathcal{W}$-algebra of a CFT. Then it carries the structure of a conformal vertex algebra. The endomorphisms appearing as coefficients in the state-field correspondence form a Lie algebra $\mathcal{A}$, and analogously for the antiholomorphic fields $\bar{\mathcal{A}}$.*

Note that in general the concept of a vertex algebra is not sufficient to capture all structure of a CFT. This is only true for so-called *holomorphic CFT*.

Definition 1.3.4 ([59], Definition 4.30)**.** Let V be the vertex algebra associated to the holomorphic states of a conformal field theory. Let $\mathcal{A}$ and $\bar{\mathcal{A}}$ be the corresponding Lie algebras. Then $\mathcal{A} \otimes \bar{\mathcal{A}}$ is called the *chiral algebra* of $\mathbb{H}$.

For a toroidal conformal field theory the chiral algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$ differs greatly depending on the point in the moduli space, which we will see in the next subsection.

1.3.1. Complex multiplication

An important feature of the minimal models ([4]) is that their space of states decomposes into only finitely many irreducible representations of the two commuting copies of the Virasoro algebra. For toroidal CFTs this would be too much to ask for, the Virasoro algebra is too small. However, there is an important class of CFTs, *rational CFTs* (see [51], [29]) that generalizes this concept by enlarging the Virasoro algebra:

Definition 1.3.5 ([59], Chapter 4)**.** Let $\mathcal{C}$ be a CFT. If its space of states $\mathbb{H}$ decomposes into a finite sum of irreducible representations of the chiral algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$, then $\mathbb{H}$ is called *rational.*

Since [4] rational CFTs with central charge $c < 1$ have been completely classified, while for general c this is open. Vafa showed in [56] and Anderson and Moore in [1] that rational CFTs have rational central charge and all fields have rational conformal weight. Whether this is sufficient is an open problem. For toroidal CFTs this is known to be true, see e.g. [57] for the full argument.

For a toroidal CFT the chiral algebra will always include the free boson algebras generated by the holomorphic and antiholomorphic $u(1)$-currents. Whether there are any additional fields depends on the structure of the charge lattice Γ. For the free boson on the circle the only candidates for additional holomorphic fields are lattice vectors $(Q, 0) \in \Gamma_R$ with $Q \neq 0$. This can happen only for $R^2 \in \mathbb{Q}$ and in precisely these cases the chiral algebra becomes large enough to render $\mathbb{H}_R$ rational. For theories with central charge $c = 2$ rationality is equivalent to $\tau, \rho \in \mathbb{Q}(\sqrt{D})$ for $D < 0$, i.e. if both elliptic curves associated to τ and ρ have complex multiplication in the same quadratic imaginary field. For these $c = 2$ rational toroidal theories Hosono, Lian, Oguiso and Yau found an interesting connection in [39] between the two complex parameters τ and ρ and the classical *Gauss product* which leads to a classification of these rational theories using class field theory. For tori of arbitrary dimension Wendland showed:

Theorem 1.3.6 ([57], Theorem 4.5.2). *Let $\mathcal{T}(\Lambda, B)$ denote a toroidal CFT with $c = d$. Then $\mathcal{C}$ is rational if and only if $G := \Lambda^T \Lambda \in \mathrm{GL}(d, \mathbb{Q})$ and $B \in \mathrm{Skew}(d) \cap \mathrm{Mat}(d, \mathbb{Q})$.*

Thus the chiral algebra changes radically depending on the modulus of the theory.

Chapter 2

Superconformal field theory

In this chapter we extend the notion of CFT from the previous chapter to superconformal field theories (SCFTs) that include fermions.

In the first section we will review the definition of an $N = (1, 1)$ and an $N = (2, 2)$ superconformal field theory. In Section 2 we review the Ising model, an example of a fermionic CFT. Section 3 treats the Dirac fermion and its interpretation as a toroidal CFT via bosonization. In Section 4 we review the definition of a toroidal SCFT and the $N = 4$ superconformal algebra and how it arises in this context ([21], [22], [23], [24], [54]). In the last section we turn to the elliptic genus, a topological invariant that appears both in geometry and SCFT and that will be a driving force in the subsequent chapters.

The main references for this chapter are [31], [57] and [58].

2.1. Definition

We will define superconformal field theories (SCFTs) as extensions of the bosonic CFTs that we saw in the first chapter. This extension includes fermions. Contrary to the bosons that we have encountered so far, fermions come with different boundary condition called spin structures. Ultimately they are only defined on a double cover of the punctured plane.

In SCFTs with a non-linear sigma model interpretation this can be illustrated as follows: Let $\varphi \in \mathcal{C}^\infty(\mathbb{C}/\Lambda_\tau, \mathbb{R}/2\pi r\mathbb{Z})$ be a string as in Subsection 1.2.1. Then the conditions (1.21) and (1.21) can be relaxed to allow periodic (P) or anti-periodic (A) boundary conditions along both cycles of the fundamental domain of the torus, e.g.:

$$\varphi(z+1) = -\varphi(z), \tag{2.1}$$

$$\varphi(z+\tau) = -\varphi(z), \tag{2.2}$$

where z is the coordinate on $\mathbb{C}$. Such a string φ is said to belong to the sector ${}_{A}\underset{A}{\square}$. All sectors ${}_{P}\underset{P}{\square}$, ${}_{A}\underset{P}{\square}$, ${}_{P}\underset{A}{\square}$ and ${}_{A}\underset{A}{\square}$ can appear in a SCFT.

Which sector we label periodic and which anti-periodic is somewhat arbitrary since it depends on the worldsheet coordinates we use. It is common to use torus and cylindrical coordinates. They can be conformally mapped into one another via $w \mapsto z = e^w$, where w is the cylinder coordinate and z the torus coordinate. Under this conformal change of coordinates a periodic field on the torus becomes anti-periodic on the cylinder and vice versa (cf. [31], Section 7.1).

Similarly to CFT the first ingredient of SCFT is the space of states. The different boundary conditions are captured as follows: The space of states of a fermionic CFT comes with a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading:

$$\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f \tag{2.3}$$

with

$$\mathbb{H}_k = \mathbb{H}_k^{NS} \oplus \mathbb{H}_k^R, \quad k \in \{b, f\}\,. \tag{2.4}$$

The vector space $\mathbb{H}_b$ is the space of purely bosonic states and $\mathbb{H}_f$ the space of purely fermionic states. NS stands for the *Neveu-Schwarz* sector and R for the *Ramond* sector. $\mathbb{H}$ posses a real structure that induces real structures on each of the four sectors $\mathbb{H}_k^s$, $k \in \{b, f\}$, $s \in \{NS, R\}$.

In the non-linear sigma model interpretation the Neveu-Schwarz sector corresponds to spacetime bosons, i.e. ${}_A\square_A$, and the Ramond-sector to spacetime fermions, i.e ${}_A\square_P$.

The space of purely bosonic states $\mathbb{H}_b$ is itself the space of states of a CFT. Our notion of fields and OPEs of Chapter 1 has to be adjusted to accommodate the different boundary conditions. On $\mathbb{H}_b$ the OPE has to be compatible with the $\mathbb{Z}_2$-grading, where NS-states are even and R-states are odd.

For an element $\Psi \in \mathbb{H}_f^{NS}$ there is a field $\Psi(z, \bar{z})$ such that the state-field correspondence $\Psi(0,0)\Omega = \Psi$ holds. The OPE extends to $\mathbb{H}_f^{NS}$ and is compatible with the $\mathbb{Z}_2$-grading on $\mathbb{H}^{NS} = \mathbb{H}_b^{NS} \oplus \mathbb{H}_f^{NS}$, where bosonic states are even and fermionic ones are odd. The sector $\mathbb{H}^{NS}$ is a fermionic representation of the OPE of $\mathbb{H}_b^{NS}$ (see Definition 1.1.5). Here fermionic means that for two odd elements pairwise locality is relaxed to *semi-locality*, i.e. commuting up to a change of sign:

$$\langle \varphi_1(z_1, \bar{z}_1) \cdots \varphi_n(z_n, \bar{z}_n) \rangle = (-1)^I \langle \varphi_{\sigma(1)}(z_{\sigma(1)}, \bar{z}_{\sigma(1)}) \cdots \varphi_{\sigma(n)}(z_{\sigma(n)}, \bar{z}_{\sigma(n)}) \rangle, \tag{2.5}$$

where $\varphi_i \in \mathbb{H}$ are all either even or odd, $\sigma \in S_n$ is a permutation and I counts the number of inversions of odd states.

States $\Psi \in \mathbb{H}_f^{NS}$ have half integer spin $h - \bar{h}$. If Ψ is a holomorphic fermionic field of dimension h in the NS-sector it has mode expansion

$$\Psi(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \Psi_r z^{r-h}. \tag{2.6}$$

The OPEs on $\mathbb{H}_b$ and $\mathbb{H}^{NS}$ can be extended to a non-local OPE $\mathbb{H}^{NS} \otimes \mathbb{H}_b^R \to \mathbb{H}_f^R\{\{z, \bar{z}\}\}$, where $f \in \mathbb{H}_f^R\{\{z, \bar{z}\}\}$ has the following behavior around $z = 0$:

$$f(z, \bar{z}) = \sum_{r \in R, n \in \mathbb{Z}} a_{rn} z^{r+n-\frac{1}{2}} \bar{z}^r, \tag{2.7}$$

with countable $R \subset \mathbb{R}$, $a_{rn} \in \mathbb{H}$ and only finitely many singular terms. The n-point functions can then be adjusted to contain arbitrary fields in $\mathbb{H}$.

A fermionic field Ψ is never single valued in the R-sector:

$$\Psi(\mathrm{e}^{2\pi i} z, \mathrm{e}^{-2\pi i} \bar{z}) = -\Psi(z, \bar{z}). \tag{2.8}$$

This is a common notation in the physics literature and means that for $t \in [0,1)$ we have $\lim_{t\to 1} \Psi(\mathrm{e}^{2\pi i t} z, \mathrm{e}^{-2\pi i t} \bar{z}) \to -\Psi(z,\bar{z})$, i.e. $\Psi(z,\bar{z})$ lives on a double cover of the punctured plane.

The two different boundary conditions in the space direction are differentiated by the Neveu-Schwarz and the Ramond sectors. The change in boundary condition in the time direction can be achieved by inserting the fermionic operator $(-1)^F$, which anti-commutes with fermions. In presence of a $u(1)$-current J_0 the traces over the different sectors are

$$Z_{NS}(\tau,z) := \mathrm{tr}_{NS}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}y^{J_0}\bar{y}^{\bar{J}_0}\right), \tag{2.9}$$

$$Z_{\widetilde{NS}}(\tau,z) := \mathrm{tr}_{NS}\left((-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}y^{J_0}\bar{y}^{\bar{J}_0}\right), \tag{2.10}$$

$$Z_{R}(\tau,z) := \mathrm{tr}_{R}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}y^{J_0}\bar{y}^{\bar{J}_0}\right), \tag{2.11}$$

$$Z_{\widetilde{R}}(\tau,z) := \mathrm{tr}_{R}\left((-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}y^{J_0}\bar{y}^{\bar{J}_0}\right). \tag{2.12}$$

The total superconformal partition function then is

$$Z(\tau,z) := \frac{1}{2}\left(Z_{NS}(\tau,z) + Z_{\widetilde{NS}}(\tau,z) + Z_R(\tau,z) + Z_{\widetilde{R}}(\tau,z)\right). \tag{2.13}$$

We now turn to a very interesting connection between CFT/SCFT and number theory functions via the (super) partition function. The partition function counts the number of states of the same conformal weight. In the case of toroidal CFTs for the vibration modes $a_n^i, \bar{a}_n^i$ this involves counting the number of partitions of a natural number. We have already seen that this enters the toroidal partition function as the Dedekind eta-function. For SCFTs other number theory functions appear as well. We will collect the relevant functions in the next definition.

Definition 2.1.1 ([57], Appendix A, [59], Definition 2.50)**.** We give the following *Jacobi theta functions* both as infinite sums and (by way of Poisson resummation) infinite products:

$$\vartheta_1(\tau,z) := i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}y^{n-\frac{1}{2}} \tag{2.14}$$

$$= iq^{\frac{1}{8}}y^{-\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)\left(1-q^{n-1}y\right)\left(1-q^ny^{-1}\right), \tag{2.15}$$

$$\vartheta_2(\tau,z) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}y^{n-\frac{1}{2}} \tag{2.16}$$

$$= q^{\frac{1}{8}}y^{-\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n-1}y\right)\left(1+q^ny^{-1}\right), \tag{2.17}$$

$$\vartheta_3(\tau,z) := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}y^{n} \tag{2.18}$$

$$= \prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n-\frac{1}{2}}y\right)\left(1+q^{n-\frac{1}{2}}y^{-1}\right), \tag{2.19}$$

$$\vartheta_4(\tau,z) := \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}}y^{n} \tag{2.20}$$

$$= \prod_{n=1}^{\infty}(1-q^n)\left(1-q^{n-\frac{1}{2}}y\right)\left(1-q^{n-\frac{1}{2}}y^{-1}\right). \tag{2.21}$$

Here $q := \exp(2\pi i\tau)$ and $y := \exp(2\pi i z)$. The ϑ_k are analytic in $q^{\frac{1}{2}}$ with $\left|q^{\frac{1}{2}}\right| < 1$ and $z \in \mathbb{C}$. From now on we use the shorthand notation $\vartheta_i(\tau) := \vartheta_i(\tau, 0)$.

In order to calculate toroidal orbifolds we will also need the transformation properties of the Jacobi theta functions under the modular group:

$$\vartheta_1(\tau+1, z) = \mathrm{e}^{\frac{\pi i}{4}}\,\vartheta_1(\tau, z), \tag{2.22}$$
$$\vartheta_2(\tau+1, z) = \mathrm{e}^{\frac{\pi i}{4}}\,\vartheta_2(\tau, z), \tag{2.23}$$
$$\vartheta_3(\tau+1, z) = \vartheta_4(\tau, z), \tag{2.24}$$
$$\vartheta_4(\tau+1, z) = \vartheta_3(\tau, z), \tag{2.25}$$
$$\vartheta_1\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = (-i)(-i\tau)^{\frac{1}{2}}\,\mathrm{e}^{\frac{\pi i z^2}{\tau}}\,\vartheta_1(\tau, z), \tag{2.26}$$
$$\vartheta_2\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\mathrm{e}^{\frac{\pi i z^2}{\tau}}\,\vartheta_4(\tau, z), \tag{2.27}$$
$$\vartheta_3\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\mathrm{e}^{\frac{\pi i z^2}{\tau}}\,\vartheta_3(\tau, z), \tag{2.28}$$
$$\vartheta_4\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\mathrm{e}^{\frac{\pi i z^2}{\tau}}\,\vartheta_2(\tau, z). \tag{2.29}$$

For more properties of the theta functions and values at special points see the very convenient compilation in Appendix A of [57].

We collect the different sectors and their boundary conditions in the following table:

sector	**boundary condition**	**field behaviour**
NS	${}_{A}\square_{A}$	$\varphi(z+1) = -\varphi(z)$ $\varphi(z+\tau) = -\varphi(z)$
$\widetilde{NS}$	${}_{P}\square_{A}$	$\varphi(z+1) = -\varphi(z)$ $\varphi(z+\tau) = \varphi(z)$
R	${}_{A}\square_{P}$	$\varphi(z+1) = \varphi(z)$ $\varphi(z+\tau) = -\varphi(z)$
$\widetilde{R}$	${}_{P}\square_{P}$	$\varphi(z+1) = \varphi(z)$ $\varphi(z+\tau) = \varphi(z)$,

So far we have defined a fermionic CFT. For a superconformal field theory the Virasoro algebra has to be enlarged to the $N = 1$ *superconformal algebra*: The NS sector of an $N = (1,1)$ *superconformal field theory* contains a copy of an $N = 1$ superconformal algebra on both the left and right moving side. The $N = 1$ superconformal algebra is generated by the bosonic Virasoro field $T(z)$ and the fermionic holomorphic field $G(z)$ of dimension $h = \frac{3}{2}$, such that

$$G(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} G_r z^{r-\frac{3}{2}}, \tag{2.30}$$

where the operators G_r satisfy

$$[L_m, G_r] = \left(r - \frac{m}{2}\right) G_{m+r}, \tag{2.31}$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \tag{2.32}$$

G is unitary, i.e. $(G_n)^\dagger = G_{-n}$ and there is an analogue field $\bar{G}(\bar{z})$ on the right moving side.

For the $N = (2,2)$ *superconformal field theory* the field $G(z)$ splits into two fields $G(z) = \frac{1}{\sqrt{2}}(G^+(z) + G^-(z))$ in the NS sector, such that $(G_n^+)^\dagger = G_{-n}^-$. The modes of $G^+(z)$ and $G^-(z)$ satisfy

$$[L_m, G_r^\pm] = \left(r - \frac{m}{2}\right) G_{m+r}^\pm, \tag{2.33}$$

$$\{G_r^+, G_s^+\} = \{G_r^-, G_s^-\} = 0, \tag{2.34}$$

$$\{G_r^+, G_s^-\} = 2L_{r+s} + (s-r)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \tag{2.35}$$

$$[L_m, J_n] = mJ_{m+n}, \quad [J_m, G_r^\pm] = \pm G_{m+r}^\pm, \quad [J_m, J_n] = \frac{c}{3} n\delta_{m+n,0}, \tag{2.36}$$

with $m, n \in \mathbb{Z}, r, s \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector and $m, n, r, s \in \mathbb{Z}$ in the Ramond sector. This defines the $N = 2$ superconformal algebra. On the antiholomorphic side $\bar{G}(\bar{z})$ splits analogously.

We will now turn to the relevant examples of SCFTs for this thesis.

2.2. Ising model

The first example of a fermionic CFT is the Ising model. It is not yet an example of an $N = (1,1)$ SCFT. The Ising model describes a single holomorphic Majorana fermion Ψ and its antiholomorphic counterpart $\bar{\Psi}$ with coupled spin structure. In statistical mechanics the Ising model describes the second order phase transition of ferromagnetism. Its partition function is given by

$$Z_{Ising}(\tau) = \frac{1}{2}\left({}_{P\bar{P}}\underset{P\bar{P}}{\boxed{}}(\tau) + {}_{A\bar{A}}\underset{P\bar{P}}{\boxed{}}(\tau) + {}_{P\bar{P}}\underset{A\bar{A}}{\boxed{}}(\tau) \pm {}_{A\bar{A}}\underset{A\bar{A}}{\boxed{}}(\tau) \right) \tag{2.37}$$

$$= \frac{1}{2}\left(\left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right| + \left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right| \pm \left|\frac{\vartheta_1(\tau)}{\eta(\tau)}\right| \right), \tag{2.38}$$

where the choice of $\pm$ depends on whether we project to the $(-1)^F = \pm 1$ eigenvectors respectively. The holomorphic fermion Ψ appears in the different sectors as

$$\begin{aligned} P: \quad & \Psi(z) = \textstyle\sum_{n \in \mathbb{Z}+\frac{1}{2}} \Psi_n z^{n-\frac{1}{2}}, \\ A: \quad & \Psi(z) = \textstyle\sum_{n \in \mathbb{Z}} \Psi_n z^{n-\frac{1}{2}}. \end{aligned} \tag{2.39}$$

The space of states of the Ising model can be constructed via a Fock space representation of the free fermion algebra similarly to the free boson on the circle of Subsection 1.2.1: Fix $r \in \left\{0, \frac{1}{2}\right\}$ and let $f = f_0 \oplus f_1$ be the Lie superalgebra where $f_0 = \mathbb{C}$ and f_1 has $\mathbb{C}$-vector space basis $\{\, \Psi_n \,|\, n \in \mathbb{Z} + r \,\}$ and relations

$$\{\Psi_n, \Psi_m\} = \delta_{m+n,0}. \tag{2.40}$$

If $r = \frac{1}{2}$ then f is in the Neveu-Schwarz sector, if $r = 0$ then f is in the Ramond sector. The universal enveloping algebra $\mathcal{F}$ of f is called the *free fermion algebra.* Choose $h \in \left\{0, \frac{1}{16}\right\}$ and let $\mathcal{I}$ be the left-handed ideal in $\mathcal{F}$ generated by

$$\Psi_{-n} \text{ for } n > 0, \quad 1_{f\otimes 1} - 1_{f\otimes 0}, \quad \Psi_0 - \frac{1}{\sqrt{2}} \text{ if } r = 0. \tag{2.41}$$

The quotient $F^h := \mathcal{F}/\mathcal{I}$ is called the *Fock space representation of the free fermion.* The representation F^0 splits into two subrepresentations F_+ with an even number of Ψ_n's and F_- with an odd number of Ψ_n's. Each of the spaces F_+, F_- and $F^{\frac{1}{16}}$ carries a representation of the Virasoro algebra at central charge $c = \frac{1}{2}$ ([59], Chapter 2).

The space of states of the Ising model now is the following modular invariant combination of the thus constructed representations:

$$\mathbb{H}_{Ising} = \left(F_+ \otimes \bar{F}_+\right) \oplus \left(F_- \otimes \bar{F}_-\right) \oplus \left(F^{\frac{1}{16}} \otimes \bar{F}^{\frac{1}{16}}\right). \tag{2.42}$$

This statement is true on the level of representations, we still need to define a real structure on the Fock spaces. This can be done similarly to the case of the free boson (cf. (1.30)). Care must be taken in the R-sector. In order to fulfill the CFT definition (cf. Chapter 1) one cannot use the same real structures on $F^{\frac{1}{16}}$ and $\bar{F}^{\frac{1}{16}}$. For details see [59].

We denote the lowest weight vectors by Ω, ε and σ, where

$$\varepsilon(z) := i : \Psi(z)\bar{\Psi}(z) : . \tag{2.43}$$

In the NS sector the free fermion has OPE

$$\Psi(z)\Psi(w) \sim \frac{1}{z-w}. \tag{2.44}$$

In the Ramond sector one uses the *twist field* σ: The field σ is the twist field defined to introduce the square-root branch cut

$$\Psi(z)\sigma(w,\bar{w}) \sim \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \frac{\mu(w,\bar{w})}{(z-w)^{\frac{1}{2}}}, \tag{2.45}$$

where μ is another twist field of same dimensions as σ, but with $(-1)^F$ eigenvalue -1. The remaining OPEs between these fields can be found by considering 4-point correlation functions (cf. [31], Section 7.5). The different sectors of the Ising model are given by:

$$\mathbb{H}_b^{NS} = \operatorname{span}_{\mathbb{C}} \{\Omega, \varepsilon\}, \tag{2.46}$$
$$\mathbb{H}_f^{NS} = \operatorname{span}_{\mathbb{C}} \{\Psi, \bar{\Psi}\}, \tag{2.47}$$
$$\mathbb{H}_b^{R} = \operatorname{span}_{\mathbb{C}} \{\sigma\}, \tag{2.48}$$
$$\mathbb{H}_f^{R} = \operatorname{span}_{\mathbb{C}} \{\mu\}. \tag{2.49}$$

2.3. Dirac fermion and bosonization

The Dirac fermion consists of two holomorphic fermions Ψ^1, Ψ^2 and their antiholomorphic counterparts $\bar{\Psi}^1, \bar{\Psi}^2$ with coupled spin structure, i.e. the partition function is given by

$$Z_{Dirac}(\tau) = \frac{1}{2}\left({}_{(A\bar{A})^2}\underset{(A\bar{A})^2}{\square}(\tau) + {}_{(P\bar{P})^2}\underset{(A\bar{A})^2}{\square}(\tau) + {}_{(A\bar{A})^2}\underset{(P\bar{P})^2}{\square}(\tau) \pm {}_{(P\bar{P})^2}\underset{(P\bar{P})^2}{\square}(\tau) \right). \tag{2.50}$$

The partition function of the Dirac fermion is equal to the $c = 1$ toroidal CFT at $R = \frac{1}{2}$:

$$\begin{aligned} Z_{Dirac}(\tau) &= \frac{1}{2}\left(\left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right|^2 \pm \left|\frac{\vartheta_1(\tau)}{\eta(\tau)}\right|^2 \right) &\quad (2.51)\\ &= Z_{circ}\left(R = \frac{1}{\sqrt{2}}\right). &\quad (2.52)\end{aligned}$$

This can be understood via bosonization: Instead of the two Majorana fermions we consider their complexifications

$$\Psi^{\pm}(z) := \frac{1}{\sqrt{2}}\left(\Psi^1(z) \pm i\Psi^2(z)\right) \tag{2.53}$$

with OPEs

$$\begin{aligned} \Psi^{\pm}(z)\Psi^{\pm}(w) &\sim 0 &\quad (2.54)\\ \Psi^{\pm}(z)\Psi^{\mp}(w) &\sim \frac{1}{z-w}. &\quad (2.55)\end{aligned}$$

We can then identify

$$\Psi^{\pm}(z) = \mathrm{e}^{\pm iX(z)}, \tag{2.56}$$

the complexified fermions with vertex operators in the bosonic theory. Note that this operator does not show up in the circle theory at $R = \frac{1}{\sqrt{2}}$ since it is also projected out of the Dirac fermion space of states by the GSO-projection that projects to the $(-1)^F = 1$ states.

The energy operator can be found via

$$\begin{aligned} :\Psi^1(z)\Psi^2(z): &= \lim_{z\to w}\frac{1}{2}\left(\mathrm{e}^{iX(z)} + \mathrm{e}^{-iX(z)}\right)\left(\mathrm{e}^{iX(z)} - \mathrm{e}^{-iX(z)}\right) &\quad (2.57)\\ &= -i\partial X(z). &\quad (2.58)\end{aligned}$$

Thus we can identify

$$\varepsilon_1 \times \varepsilon_2 \longleftrightarrow -:\Psi^1_{\frac{1}{2}}\bar{\Psi}^1_{\frac{1}{2}}\Psi^2_{\frac{1}{2}}\bar{\Psi}^2_{\frac{1}{2}} :=: \Psi^1_{\frac{1}{2}}\Psi^2_{\frac{1}{2}}\bar{\Psi}^1_{\frac{1}{2}}\bar{\Psi}^2_{\frac{1}{2}} :\longleftrightarrow -:\partial X(z)\bar{\partial}\bar{X}(\bar{z}):. \tag{2.59}$$

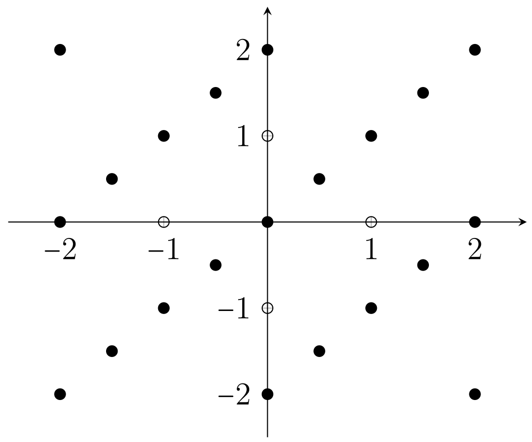
Also note that $\mathbb{H}_{1,1}$ is one-dimensional at $R = \frac{1}{\sqrt{2}}$.

In the torus theories we had to use cocycle factors to implement locality. This is also true for the bosonized Dirac fermion $\mathbb{H}_{Dirac} = \mathbb{H}_{circ}\left(R = \frac{1}{\sqrt{2}}\right)$. Note, however, that the complexified fermions $\Psi^{\pm}(z)$ do not survive the GSO-projection. If we want to include them we can go to a cover theory. For the complexified fermions we do not need cocycle factors. Their absence implements semi-locality which is precisely what we want for fermions.

The charge lattice

$$\Gamma_{\frac{1}{\sqrt{2}}} = \left\{ \left(\frac{n}{2} + m, \frac{n}{2} - m \right) \middle| n, m \in \mathbb{Z} \right\} \tag{2.60}$$

looks like



where complexified fermions are also included with unfilled circles. They can be included by allowing $m \in \frac{1}{2}\mathbb{Z}$ which extends the theory to a non-local covering theory. The states of the NS sector of the theory with $(-1)^F = \pm 1$ eigenvalues are given respectively by $\{ n \in 2\mathbb{Z}, m \in \mathbb{Z} \}$ and $\{ n \in 2\mathbb{Z}+1, m \in \mathbb{Z} + \frac{1}{2} \}$. The states of the R sector with $(-1)^F = \pm 1$ are given respectively by $\{ n \in 2\mathbb{Z}+1, m \in \mathbb{Z} \}$ and $\{ n \in 2\mathbb{Z}, m \in \mathbb{Z} + \frac{1}{2} \}$ ([31], Section 8.2).

Using the complexified fermion we can define a $u(1) - current$

$$J(z) :=: \Psi^+(z)\Psi^-(z) : . \tag{2.61}$$

With respect to its zero mode J_0 the superconformal partition function is

$$Z_{Dirac}(\tau, z) = \frac{1}{2}\left(\left| \frac{\vartheta_3(\tau,z)}{\eta(\tau)} \right|^2 + \left| \frac{\vartheta_4(\tau,z)}{\eta(\tau)} \right|^2 + \left| \frac{\vartheta_2(\tau,z)}{\eta(\tau)} \right|^2 \pm \left| \frac{\vartheta_1(\tau,z)}{\eta(\tau)} \right|^2 \right). \tag{2.62}$$

The ground state of the R-sector can be read off from this to have conformal weight $(h, \bar{h}) = (\frac{1}{8}, \frac{1}{8})$ and charge $(Q, \bar{Q}) = (\frac{-1}{2}, \frac{-1}{2})$. Up to a complex multiple the only field in the circle theory with this property is

$$\sigma(z) :=: \mathrm{e}^{-i\frac{1}{2}X(z) - i\frac{1}{2}\bar{X}(\bar{z})} : . \tag{2.63}$$

The OPEs with the complexified fermions are

$$\Psi^+(z)\sigma(w) \sim (z-w)^{-\frac{1}{2}}\, \mathrm{e}^{i\frac{1}{2}X(w) - i\frac{1}{2}\bar{X}(\bar{w}) + \mathcal{O}(z-w)}, \tag{2.64}$$

$$\Psi^-(z)\sigma(w) \sim (z-w)^{\frac{1}{2}}\, \mathrm{e}^{-i\frac{3}{2}X(w) - i\frac{1}{2}\bar{X}(\bar{w}) - \mathcal{O}(z-w)} . \tag{2.65}$$

While the field $\mathrm{e}^{i\frac{1}{2}X(z)+i\frac{1}{2}\bar{X}(\bar{z})}$ seems to appear in the superconformal trace as $:\Psi_0\bar{\Psi}_0\sigma:$ it appears in the conformal trace as a Virasoro ground state itself and gives the square root cut with one of the complexified fermions:

$$\Psi^-(z)\,\mathrm{e}^{i\frac{1}{2}X(w)+i\frac{1}{2}\bar{X}(\bar{w})}\sim(z-w)^{-\frac{1}{2}}\,\mathrm{e}^{-i\frac{1}{2}X(w)+i\frac{1}{2}\bar{X}(\bar{w})-\mathcal{O}(z-w)}\,. \tag{2.66}$$

2.4. Toroidal SCFT

We now turn to the superconformal versions of the toroidal CFTs.

Definition 2.4.1 ([57], Def 4.1.3)**.** An $N=(1,1)$ SCFT $\mathcal{C}$ with central charge $c=\frac{3d}{2}$ for $d\in\mathbb{N}$ is called *toroidal SCFT*, if the following holds: $\mathcal{C}=\mathcal{C}_b\otimes\mathcal{C}_f$, where $\mathcal{C}_b$ is a toroidal CFT with central charge d and $\mathcal{C}_f$ is a fermionic CFT. Each generator j of the $u(1)^d_1$ current algebra in $\mathcal{C}_b$ has a superpartner Ψ in $\mathcal{C}_f$ of conformal weight $(h;h)=\left(\frac{1}{2};0\right)$, and analogously on the right moving side.

Let $T^d=\mathbb{R}^d/\Gamma$. For every $u(1)$-current j^k there is a free fermion superpartner Ψ^k and the toroidal SCFT is the tensor product of the toroidal CFT and fermions with coupled spin structures, leading to the partition function

$$Z_\Gamma(\tau)\cdot\frac{1}{2}\left({}_{(A\bar{A})^d}\underset{(A\bar{A})^d}{\square}(\tau)+{}_{(P\bar{P})^d}\underset{(A\bar{A})^d}{\square}(\tau)+{}_{(A\bar{A})^d}\underset{(P\bar{P})^d}{\square}(\tau)+{}_{(P\bar{P})^d}\underset{(P\bar{P})^d}{\square}(\tau)\right). \tag{2.67}$$

For even $d\in2\mathbb{Z}$ we can complexify pairs of fermions to Dirac fermions as in the previous section. Furthermore we complexify the $u(1)$-currents

$$\Psi^{(k)}_\pm:=\frac{1}{\sqrt{2}}\left(\Psi^{2k-1}\pm i\Psi^{2k}\right),\quad j^{(k)}_\pm:=\frac{1}{\sqrt{2}}\left(j^{2k-1}\pm ij^{2k}\right)\text{ for }k\in\left\{1,\dots,\frac{d}{2}\right\}. \tag{2.68}$$

We can realize the fields of the $N=2$ superconformal algebra using the free fields of the toroidal SCFT:

$$J(z):=\sum_{k=1}^{\frac{d}{2}}:\Psi^{(k)}_+\Psi^{(k)}_-:,\quad G^\pm(z):=\sqrt{2}\sum_{k=1}^{\frac{d}{2}}:\Psi^{(k)}_\pm j^{(k)}_\mp:. \tag{2.69}$$

Together with the Virasoro field $T(z)$ they make up the $N=2$ superconformal algebra at $c=\frac{3d}{2}$. The commutator and anti-commutator relations are also encoded by their OPEs:

$$T(z)T(w)\sim\frac{\frac{c}{2}}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}, \tag{2.70}$$

$$T(z)J(w)\sim\frac{J(w)}{(z-w)^2}+\frac{\partial J(w)}{z-w},\quad J(z)J(w)\sim\frac{\frac{c}{3}}{(z-w)^2}, \tag{2.71}$$

$$T(z)G^\pm(w)\sim\frac{\frac{3}{2}G^\pm(w)}{(z-w)^2}+\frac{\partial G^\pm(w)}{z-w},\quad J(z)G^\pm(w)\sim\pm\frac{G^\pm(w)}{z-w}, \tag{2.72}$$

$$G^\pm(z)G^\mp(w)\sim\frac{\frac{c}{3}}{(z-w)^3}\pm\frac{J(w)}{(z-w)^2}+\frac{T(w)\pm\frac{1}{2}\partial J(w)}{z-w},G^\pm(z)G^\pm(w)\sim0. \tag{2.73}$$

For $d = 4$ the $N = 2$ superconformal algebra can be further extended to the $N = 4$ superconformal algebra ([57], 3.2.1):

$$J^3 = \frac{1}{2}\Big(:\Psi_+^{(1)}\Psi_-^{(1)}: + :\Psi_+^{(2)}\Psi_-^{(2)}:\Big), \quad J^\pm = \pm :\Psi_\pm^{(1)}\Psi_\pm^{(2)}:, \tag{2.74}$$

$$G^\pm = \sqrt{2}\Big(:\Psi_\pm^{(1)} j_\mp^{(1)}: + :\Psi_\pm^{(2)} j_\mp^{(2)}:\Big), \quad G'^\pm = \Big(:\Psi_\mp^{(1)} j_\mp^{(2)}: - :\Psi_\mp^{(2)} j_\mp^{(1)}:\Big), \tag{2.75}$$

and the Virasoro field $T(z)$.

The superconformal partition function of a toroidal SCFT is given by

$$\begin{aligned} &Z_\Gamma(\tau, z) := \\ &Z_\Gamma(\tau)\cdot\frac{1}{2}\left({}_{(A\bar{A})^d}\boxed{}_{(A\bar{A})^d}(\tau,z) + {}_{(P\bar{P})^d}\boxed{}_{(A\bar{A})^d}(\tau,z) + {}_{(A\bar{A})^d}\boxed{}_{(P\bar{P})^d}(\tau,z) + {}_{(P\bar{P})^d}\boxed{}_{(P\bar{P})^d}(\tau,z)\right) \\ &= Z_\Gamma(\tau)\cdot\frac{1}{2}\left(\left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right|^d + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right|^d + \left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right|^d \pm \left|\frac{\vartheta_1(\tau)}{\eta(\tau)}\right|^d\right). \end{aligned} \tag{2.76}$$

The mirror symmetry automorphism maps

$$(T, J, G^+, G^-) \longmapsto (T, -J, G^-, G^+) \tag{2.77}$$

and for $d = 2$ again identifies $\mathcal{T}(\tau,\rho)$ with $\mathcal{T}(\rho,\tau)$. Since the torus theory parameter Γ only enters in the bosonic part of the theory the superconformal toroidal SCFT depends on the same parameters as toroidal CFT.

2.5. Elliptic genus

The *elliptic genus* is a topological invariant that appears both in geometry and in SCFT. In this section we review its SCFT definition as well as its geometric definition. The main references for this section are Section 2.4 of [58] and Section 3.1 of [57].

Definition 2.5.1 ([20])**.** Let $\mathbb{H}^R$ be the Ramond space of states of an $N = (2,2)$ superconformal field theory with central charges c and $\bar{c}$. Then for $q := \exp(2\pi i\tau)$ with $\tau \in \mathbb{C}, \Im\tau > 0$ and $y := \exp(2\pi i z)$ for $z \in \mathbb{C}$

$$\mathcal{E}(\tau, z) := \mathrm{tr}_{\mathbb{H}^R}\left((-1)^{J_0 - \bar{J}_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} y^{J_0}\right) \tag{2.78}$$

is the *conformal field theoretic elliptic genus.*

Proposition 2.5.2 ([14], [61])**.** *Let $\mathbb{H}^R$ be the Ramond space of states of an $N = (2,2)$ superconformal field theory at central charge c, $\bar{c}$. Then $\mathcal{E}(\tau, z)$ is holomorphic in τ, bounded when $\tau \to i\infty$ and transforms covariantly under modular transformation*

$$\mathcal{E}(\tau+1, z) = \mathcal{E}(\tau, z), \quad \mathcal{E}(-\frac{1}{\tau}, \frac{z}{\tau}) = \mathrm{e}^{2\pi i \frac{c}{6}\frac{z^2}{\tau}}\,\mathcal{E}(\tau, z). \tag{2.79}$$

If in addition $c = \bar{c} \in 3\mathbb{N}$ and all eigenvalues of J_0 and $\bar{J}_0$ in the R-sector lie in $\frac{c}{6} + \mathbb{Z}$, then

$$\mathcal{E}(\tau, z+1) = (-1)^{\frac{c}{3}} \mathcal{E}(\tau, z), \quad \mathcal{E}(\tau, z+\tau) = q^{-\frac{c}{6}} \cdot y^{-\frac{c}{3}} \cdot \mathcal{E}(\tau, z). \tag{2.80}$$

In other words, $\mathcal{E}(\tau, z)$ is a weak Jacobi form (with a character, if $\frac{c}{3}$ is odd) of weight 0 and index $\frac{c}{6}$.

The additional assumptions on the central charges and the eigenvalues of J_0 and $\bar{J}_0$ are expected to hold for SCFTs arising as non-linear sigma models on Calabi-Yau manifolds.

On the other hand since Hirzebruch ([36]) the elliptic genus is known to topologists as a ring homomorphism from the cobordism ring of smooth oriented compact manifolds into a ring of modular forms ([47], [37]). For a D-dimensional Calabi-Yau manifold X the associated elliptic genus $\mathcal{E}_X(\tau, z)$ is a modular form obeying the transformation properties of Proposition 2.5.2 with $c = 3D$. It interpolates between the standard topological invariants, the Euler characteristic $\chi(X)$, the signature $\sigma(X)$ and the topological Euler characteristic $\chi(\mathcal{O}_X)$.

In order to understand the connection between the CFT and the geometric elliptic genus we first turn to the Hirzebruch genus and its connection to the Atiyah-Singer Index Theorem ([2]). For $y \in \mathbb{C}$ the Hirzebruch χ_y-genus is defined by

$$\chi_y(X) := \sum_{p,q=0}^{D} (-1)^q y^p h^{p,q}(X), \tag{2.81}$$

where $h^{p,q}(X)$ are the Hodge numbers of X. The Hirzebruch genus also interpolates between the standard topological invariants:

$$\chi(X) = \chi_{-1}(X), \quad \sigma(X) = \chi_{+1}(X), \quad \chi_0(X). \tag{2.82}$$

For any complex vector bundle E on X and a formal variable x, define the following shorthand notations:

$$\Lambda_x E := \bigoplus_p x^p \Lambda^p E, \quad S_x E := \bigoplus_p x^p S^p E, \tag{2.83}$$

where Λ is the exterior and S the symmetric algebra. The Chern character of such a formal power series in x whose coefficients are complex vector bundles F_p is

$$ch(\sum_p x^p F_p) := \sum_p x^p ch(F_p). \tag{2.84}$$

Then the Hirzebruch-Riemann-Roch formula ([35]), a special case of the Atiyah-Singer Index Theorem yields

$$\chi_y(X) = \int_X Td(X) ch(\Lambda_y T^*), \tag{2.85}$$

where $Td(X)$ denotes the Todd genus and $T := T^{1,0}X$ is the holomorphic tangent bundle of X. The generalization of this formula motivates the following geometric definition of the elliptic genus:

Definition 2.5.3 ([37], [60], [46]). Let X denote a compact complex D-dimensional manifold with holomorphic tangent bundle $T := T^{1,0}X$. Set

$$\mathbb{E}_{q,-y} := y^{-\frac{D}{2}} \bigotimes_{n=1}^{\infty} \left(\Lambda_{-yq^{n-1}} T^* \otimes \Lambda_{-y^{-1}q^n} T \otimes S_{q^n} T \otimes S_{q^n} T^* \otimes S_{q^n} T \right), \tag{2.86}$$

viewed as a formal power series with variables $y^{\pm\frac{1}{2}}, q$, whose coefficients are characteristic classes on X. Then with $q := \exp(2\pi i \tau)$ and $y = \exp(2\pi i z)$, the holomorphic Euler characteristic of $\mathbb{E}_{q,-y}$,

$$\mathcal{E}_X(\tau, z) := \int_X Td(X) ch(\mathbb{E}_{q,-y}) \in y^{-\frac{D}{2}} \cdot \mathbb{Z}[[y^{\pm 1}, q]] \tag{2.87}$$

is the *geometric elliptic genus of* X.

By [37], [60], [46] the geometric elliptic genus yields a well-defined function in $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ and in $z \in \mathbb{C}$. If X is Calabi-Yau, then the geometric elliptic genus obeys the same properties as in Proposition 2.5.2 ([7]). It also interpolates between the standard topological invariants: We have $\mathcal{E}_X(\tau, z = 0) \overset{\tau \to i\infty}{\longrightarrow} y^{-\frac{D}{2}} \chi_{-y}(X)$ and

$$\mathcal{E}_X(\tau, z = 0) = \chi(X), \quad \mathcal{E}_X(\tau, z = \frac{1}{2}) = (-1)^{\frac{D}{2}} \sigma(X) + \mathcal{O}(q), \tag{2.88}$$

$$q^{\frac{D}{4}} \mathcal{E}_X \left(\tau, z = \frac{\tau + 1}{2} \right) = (-1)^{\frac{D}{2}} \chi(\mathcal{O}_X) + \mathcal{O}(q). \tag{2.89}$$

One expects the CFT elliptic genus coming from a non-linear sigma model construction on a target manifold X to be equal to the geometric elliptic genus of this manifold, i.e.

$$\mathcal{E}_X(\tau, z) = \operatorname{tr}_{\mathbb{H}^R} \left((-1)^{J_0 - \bar{J}_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} y^{J_0} \right), \tag{2.90}$$

which would be a generalization of the McKean-Singer formula ([50]). As noted in the introduction the construction of the non-linear sigma model is in general not mathematically precise. So in particular a proof of Conjecture 2.90 is out of reach. There is, however, some evidence. For example, in the mathematically understood case of the target manifold being a torus both the CFT and the geometric elliptic genus vanish. Furthermore, the compatibility of the elliptic genus with the orbifold procedure (cf. Chapter 3) was proved in [8], [25].

In Chapter 4 we will see how the elliptic genus also arises in the setting of the chiral de Rham complex. Here the results of [6], [8] give further evidence for the conjecture.

Chapter 3

Orbifold construction

The orbifold construction ([13], [18], [15], [16], [27]) enables one to create new theories by modding out symmetries of an existing theory. For us this means in particular using a $\mathbb{Z}_2$-orbifold of the toroidal CFT/SCFTs that we saw in the last chapters. If a CFT/SCFT has an interpretation as a non-linear sigma model on a manifold and the symmetry of the CFT/SCFT comes from a symmetry of the manifold, the resulting orbifold theory should correspond to the minimal resolution of the (possibly singular) orbifold target space. Interestingly, blowing up singularities naturally appears in the orbifold construction by imposing a modular invariant partition function.

In Section 3.1 we review the general orbifold procedure for CFTs and as our main example the $\mathbb{Z}_2$-orbifold of the toroidal CFT. We extend this to SCFTs in Section 3.2 and again restrict to the $\mathbb{Z}_2$-orbifold of the toroidal SCFT. In the moduli space of all SCFTs with central charge c there are intersection points between the toroidal theories and their orbifold. We review some of these in Section 3.3.

The main sources for this chapter are Section 8.3 of [31] and Chapter 5 of [57], which we supplement with additional examples.

3.1. CFT orbifold construction

Let $\mathcal{C}$ be a CFT with space of states $\mathbb{H}$ in the sense of Definition 1.1.6. Let G be a finite symmetry group acting on $\mathcal{C}$ that leaves the Virasoro fields invariant. We want to build the orbifold CFT $\mathcal{C}/G$. A general principle in mathematics is that a space can be described by the sheaf of functions on it. When considering the process of orbifolding and blowing up a manifold, one asks what functions or differential forms are defined on the orbifold. The first step in order to find functions or differential forms on the resolution is to project to the G-invariants of the original manifold. The same is true for CFT. Let $P = \frac{1}{|G|}\sum_{g\in G} g$ be the projector to the G-invariant states of the space of states $\mathbb{H}$ of $\mathcal{C}$. When inserting P into the partition function we use the following shorthand notation, analogous to the different boundary conditions of fermions of Chapter 2:

$$ {}_{g}\square_{\mathbb{1}}(\tau) := \mathrm{tr}_{\mathbb{H}}\left(g q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right). \tag{3.1}$$

The partition function for the invariant states then is

$$Z^{inv}(\tau) := \mathrm{tr}_{\mathbb{H}}(Pq^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}) = \sum_{g\in G}\frac{1}{|G|}\, g\underset{\mathbb{1}}{\square}(\tau). \tag{3.2}$$

For non-trivial G this function will never be modular invariant. To regain modular invariance we have to add additional G-invariant states, so called *twisted states*. All $g\underset{\mathbb{1}}{\square}$ are traces over the same pre-Hilbert space $\mathbb{H}$ with g inserted into the trace. For the twisted states we need a new pre-Hilbert space.

The necessity of these states can also be illustrated using the world-sheet interpretation of $g\underset{\mathbb{1}}{\square}$: Let $\varphi\in\mathcal{C}^\infty(\mathbb{C}/\Lambda_\tau, X)$ be a smooth map from the world-sheet torus to a manifold X. On the orbifold X/G maps that are displaced by a group element along each period are also defined:

$$g\underset{\mathbb{1}}{\square}:\quad \begin{array}{c}\varphi(z+1)=g\varphi(z)\\ \varphi(z+\tau)=\varphi(z)\end{array}, \tag{3.3}$$

the twist in the time direction. The sector $\mathbb{1}\underset{\mathbb{1}}{\square}$ represents the strings defined on the original theory. We also need to consider the twist in the space direction

$$\mathbb{1}\underset{h}{\square}:\quad \begin{array}{c}\varphi(z+1)=\varphi(z)\\ \varphi(z+\tau)=h\varphi(z)\end{array}, \tag{3.4}$$

as these maps are also defined on the orbifold X/G.

We already know the Hilbert space interpretation of $g\underset{\mathbb{1}}{\square}$; we still need the one for $\mathbb{1}\underset{h}{\square}$ and more generally $g\underset{h}{\square}$. We call the space of states with property (3.4) the *h-twisted space of states* and denote it by $\mathbb{H}_h$. Whether one can construct such a twisted sector is not clear in general. Once a twisted sector is constructed, one still has to extend the G-action to this space to find $g\underset{h}{\square}$. This may not be unique, the choices are classified by $\mathrm{H}^2(G,U(1))$ and called *discrete torsion* ([55]). For $G=\mathbb{Z}_m$ this cohomology group is trivial.

The trace over the h-twisted states is

$$\mathbb{1}\underset{h}{\square}(\tau) := \mathrm{tr}_{\mathbb{H}_h}(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}). \tag{3.5}$$

Here we also need to project to the G-invariant states so we need to define $g\underset{h}{\square}(\tau)$ and do so analogously to the untwisted sector:

$$g\underset{h}{\square}(\tau) := \mathrm{tr}_{\mathbb{H}_h}(gq^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}). \tag{3.6}$$

An element φ in the twisted sector $\mathbb{H}_h$ also obeys

$$\forall g\in G:\quad g\varphi(z+\tau) = (ghg^{-1})g\varphi(z), \tag{3.7}$$

so we have to identify $\mathbb{H}_h \cong \mathbb{H}_{ghg^{-1}}$ via the induced map g. For an element φ with boundary condition $g\underset{h}{\square}$ by formulas (3.3) and (3.4) we have $gh\varphi = hg\varphi$. So for non-commuting pairs g, h the action of g on $\mathbb{H}_h$ cannot be defined in a sensible way. Thus one restricts to commuting g, h. The correct projector in the twisted sector $\mathbb{H}_h$ then is $P_h := \frac{1}{|G|}\sum_{g\in G:[g,h]=0}$ ([18]). Taking all this into account we arrive at the orbifold partition function

$$Z^{G-orb}(\tau) := \sum_{h\in G} \mathrm{tr}_{\mathbb{H}_h}\left(P_h q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) = \frac{1}{|G|}\sum_{\substack{g,h\in G,\\ [g,h]=0}} g\underset{h}{\square}(\tau). \tag{3.8}$$

When faced with the challenge of constructing the space of states for the twisted sectors, the action of the modular group $\mathrm{PSL}(2;\mathbb{Z})$ proves to be a powerful tool: It relates the traces over spaces in the twisted sectors with those in the untwisted sectors ([31], 8.3):

$$g\underset{h}{\square}\left(\frac{-1}{\tau}\right) = h^{-1}\underset{g}{\square}(\tau), \tag{3.9}$$

$$g\underset{h}{\square}(\tau+1) = gh\underset{h}{\square}(\tau). \tag{3.10}$$

From this the modular invariance of the G-orbifold partition function follows.

We turn to the first example of an orbifold:

Example 3.1.1 (Shift-orbifold, [57], 5.1)**.** Let $\Gamma \subset \mathbb{R}^{d,d}$ be the charge lattice of a toroidal CFT $\mathcal{T}$ of central charge $c = d$ with space of states $\mathbb{H}$. Let $\Delta \in \mathbb{R}^{d,d}$ be a vector such that there is a natural number $D \in \mathbb{N}$ for which $D \cdot \Delta \in \Gamma$ is a lattice vector of Γ. This implies that for any lattice vector $p \in \Gamma$ the scalar product $p \cdot \Delta$ only takes values in $\frac{1}{D}\mathbb{Z}$. Thus for $p \in \Gamma$ we can identify $\mathrm{e}^{2\pi i\Delta\cdot p}$ with the Dth roots of unity. Let $|p\rangle \in \mathbb{H}$ be the lowest weight vector of the free boson algebra corresponding to the charge vector $p \in \Gamma$. We define an action on $|p\rangle$ as

$$T_\Delta|p\rangle := e^{2\pi i p\cdot\Delta}|p\rangle. \tag{3.11}$$

We extend this to all of $\mathbb{H}$ such that it leaves the vibration modes $a_n^i, \bar{a}_n^i$ invariant. Because of $Dp\cdot\Delta \in \mathbb{Z}$ we have $T_\Delta^D = \mathbb{1}$ and we can identify the group generated by T_Δ with $\mathbb{Z}_D$. The action of this group induces a partition on the lattice vectors:

$$\Gamma^m := \left\{p \in \Gamma \,\middle|\, p\cdot\Delta \in \mathbb{Z} + \frac{m}{D}\right\}. \tag{3.12}$$

Let

$$Z_{\Gamma^m}(\tau) := \sum_{(p_l;p_r)\in\Gamma^m} q^{\frac{1}{2}p_l^2}\bar{q}^{\frac{1}{2}p_r^2}, \tag{3.13}$$

then

$$T_\Delta\underset{\mathbb{1}}{\square}(\tau) = \frac{1}{|\eta(\tau)|^{2d}}\sum_{m=0}^{D-1} \mathrm{e}^{2\pi i\frac{m}{D}} Z_{\Gamma^m}(\tau). \tag{3.14}$$

For the twisted sectors we also need

$$\Gamma_{l\Delta}^{m} := \left\{ p \in \Gamma + l\Delta \,\middle|\, p \cdot \Delta \in \mathbb{Z} + \frac{m}{D} \right\} \tag{3.15}$$

and via modular transformation properties (3.9) and (3.10) find

$$T_\Delta^k \underset{T_\Delta^l}{\boxed{}}(\tau) = \frac{1}{|\eta(\tau)|^{2d}} \sum_{m=0}^{D-1} \mathrm{e}^{2\pi i \frac{km}{D}}\, Z_{\Gamma_{l\Delta}^m}(\tau). \tag{3.16}$$

In every twisted sector we again project to vectors with integer scalar product with Δ. Since the $u(1)$-currents are left invariant the shift orbifold theory is again by definition a toroidal CFT with charge lattice

$$\tilde{\Gamma} = \{ p \in \Gamma + l\Delta \,|\, l \in \{0, \ldots, D-1\}, \quad p \cdot \Delta \in \mathbb{Z} \}. \tag{3.17}$$

Thus the charge lattice is shifted and hence the name *shift orbifold*.

Remark 3.1.2. It is not hard to see that for the free boson on the circle $\mathcal{T}(R=1)$ and the half-lattice vector $\Delta = \left(\frac{\sqrt{2}}{4}; \frac{\sqrt{2}}{4}\right)$ the shift orbifold gives $\mathcal{T}\left(R = \frac{1}{2}\right)$.

3.1.1. $\mathbb{Z}_2$-orbifold of toroidal CFT

Let $T^d = \mathbb{R}^d/\Lambda$ be a torus. The $\mathbb{Z}_2$-action on $\mathbb{R}^d$ induced by

$$\begin{array}{rccc} \kappa: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & (x_1, \ldots, x_d) & \longmapsto & (-x_1, \ldots, -x_d) \end{array} \tag{3.18}$$

descends to the torus T^d. We want to investigate the resulting $\mathbb{Z}_2$-orbifold on the toroidal CFT $\mathcal{T}$: On the $u(1)$-current κ acts as $\kappa : j^{(k)} \mapsto -j^{(k)}$ and because of (1.38) on the charges as $(p_l; p_r) \mapsto (-p_l; -p_r)$. This leads to the following partition function in the invariant sector:

$$Z_{\mathcal{T}}^{\mathbb{Z}_2, inv}(\tau) = \frac{1}{2}\left(\mathbb{1}\underset{\mathbb{1}}{\boxed{}}(\tau) + \kappa\underset{\mathbb{1}}{\boxed{}}(\tau) \right) \tag{3.19}$$

where

$$\mathbb{1}\underset{\mathbb{1}}{\boxed{}}(\tau) = Z_{\mathcal{T}}(\tau) = \frac{1}{|\eta(\tau)|^{2d}} Z_\Gamma(\tau), \tag{3.20}$$

$$\kappa\underset{\mathbb{1}}{\boxed{}}(\tau) = \mathrm{tr}_{\mathbb{H}}\, \kappa q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} = \sum_i \kappa q^{h_i - \frac{c}{24}} \bar{q}^{\bar{h}_i - \frac{\bar{c}}{24}} \langle \varphi_i, \kappa\varphi_i \rangle \tag{3.21}$$

$$= q^{-\frac{d}{24}} \bar{q}^{-\frac{d}{24}} \left(\prod_{n=1}^{\infty} \frac{1}{1+q^n} \frac{1}{1+\bar{q}^n} \right)^d = 2^d \left| \frac{\eta(\tau)}{\vartheta_2(\tau)} \right|^d. \tag{3.22}$$

Here $\{ \varphi_i \}$ is an orthonormal basis with respect to the scalar product (1.31). Note that when κ is inserted there are no winding or momentum modes, i.e. this sector is independent of the charge lattice Γ and built only upon $0 \in \Gamma$. This can also be seen from property (3.3). The twisted sectors follow from modular transformation properties

$$\mathbb{1}\underset{\kappa}{\square}(\tau) = \kappa\underset{\mathbb{1}}{\square}\left(\frac{-1}{\tau}\right) = 2^d \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|^d , \tag{3.23}$$

$$\kappa\underset{\kappa}{\square}(\tau) = \mathbb{1}\underset{\kappa}{\square}(\tau+1) = 2^d \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|^d . \tag{3.24}$$

This leads to the following theorem known since [31], for a detailed proof see [57]:

Theorem 3.1.3 ([57], Theorem 5.2.1)**.** *The $\mathbb{Z}_2$-orbifold partition function of a toroidal CFT $\mathcal{T}$ with charge lattice Γ is given by*

$$Z_{\mathcal{T}}^{\mathbb{Z}_2}(\tau) = \frac{1}{2}\left(Z_{\mathcal{T}}(\tau) + 2^d \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right|^d + 2^d \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|^d + 2^d \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|^d \right). \tag{3.25}$$

The partition function $Z_{\mathcal{T}}^{\mathbb{Z}_2}(\tau)$ of the $\mathbb{Z}_2$-orbifold still depends on the toroidal partition function $Z_{\mathcal{T}}(\tau)$ and the $\mathbb{Z}_2$-orbifold depends on the same parameters as the toroidal one. Note that this dependency only enters in the sector $\mathbb{1}\underset{\mathbb{1}}{\square}$.

Similar to the fermions in the Ising model we can also interpret the $\mathbb{Z}_2$-orbifold in terms of different boundary conditions of the free boson ([31], 6.2). To explain this, we restrict to $d = c = 1$. First, note the leading term in the twisted sector:

$$\mathbb{1}\underset{\kappa}{\square}(\tau) = 2\left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| = 2 \cdot q^{\frac{1}{16}-\frac{1}{24}}\bar{q}^{\frac{1}{16}-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1-q^{n-\frac{1}{2}}\right)\left(1-\bar{q}^{n-\frac{1}{2}}\right). \tag{3.26}$$

This hints towards a representation space built upon a lowest weight vector of conformal weight $\left(\frac{1}{16}, \frac{1}{16}\right)$ using modes of half-integer weight. We will now investigate this in terms of boundary conditions. Similar to the Majorana fermion of the superconformal theories (Chapter 2) the holomorphic $u(1)$-current comes in two different boundary conditions:

$$\begin{aligned} j(z) &= \textstyle\sum_{n\in\mathbb{Z}} a_n z^{n-1} \quad (P), \\ j(z) &= \textstyle\sum_{n\in\mathbb{Z}+\frac{1}{2}} a_n z^{n-1} \quad (A), \end{aligned} \tag{3.27}$$

with $[a_n, a_m] = n\delta_{n+m,0}$ in both cases. Note that this analogy to the boundary conditions of superconformal field theories is only applicable to the $\mathbb{Z}_2$-orbifolds.

We want to construct the twisted space of states. For this it is useful to introduce a twist field $\sigma(z)$ that implements the square-root cut in (A)

$$a(z)\sigma(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\mu(w), \tag{3.28}$$

where μ is another twist field. Let h_σ, h_μ be the respective conformal weights of the fields σ and μ. Then $h_\sigma + \frac{1}{2} = h_\mu$ because of compatibility of the OPE with the conformal grading. By calculating the expectation value of the Virasoro field in the twisted sector we can deduce $h_\sigma = \frac{1}{16}$. This is compatible with the leading term in $\mathbb{1}\underset{\kappa}{\square}(\tau)$! The Fock space representation $\tilde{B}^{(\frac{1}{16})}$ is now built upon a lowest weight vector of weight $\frac{1}{16}$ similarly to B^Q but using $a_n, n \in \mathbb{Z} + \frac{1}{2}$ ([59], Chapter 3). The Virasoro algebra at central charge $c = 1$ acts as

$$L_n := \frac{1}{2} \sum_{k \in \mathbb{Z}+\frac{1}{2}} a_{n+k} a_{-k} \quad \text{for } n \neq 0 \quad \text{and } L_0 := \sum_{k \in \mathbb{N}+\frac{1}{2}} a_k a_{-k} + \frac{1}{16}. \tag{3.29}$$

Interestingly $\frac{1}{16}$ is the only lowest weight that turns $\tilde{B}$ into a Virasoro representation. On the antiholomorphic side we proceed analogously. In Subsection 3.3.1 we will be able to better understand the field associated to the twisted ground states of weight $(h, \bar{h}) = (\frac{1}{16}, \frac{1}{16})$ at certain points in the moduli space of toroidal CFTs.

3.2. SCFT orbifold

For the orbifold construction for SCFTs we have to take the group action on the different fermionic boundary conditions into account. For these we use the following shorthand notations:

$$A\underset{A}{\square} := (A\bar{A})^d \underset{(A\bar{A})^d}{\square}, \quad P\underset{A}{\square} := (P\bar{P})^d \underset{(A\bar{A})^d}{\square}, \quad A\underset{P}{\square} := (A\bar{A})^d \underset{(P\bar{P})^d}{\square}, \quad P\underset{P}{\square} := (P\bar{P})^d \underset{(P\bar{P})^d}{\square}. \tag{3.30}$$

The NS-sector $A\underset{A}{\square}$ takes the following form under the $\mathbb{Z}_2$-orbifold

$$\mathbb{1}\underset{\mathbb{1}}{\square} : A\underset{A}{\square} \longmapsto A\underset{A}{\square}, \tag{3.31}$$

$$\kappa\underset{\mathbb{1}}{\square} : A\underset{A}{\square} \longmapsto P\underset{A}{\square}, \tag{3.32}$$

$$\mathbb{1}\underset{\kappa}{\square} : A\underset{A}{\square} \longmapsto A\underset{P}{\square}, \tag{3.33}$$

$$\kappa\underset{\kappa}{\square} : A\underset{A}{\square} \longmapsto P\underset{P}{\square} \tag{3.34}$$

and analogously for the other sectors. For the free fermion theories with coupled spin structure, e.g. the Ising model and Dirac fermion, this means that the $\mathbb{Z}_2$-orbifold simply exchanges sectors, but does not change the overall partition function. When using the modular group to find the (super) partition function of the twisted sector, we also need to know how it acts on the different boundary conditions:

$$\begin{aligned} T: &\quad (\tau, z) \mapsto (\tau + 1, z): \quad A\underset{A}{\square} \leftrightarrow P\underset{A}{\square}, \quad A\underset{P}{\square} \circlearrowleft, \quad P\underset{P}{\square} \circlearrowleft, \\ S: &\quad (\tau, z) \mapsto \left(\tfrac{-1}{\tau}, \tfrac{z}{\tau}\right): \quad P\underset{A}{\square} \leftrightarrow A\underset{P}{\square}, \quad A\underset{A}{\square} \circlearrowleft, \quad P\underset{P}{\square} \circlearrowleft. \end{aligned} \tag{3.35}$$

For our first example note the similarity between the different spin structure of the fermion in the Ising model and the $\mathbb{Z}_2$-orbifold of the toroidal CFT of the previous section. The Ising model can also be interpreted as the $\mathbb{Z}_2$-orbifold of two fermions $\Psi, \bar{\Psi}$ with action $\Psi \mapsto -\Psi, \bar{\Psi} \mapsto -\bar{\Psi}$ in the ${}_{A}\underset{A}{\square}$ sector.

Example 3.2.1 (Ising spin orbifold, [31], 7.5)**.** Instead of only considering different spin structures on the fermion, we can do the same for the Ising spin σ. This means taking the orbifold of the action of $\kappa : \sigma \mapsto -\sigma$. For the Ising model with Ising spin σ we have

$$Z_{Ising}(\tau) = \frac{1}{2}\left({}_{A\bar{A}}\underset{A\bar{A}}{\square} + {}_{P\bar{P}}\underset{A\bar{A}}{\square} + {}_{A\bar{A}}\underset{P\bar{P}}{\square} + {}_{P\bar{P}}\underset{P\bar{P}}{\square} \right)(\tau), \tag{3.36}$$

which gives periodic boundary conditions in both space and time direction for the Ising spin and is the starting point for this orbifold, i.e. ${}_{\mathbb{1}}\underset{\mathbb{1}}{\square}$. The other sectors are

$$ {}_{\kappa}\underset{\mathbb{1}}{\square}(\tau) = \frac{1}{2}\left({}_{A\bar{A}}\underset{A\bar{A}}{\square} + {}_{P\bar{P}}\underset{A\bar{A}}{\square} - {}_{A\bar{A}}\underset{P\bar{P}}{\square} - {}_{P\bar{P}}\underset{P\bar{P}}{\square} \right)(\tau), \tag{3.37}$$

$$ {}_{\mathbb{1}}\underset{\kappa}{\square}(\tau) = {}_{\kappa}\underset{\mathbb{1}}{\square}\left(\frac{-1}{\tau}\right) = \frac{1}{2}\left({}_{A\bar{A}}\underset{A\bar{A}}{\square} - {}_{P\bar{P}}\underset{A\bar{A}}{\square} + {}_{A\bar{A}}\underset{P\bar{P}}{\square} - {}_{P\bar{P}}\underset{P\bar{P}}{\square} \right)(\tau), \tag{3.38}$$

$$ {}_{\kappa}\underset{\kappa}{\square}(\tau) = {}_{\mathbb{1}}\underset{\kappa}{\square}(\tau+1) = \frac{1}{2}\left(-{}_{A\bar{A}}\underset{A\bar{A}}{\square} + {}_{P\bar{P}}\underset{A\bar{A}}{\square} + {}_{A\bar{A}}\underset{P\bar{P}}{\square} - {}_{P\bar{P}}\underset{P\bar{P}}{\square} \right)(\tau). \tag{3.39}$$

This produces the orbifold partition function

$$Z^{\mathbb{Z}_2}_{Ising}(\tau) = \frac{1}{2}\left({}_{A\bar{A}}\underset{A\bar{A}}{\square} + {}_{P\bar{P}}\underset{A\bar{A}}{\square} + {}_{A\bar{A}}\underset{P\bar{P}}{\square} - {}_{P\bar{P}}\underset{P\bar{P}}{\square} \right)(\tau), \tag{3.40}$$

implementing the duality $\sigma \leftrightarrow \mu$.

Because of bosonization there is a particular nice interpretation of the $\mathbb{Z}_2$-orbifold of the Dirac fermion in terms of a shift orbifold:

Example 3.2.2 (Dirac fermion $\mathbb{Z}_2$-orbifold, [31], 8.2)**.** For the Dirac fermion we can also understand the $\mathbb{Z}_2$-orbifold via bosonization. In Section 2.3 we identified

$$\mathcal{C}_{Dirac} = \mathcal{T}\left(R = \frac{1}{\sqrt{2}}\right), \tag{3.41}$$

using

$$\Psi^{\pm}(z) := \frac{1}{\sqrt{2}}\left(\Psi^1(z) \pm i\Psi^2(z)\right) = \mathrm{e}^{\pm iX(Z)}. \tag{3.42}$$

The effect of the $\mathbb{Z}_2$-orbifold $\Psi^{\pm}(z) \mapsto -\Psi^{\pm}(z)$ translates into a shift orbifold in the bosonization. Since in $\mathcal{C}_{Dirac}$ we already considered projections in the different boundary sectors, fields with odd number of $\Psi^{\pm}$ are already modded out. The shift orbifold acts trivially on $\mathcal{T}\left(R = \frac{1}{\sqrt{2}}\right)$. The twist field ground states remain to have $(J_0, \bar{J}_0)$-charge $(\pm\frac{1}{2}, \pm\frac{1}{2})$ and conformal weight $(\frac{1}{8}, \frac{1}{8})$ and are given by a linear combination of

$$\mathrm{e}^{\pm\frac{i}{2}X(z)\pm\frac{i}{2}\bar{X}(\bar{z})}. \tag{3.43}$$

On the other hand, if we decide to implement a $\mathbb{Z}_2$-action on the twist fields as in the previous Ising spin example, this is a different shift orbifold and the twist fields get modded out. The new twisted sector is build on $(J_0, \bar{J}_0)$-charges $(\pm\frac{1}{2}, \mp\frac{1}{2})$. Again, this implements a duality of the model and exchanges between the projection to the different $(-1)^F$ eigenvalues. In terms of two Ising spin and disorder operators it exchanges

$$\sigma^1 \times \sigma^2, \qquad \mu^1 \times \mu^2 \quad \longleftrightarrow \quad \sigma^1 \times \mu^2, \qquad \mu^1 \times \sigma^2. \tag{3.44}$$

3.2.1. $\mathbb{Z}_2$-orbifold of toroidal SCFT

For the toroidal SCFT $\mathcal{T}$ on $T^d = \mathbb{R}^d/\Gamma$ we have to consider both the CFT orbifold and the exchange of boundary condition sectors. The NS-part of the SCFT $\mathbb{Z}_2$-orbifold partition function can be calculated using (3.31):

$$\begin{aligned} 2 \cdot Z_{\mathcal{T}}^{\mathbb{Z}_2,NS}(\tau, z) = {}&\mathbb{1}\underset{\mathbb{1}}{\boxed{}}(\tau)\, A\underset{A}{\boxed{}}(\tau, z) + \kappa\underset{\mathbb{1}}{\boxed{}}(\tau)\, P\underset{A}{\boxed{}}(\tau, z) \\ &+ \mathbb{1}\underset{\kappa}{\boxed{}}(\tau)\, A\underset{P}{\boxed{}}(\tau, z) + \kappa\underset{\kappa}{\boxed{}}(\tau)\, P\underset{P}{\boxed{}}(\tau, z). \end{aligned} \tag{3.45}$$

Theorem 3.2.3 ([57], Theorem 5.2.2)**.** *The NS-part of the $\mathbb{Z}_2$-orbifold partition function of a toroidal SCFT $\mathcal{T}$ given by a charge lattice $\Gamma$is*

$$\begin{aligned} 2 \cdot Z_{\mathcal{T}}^{\mathbb{Z}_2,NS}(\tau, z) = {}& Z_{\mathcal{T}}(\tau) \left|\frac{\vartheta_3(\tau, z)}{\eta(\tau)}\right|^d + 2^d \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right|^d \left|\frac{\vartheta_4(\tau, z)}{\eta(\tau)}\right|^d \\ &+ 2^d \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|^d \left|\frac{\vartheta_2(\tau, z)}{\eta(\tau)}\right|^d + 2^d \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|^d \left|\frac{\vartheta_1(\tau, z)}{\eta(\tau)}\right|^d. \end{aligned} \tag{3.46}$$

For $d \in 2\mathbb{Z}$ using spectral flow we find the elliptic genus of Section 2.5

$$\mathcal{E}_{T^d}^{\mathbb{Z}_2}(\tau, z) = 2^{d-1} \cdot \left(\frac{\vartheta_2(\tau, z)^{\frac{d}{2}}}{\vartheta_2(\tau)^{\frac{d}{2}}} + \frac{\vartheta_3(\tau, z)^{\frac{d}{2}}}{\vartheta_3(\tau)^{\frac{d}{2}}} + \frac{\vartheta_4(\tau, z)^{\frac{d}{2}}}{\vartheta_4(\tau)^{\frac{d}{2}}} \right). \tag{3.47}$$

Example 3.2.4 ($\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $E \times E \times E$)**.** Let $E = \mathbb{C}/\Gamma$ be an elliptic curve. Consider a toroidal SCFT $\mathcal{T}$ on the product $E \times E \times E$ and a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action generated by

$$\begin{array}{ccccccc} & & E & \times & E & \times & E \\ \kappa_1 & : & + & & - & & -. \\ \kappa_2 & : & - & & - & & + \end{array} \tag{3.48}$$

The different sectors of the bosonic orbifold are given by

$$\mathbb{1}\underset{\mathbb{1}}{\square}(\tau) = \left(Z_{\mathcal{T}}(\tau)\right)^3, \tag{3.49}$$

$$\kappa_1\underset{\mathbb{1}}{\square}(\tau) = Z_{\mathcal{T}}(\tau)\cdot 2^4\cdot\left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right|^4 = \kappa_2\underset{\mathbb{1}}{\square}(\tau) = \kappa_1\circ\kappa_2\underset{\mathbb{1}}{\square}(\tau), \tag{3.50}$$

$$\mathbb{1}\underset{\kappa_1}{\square}(\tau) = \kappa_1\underset{\mathbb{1}}{\square}\left(\frac{-1}{\tau}\right) = Z_{\mathcal{T}}(\tau)\cdot 2^4\cdot\left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|^4 = \mathbb{1}\underset{\kappa_2}{\square}(\tau) = \mathbb{1}\underset{\kappa_1\circ\kappa_2}{\square}(\tau), \tag{3.51}$$

$$\kappa_1\underset{\kappa_1}{\square}(\tau) = \mathbb{1}\underset{\kappa_1}{\square}(\tau+1) = Z_{\mathcal{T}}(\tau)\cdot 2^4\cdot\left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|^4 = \kappa_2\underset{\kappa_2}{\square}(\tau) = \kappa_1\circ\kappa_2\underset{\kappa_1\circ\kappa_2}{\square}(\tau), \tag{3.52}$$

$$\kappa_2\underset{\kappa_1}{\square}(\tau) = 16\cdot 2^2\cdot\left|\frac{\eta(\tau)^3}{\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau)}\right|^2 = 16, \tag{3.53}$$

etc., implying

$$Z^{\mathbb{Z}_2^2}_{E\times E\times E}(\tau) = \frac{1}{4}\left(\left(Z_{\mathcal{T}}(\tau)\right)^3 + 16\cdot 3\cdot Z_{\mathcal{T}}(\tau)\left(\left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right|^4 + \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|^4 + \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|^4\right) + 16\cdot 6\right). \tag{3.54}$$

For the SCFT we consider the NS-sector in the different orbifold sectors by applying $\mathbb{Z}_2\times\mathbb{Z}_2$ group elements and the modular group element S to go to the twisted sectors, e.g.:

$$\mathbb{1}\underset{\mathbb{1}}{\overset{NS}{\square}}(\tau,z) = \mathbb{1}\underset{\mathbb{1}}{\square}(\tau)\cdot (A\bar{A})^2\underset{(A\bar{A})^2}{\square}(\tau,z)\cdot (A\bar{A})^2\underset{(A\bar{A})^2}{\square}(\tau,z)\cdot (A\bar{A})^2\underset{(A\bar{A})^2}{\square}(\tau,z) \tag{3.55}$$

$$\kappa_1\underset{\mathbb{1}}{\overset{NS}{\square}}(\tau,z) = \kappa_1\underset{\mathbb{1}}{\square}(\tau)\cdot (A\bar{A})^2\underset{(A\bar{A})^2}{\square}(\tau,z)\cdot (P\bar{P})^2\underset{(A\bar{A})^2}{\square}(\tau,z)\cdot (P\bar{P})^2\underset{(A\bar{A})^2}{\square}(\tau,z) \tag{3.56}$$

$$\mathbb{1}\underset{\kappa_1}{\overset{NS}{\square}}(\tau,z) = \mathbb{1}\underset{\kappa_1}{\square}(\tau)\cdot (A\bar{A})^2\underset{(A\bar{A})^2}{\square}(\tau,z)\cdot (A\bar{A})^2\underset{(P\bar{P})^2}{\square}(\tau,z)\cdot (A\bar{A})^2\underset{(P\bar{P})^2}{\square}(\tau,z) \tag{3.57}$$

$$\kappa_1\underset{\kappa_1}{\overset{NS}{\square}}(\tau,z) = \kappa_1\underset{\kappa_1}{\square}(\tau)\cdot (A\bar{A})^2\underset{(A\bar{A})^2}{\square}(\tau,z)\cdot (P\bar{P})^2\underset{(P\bar{P})^2}{\square}(\tau,z)\cdot (P\bar{P})^2\underset{(P\bar{P})^2}{\square}(\tau,z) \tag{3.58}$$

$$\kappa_2\underset{\kappa_1}{\overset{NS}{\square}}(\tau,z) = \kappa_2\underset{\kappa_1}{\square}(\tau)\cdot (P\bar{P})^2\underset{(A\bar{A})^2}{\square}(\tau,z)\cdot (P\bar{P})^2\underset{(P\bar{P})^2}{\square}(\tau,z)\cdot (A\bar{A})^2\underset{(P\bar{P})^2}{\square}(\tau,z). \tag{3.59}$$

The NS-part of the SCFT partition function is

$$
\begin{aligned}
4 \cdot Z^{\mathbb{Z}_2^2, NS}_{E\times E\times E}(\tau, z) = (Z_{\mathcal{T}}(\tau))^3 \cdot \left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^6 &+ 16\cdot 3\cdot Z_{\mathcal{T}}(\tau)\cdot \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right|^4 \left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^2 \left|\frac{\vartheta_4(\tau,z)}{\eta(\tau)}\right|^4 \\
&+ 16\cdot 3\cdot Z_{\mathcal{T}}(\tau)\cdot \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|^4 \left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^2 \left|\frac{\vartheta_2(\tau,z)}{\eta(\tau)}\right|^4 \\
&+ 16\cdot 3\cdot Z_{\mathcal{T}}(\tau)\cdot \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right|^4 \left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^2 \left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4 \\
&+ 16\cdot 6\cdot \left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^2 \left|\frac{\vartheta_2(\tau,z)}{\eta(\tau)}\right|^2 \left|\frac{\vartheta_4(\tau,z)}{\eta(\tau)}\right|^2 .
\end{aligned}
\tag{3.60}
$$

Using spectral flow we find the elliptic genus

$$
\mathcal{E}^{\mathbb{Z}_2^2}_{E\times E\times E}(\tau, z) = Z_{\tilde{R}}(\tau, z, \bar{\tau}, \bar{z} = 0) = 48 \cdot \frac{\vartheta_2(\tau,z)\vartheta_3(\tau,z)\vartheta_4(\tau,z)}{\eta(\tau)^3}. \tag{3.61}
$$

3.3. Intersection point of $\mathbb{Z}_2$-orbifold and torus models

The moduli spaces of the toroidal CFTs and their $\mathbb{Z}_2$-orbifolds that we have encountered so far are both components of the moduli space of unitary CFTs at integer central charges $c \geq 1$. These two components intersect at special points in the moduli space. Here we have an interpretation for the twist fields in terms of vertex operators of a toroidal CFT which we understand very well. We want to use this to deduce the OPEs of the twist fields of the $\mathbb{Z}_2$-orbifold models. We will review this for different central charges.

It is important to remark that in order to identify two theories it is not sufficient to check equality of their partition function. Instead, here, one uses enhanced $SU(2)$-symmetry to show the equivalence of two CFTs.

For the $\mathbb{Z}_2$-orbifold of toroidal theories this was investigated by K.-I. Kobayashi and Sakamoto in [44], in particular for $d = 1, \ldots, 4$. For more intersection points of different orbifolds see [19] and [57].

Away from the intersection point one hopes to determine OPEs using deformation theory. For an explanation how this should work in principle see Section 2.2 of [57].

3.3.1. $c = 1$

The moduli space of the free boson compactified on the circle introduced in Subsection 1.2.1 can be identified with $\mathbb{R}_{\geq 1}$. For every such $c = 1$ toroidal CFT one also has its $\mathbb{Z}_2$-orbifold, which depends on $R \in \mathbb{R}_{\geq 1}$, too. These two lines lie in the moduli space of unitary CFTs of central charge $c = 1$. In fact, apart from possibly additional isolated non-rational theories without deformations they make up all of this moduli space ([17],[32],[43],[59]).

We want to find the value of R for which the two lines intersect. For this we follow the treatment in Section 8.7 of [31].

At the self-dual point $R = 1$ of the duality $\mathcal{T}(R) = \mathcal{T}\left(\frac{1}{R}\right)$ (cf. (1.46)) the charge lattice arranges itself in such a way that the theory has two additional $(1,0)$- and $(0,1)$-fields apart from the generic fields $j(z)$ and $\bar{j}(\bar{z})$. We denote them by

$$J^{\pm}(z) := \mathrm{e}^{\pm i\sqrt{2}X(z)}, \quad \bar{J}^{\pm}(\bar{z}) := \mathrm{e}^{\pm i\sqrt{2}\bar{X}(\bar{z})} \tag{3.62}$$

and relabel

$$J^3(z) := i\partial X(z) = j(z), \quad \bar{J}^3(\bar{z}) := i\bar{\partial}\bar{X}(\bar{z}) = \bar{j}(\bar{z}). \tag{3.63}$$

The triplet $\{J^{\pm}(z), J^3(z)\}$ extends the $u(1)$-currents to a $SU(2)$ affine Kac-Moody-algebra. Define the fields $J^1(z), J^2(z)$ respectively $\bar{J}^1(z), \bar{J}^2(z)$ by interpreting the fields $J^{\pm}(z)$ and $\bar{J}^{\pm}(\bar{z})$ as complexifications

$$J^{\pm}(z) = \frac{1}{\sqrt{2}}\left(J^1(z) \pm iJ^2(z)\right), \quad \bar{J}^{\pm}(\bar{z}) = \frac{1}{\sqrt{2}}\left(\bar{J}^1(z) \pm i\bar{J}^2(z)\right). \tag{3.64}$$

This complexification is with respect to the real structure (1.30) with J^1 and J^2 being real.

We can define two orbifolds by mapping $X \mapsto -X$ or $X \mapsto X + \frac{\pi}{\sqrt{2}}$. Their effects on the $SU(2)$ fields are:

$$\begin{array}{rcllll} X \mapsto -X & : & J^1 \mapsto J^1, & J^2 \mapsto -J^2, & J^3 \mapsto -J^3 & , \\ X \mapsto X + \frac{\pi}{\sqrt{2}} & : & J^1 \mapsto -J^1, & J^2 \mapsto -J^2, & J^3 \mapsto J^3 & . \end{array} \tag{3.65}$$

The first action is the same as κ of the $\mathbb{Z}_2$-orbifold of toroidal CFT in (3.18), the second is the shift orbifold of Example 3.1.1 with half-lattice vector $\Delta = \left(\frac{\sqrt{2}}{4}; \frac{\sqrt{2}}{4}\right)$. These two orbifolds are equivalent via $SU(2)$ symmetry. By Remark 3.1.2 the resulting toroidal CFT is $\mathcal{T}\left(R = \frac{1}{2}\right)$. Thus we have identified

$$\mathcal{T}^{\mathbb{Z}_2}(R = 1) = \mathcal{T}^{shift}(R = 1) = \mathcal{T}\left(R = \frac{1}{2}\right). \tag{3.66}$$

At this point in the moduli space of $c = 1$ theories the $(\frac{1}{16}, \frac{1}{16})$-twist fields arrange themselves to fit into the charge lattice at $R = \frac{1}{2}$. The twist fields are linear combinations of

$$W_{\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)}(z) \quad \text{and} \quad W_{\left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}\right)}(z). \tag{3.67}$$

Note that $J^3(z)$ of the original theory $\mathcal{T}(R = 1)$ is rotated by the $SU(2)$ symmetry to $J^1(z)$ in the shift orbifold. We find the OPE of $u(1)$-currents with the twist field

$$\begin{aligned} J^1(z)W_{\left(\pm\frac{\sqrt{2}}{4}, \pm\frac{\sqrt{2}}{4}\right)}(w) &= \frac{1}{\sqrt{2}}\left(e^{i\sqrt{2}X(z)} + e^{-i\sqrt{2}X(z)}\right)e^{\pm i\frac{\sqrt{2}}{4}X(w) \pm i\frac{\sqrt{2}}{4}\bar{X}(\bar{w})} \\ \sim \frac{1}{\sqrt{2}}\Big((z-w)^{\pm\frac{1}{2}}e^{i\sqrt{2}X(z) \pm i\frac{\sqrt{2}}{4}X(w) \pm i\frac{\sqrt{2}}{4}\bar{X}(\bar{w})} &+ (z-w)^{\mp\frac{1}{2}}e^{-i\sqrt{2}X(z) \pm i\frac{\sqrt{2}}{4}X(w) \pm i\frac{\sqrt{2}}{4}\bar{X}(\bar{w})}\Big), \end{aligned} \tag{3.68}$$

implementing the square-root cut of (3.28).

3.3.2. $c = 3$

We want to find an intersection point of the $c = 3$ toroidal SCFT and its $\mathbb{Z}_2$-orbifold. Before involving the Dirac fermion, we analyze the bosonic $c = 2$ part first.

Again, we need a point in the moduli space with enhanced symmetry. The tensor product of two circles at $R = 1$ has two copies of the additional $(1,0)$- and $(0,1)$-fields of the previous subsection. We use

$$\mathcal{T}(\tau = i, \rho = i) = \mathcal{T}(R = 1) \otimes \mathcal{T}(R = 1) \tag{3.69}$$

as the starting point. On both factors we can define the two actions (3.65) and they do not mix the two $\mathcal{T}(R = 1)$ factors. Hence the shift vector is the half-lattice vector

$$\Delta = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}; \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right) \tag{3.70}$$

and one finds ([44],[19])

$$\mathcal{T}^{\mathbb{Z}_2}(\tau = i, \rho = i) = \mathcal{T}^{shift}(\tau = i, \rho = i) = \mathcal{T}(\tau = i, \rho = 2i) = \mathcal{T}(R = \sqrt{2}) \otimes \mathcal{T}(R = \sqrt{2}). \tag{3.71}$$

In the superconformal case we have to include the additional factor of the Dirac fermion. For the Dirac fermion we have the handy description via bosonization $\mathcal{C}_{Dirac} = \mathcal{T}\left(\frac{1}{\sqrt{2}}\right)$ at our disposal. As seen in Example 3.2.2 the $\mathbb{Z}_2$-orbifold translates into a shift orbifold, which leaves the theory invariant. In total, the $\mathbb{Z}_2$ orbifold of

$$\mathcal{T}(\tau = i, \rho = i) \otimes \mathcal{T}\left(\frac{1}{\sqrt{2}}\right) \tag{3.72}$$

can be identified with

$$\mathcal{T}(\tau = i, \rho = 2i) \otimes \mathcal{T}\left(\frac{1}{\sqrt{2}}\right) = \mathcal{T}(R = \sqrt{2}) \otimes \mathcal{T}(R = \sqrt{2}) \otimes \mathcal{T}\left(\frac{1}{\sqrt{2}}\right). \tag{3.73}$$

The four NS-sector twist fields of conformal weight $(\frac{1}{4}, \frac{1}{4})$ and fermionic charge $(\frac{1}{2}, \frac{1}{2})$ are linear combinations of

$$W_{(\frac{1}{2},0,\frac{1}{2};\frac{1}{2},0,\frac{1}{2})}(z), \tag{3.74}$$

$$W_{(-\frac{1}{2},0,\frac{1}{2};-\frac{1}{2},0,\frac{1}{2})}(z), \tag{3.75}$$

$$W_{(0,\frac{1}{2},\frac{1}{2};0,\frac{1}{2},\frac{1}{2})}(z), \tag{3.76}$$

$$W_{(0,-\frac{1}{2},\frac{1}{2};0,-\frac{1}{2},\frac{1}{2})}(z). \tag{3.77}$$

Note that not all OPEs between these fields are non-singular! Just as in the $c = 1$ case this implements square-root cuts in the OPEs with the free bosons and the free fermions.

Since we now have an interpretation of the twist fields as vertex operators in a toroidal CFT/SCFT we can calculate OPEs using (1.45).

Chapter 4

Chiral de Rham complex

From a mathematical point of view it would be desirable to have a functor from Calabi-Yau manifolds to conformal field theories that makes the notion of non-linear sigma model precise. At the moment this seems to be out of reach. But restricting to the mathematically better understood vertex operator algebra part of the CFT may be more promising. This is exactly what Malikov, Schechtman and Vaintrob ([49]) attempted when defining the *chiral de Rham complex* Ω_X^{ch}.

In this chapter we will review their definition and main results as well as further results by Borisov and Libgober ([6], [7]) dealing with sheaf cohomology and relating an operator trace over the chiral de Rham cohomology to the elliptic genus of the Calabi-Yau manifold. We will see that their result for hypersurfaces in toric varieties is not appropriate for our quest to determine the chiral de Rham complex of tori and their $\mathbb{Z}_2$-orbifolds.

4.1. Local chiral de Rham complex on $\mathbb{C}^D$

In [49] the authors Malikov, Schechtman and Vaintrob defined a sheaf of vertex algebras Ω_X^{ch} by gluing local $(bc - \beta\gamma)$-systems. While they define this in a more general setting we will restrict to compact complex manifolds here. We start with the local models on $\mathbb{C}^D$ and follow the original paper [49].

Example 4.1.1 ($(bc - \beta\gamma)$-system)**.** The $(bc - \beta\gamma)$-system[1] is a tensor product of a Heisenberg and a Clifford vertex algebra. Let $D \in \mathbb{N}$ and consider the super Lie algebra A_D with $\mathbb{C}$-vector space basis $\{ 1, a_n^i, b_n^i, \Phi_n^i, \Psi_n^i \mid n \in \mathbb{Z}, i \in \{ 1, \dots, D \} \}$, where a_n^i, b_n^i are even and Φ_n^i, Ψ_n^i are odd and 1 is central. The only non-trivial relations are

$$[a_n^i, b_m^j] = \delta_{i,j}\delta_{n+m,0}, \tag{4.1}$$

$$\{ \Phi_n^i, \Psi_m^j \} = \delta_{i,j}\delta_{n+m,0} \tag{4.2}$$

for $n, m \in \mathbb{Z}$ and $i, j \in \{ 1, \dots, D \}$.

We will construct a vertex algebra for the Lie algebra A_D similarly to the one for the free boson algebra in Subsection 1.2.1: Let $\tilde{A}_D$ be the universal enveloping algebra of A_D.

[1]This name is due to historical reasons. Originally the generators were named b, c, β and γ. Instead we use the letters a,b,Φ and Ψ for the generators, in accordance with the notation in [49].

Let $\mathcal{I}$ be the left-handed ideal in $\tilde{A}_D$ generated by

$$a^i_n, b^i_m, \Psi^i_n, \Phi^i_m \quad \text{for } n \le 0, m < 0 \quad \text{and } 1_{A_D^{\otimes 1}} - 1_{A_D^{\otimes 0}}. \tag{4.3}$$

Then the quotient $\Omega_D := \tilde{A}_D/\mathcal{I}$ is a representation of $\tilde{A}_D$. Its vector space will be the underlying vector space of the $(bc-\beta\gamma)$-system vertex algebra. The vertex algebra structure is defined by associating fields for $i = 1, \dots, D$ in the following way:

$$a^i(z) := a^i_1(z) := \sum_{n\in\mathbb{Z}} a^i_n z^{n-1}, \tag{4.4}$$

$$b^i(z) := b^i_0(z) := \sum_{n\in\mathbb{Z}} b^i_n z^n, \tag{4.5}$$

$$\Psi^i(z) := \Psi^i_1(z) := \sum_{n\in\mathbb{Z}} \Psi^i_n z^{n-1}, \tag{4.6}$$

$$\Phi^i(z) := \Phi^i_0(z) := \sum_{n\in\mathbb{Z}} \Phi^i_n z^n \tag{4.7}$$

to the modes $a^i_1, b^i_0, \Psi^i_1, \Phi^i_0$ respectively with non-regular OPEs

$$a^i(z)b^j(w) \sim \frac{\delta_{ij}}{z-w}, \quad \Phi^i(z)\Psi^j(w) \sim \frac{\delta_{ij}}{z-w}. \tag{4.8}$$

All other fields are found using derivatives and normal ordered products. Note that we have identified a^i, b^i, Ψ^i and Φ^i with a^i_0, b^i_0, Ψ^i_0 and Φ^i_0 respectively. We will follow this convention throughout this text.

There are four distinguished fields of importance to us:

$$T^{top}(z) := \sum_{j=1}^{D}(:\partial b^j a^j:(z) + :\partial\Phi^j\Psi^j:(z)) = \sum_{n\in\mathbb{Z}} L^{top}_n z^{n-2}, \tag{4.9}$$

$$J(z) := \sum_{j=1}^{D} :\Phi^j\Psi^j:(z), \tag{4.10}$$

$$Q(z) := \sum_{j=1}^{D} :a^j\Phi^j:(z), \tag{4.11}$$

$$G(z) := \sum_{j=1}^{D} :\Psi^j\partial b^j:(z) \tag{4.12}$$

with OPEs:

$$T^{top}(z)T^{top}(w) \sim \frac{2T^{top}(w)}{(z-w)^2} + \frac{\partial T^{top}(w)}{z-w}, \tag{4.13}$$

$$T^{top}(z)J(w) \sim \frac{\frac{c}{3}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad J(z)J(w) \sim \frac{\frac{c}{3}}{(z-w)^2}, \tag{4.14}$$

$$T^{top}(z)Q(w) \sim \frac{Q(w)}{(z-w)^2} + \frac{\partial Q(w)}{z-w}, \quad Q(z)Q(w) \sim 0, \quad J(z)Q(w) \sim \frac{Q(w)}{z-w}, \tag{4.15}$$

$$T^{top}(z)G(w) \sim \frac{2G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}, \quad G(z)G(w) \sim 0, \quad J(z)G(w) \sim -\frac{G(w)}{z-w}, \tag{4.16}$$

$$Q(z)G(w) \sim \frac{\frac{c}{3}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T^{top}(w)}{z-w}, \tag{4.17}$$

where $c = 3D$. The Lie algebra generated by the modes of the fields $T^{top}(z), J(z), Q(z)$ and $G(z)$ is called the $N = 2$ *topological superconformal algebra*. It is related to the $N = 2$ superconformal algebra of Chapter 2 via the topological twist

$$T(z) = T^{top}(z) - \frac{1}{2}\partial J(z), \quad G^+(z) = Q(z), \quad G^-(z) = G(z), \quad J(z) = J(z). \tag{4.18}$$

Note that the zero and one modes of the fields $T(z)$ and $T^{top}(z)$ are related by $L_0 = L_0^{top} + \frac{1}{2}J_0$ and $L_1 = L_1^{top}$.

The vertex algebra Ω_D is also a complex. This grading is given by the fermionic charge operator J_0, which induces

$$\Omega_D = \oplus_{p\in\mathbb{Z}}\Omega_D^p \quad \text{with} \quad \Omega_D^p := \{\, \omega \in \Omega_D \mid J_0\omega = p\omega \,\}, \tag{4.19}$$

and the differential given by

$$d := -Q_0 = -\sum_{i,n} : a_n^i \Phi_{-n}^i : . \tag{4.20}$$

Definition 4.1.2 ([49], 2.2)**.** We define the *algebraic chiral de Rham complex of* $\mathbb{C}^D$ to be the vertex algebra Ω_D of the $(bc - \beta\gamma)$-system of the previous example.

The connection to the usual algebraic de Rham complex $\Omega(\mathbb{C}^D) = \bigoplus_{p=0}^D \Omega^p(\mathbb{C}^D)$ is the following: We write

$$\Omega(\mathbb{C}^D) = \mathbb{C}[b_0^1, \dots, b_0^D] \otimes \Lambda(\Phi_0^1, \dots \Phi_0^D) \tag{4.21}$$

where we identified the coordinate functions with $b_0^1, \dots b_0^D$ and their differentials with $\Phi_0^1, \dots \Phi_0^D$ and Λ is the exterior algebra. The classical de Rham differential is

$$d_{dR} = \sum_i a_0^i \Phi_0^i, \tag{4.22}$$

where we identified a_0^i with the vector field $\partial_{b_0^i}$. We thus see how the algebraic de Rham complex $\Omega(\mathbb{C}^D)$ sits inside the chiral de Rham complex Ω_D as the conformal weight zero part. Moreover:

Theorem 4.1.3 ([49], 2.4)**.** *The embedding of complexes*

$$i : (\Omega(\mathbb{C}^D), d_{dR}) \longrightarrow (\Omega_D, d) \tag{4.23}$$

is compatible with the differentials and a quasi-isomorphism.

Proof. We write

$$d = \sum_{i,n\neq 0} : a_n^i \Phi_{-n}^i : + d_{dR}. \tag{4.24}$$

Since the first summand commutes with the conformal weight zero space the inclusion is compatible with the differentials.
From OPE (4.17) follows $[G_0, d] = L_0^{top}$. Let $v \in \ker d$ such that $v \notin \Omega(\mathbb{C}^D)$, i.e. $L_0 v = \lambda v$ with $\lambda \neq 0$. Then we have

$$v = d\frac{1}{\lambda}G_0 v \in \operatorname{im} d \tag{4.25}$$

and conclude that the inclusion is a quasi-isomorphism. □

So far we have worked in the realm of algebraic geometry, allowing only polynomial functions in the coordinates. We now want to move to complex geometry where the structure sheaf is the sheaf of holomorphic functions, i.e. locally convergent power series. Thus, in order to define the chiral de Rham complex here we need to allow convergent power series $\mathbb{C}\{b_0^1, \dots, b_0^D\}$ in the coordinate functions. Let

$$\widehat{\Omega}_D := \mathbb{C}\left\{b_0^1, \dots, b_0^D\right\} \otimes_{\mathbb{C}[b_0^1,\dots,b_0^D]} \Omega_D \tag{4.26}$$

be the vector space underlying the vertex algebra. We keep the vertex algebra structure on Ω_D and only need to define the field associated to a power series $f \in \mathbb{C}\{b_0^1, \dots, b_0^D\}$. For this we write

$$b^i(z) = b_0^i + \Delta b^i(z). \tag{4.27}$$

We then define the field $f(z) = f(b^1(z), \dots, b^D(z))$ via the Taylor formula

$$f(b^1(z), \dots, b^D(z)) = \sum_{n\in\mathbb{Z}} f_n z^n \tag{4.28}$$

$$:= \sum_{(i_1,\dots,i_D)\in\mathbb{N}_0^D} \Delta b^1(z)^{i_1} \cdot \ldots \cdot \Delta b^D(z)^{i_D} \partial^{(i_1,\dots,i_D)} f(b_0^1, \dots, b_0^D), \tag{4.29}$$

where

$$\partial^{(i_1,\dots,i_D)} = \frac{\partial_{b^1}^{i_1}}{i_1!} \cdot \ldots \cdot \frac{\partial_{b^D}^{i_D}}{i_D!}. \tag{4.30}$$

Let $x \in \widehat{\Omega}_D$ be of fixed conformal weight. Since the $\Delta b^i(z)$ have no zero mode there are only finitely many summands in any f_k that do not annihilate x and $f_k x$ is again in $\widehat{\Omega}_D$. Moreover, for $k << 0$, the mode f_k will annihilate x. The power series $f(z)$ is indeed in $\mathrm{End}(\widehat{\Omega}_D)[[z, z^{-1}]]$ and a field ([49], 3.1). In particular $f_k\mathbb{1} \in \widehat{\Omega}_D$, i.e. a combination of a power series in the b_0^i and a polynomial in the other variables.

Definition 4.1.4 ([49], 3.1)**.** The vertex algebra $\widehat{\Omega}_D$ is the *(holomorphic) chiral de Rham complex* of $\mathbb{C}^D$.

Remark 4.1.5. Note that in the literature it depends on the context whether the algebraic, holomorphic or $\mathcal{C}^\infty$ chiral de Rham complex is meant by *chiral de Rham complex.* Unless stated otherwise we will always mean the holomorphic chiral de Rham complex.

4.2. Chiral de Rham complex sheaf

We can now turn to the sheaf version of the chiral de Rham complex. Let X be a compact D-dimensional complex manifold covered by $X = \bigcup_i U_i$. Choose local holomorphic coordinates $x^1, \dots, x^D$ on U_i. Again, we identify geometric objects with those of the $(bc - \beta\gamma)$-system via the map

$$(x^i, \partial_{x^i}, dx^i, \partial_{dx^i}) \longmapsto (b^i, a^i, \Phi^i, \Psi^i), \tag{4.31}$$

which induces a super Lie algebra homomorphism. Denote by $\mathbb{C}\{b_0^1,\dots,b_0^D\}$ the convergent power series on U_i. We define the local sections of the chiral de Rham complex on U_i to be

$$\Omega^{ch}(U_i) := \mathbb{C}\left\{b_0^1,\dots,b_0^D\right\} \otimes \Omega_D = \widehat{\Omega}_D, \tag{4.32}$$

since in a complex-analytic setting we need to allow convergent power series. The great insight of Malikov, Schechtman and Vaintrob was to find the correct translation laws for all elements of the vertex algebra. Consider an invertible coordinate transformation

$$\tilde{b}^i = g^i(b^1,\dots,b^D), \quad b^i = f^i(\tilde{b}^1,\dots,\tilde{b}^D), \tag{4.33}$$

where $g^i \in \mathbb{C}\{b_0^1,\dots,b_0^D\}$ and $f^i \in \mathbb{C}\left\{\tilde{b}_0^1,\dots,\tilde{b}_0^D\right\}$. The transformation laws for the geometric objects are well known

$$\tilde{\Phi}^i = \frac{\partial g^i}{\partial b^j}\Phi^j, \tag{4.34}$$

$$\partial_{\tilde{b}^i} = \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{b^j} + \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\Phi^r \partial_{\Phi^k}, \tag{4.35}$$

$$\partial_{\tilde{\Phi}^i} = \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{\Phi^j}. \tag{4.36}$$

They inspire the transformations of the associated fields and thus their modes:

$$\tilde{b}^i(z) := g^i(b)(z), \tag{4.37}$$

$$\tilde{\Phi}^i(z) := \left(\frac{\partial g^i}{\partial b^j}\Phi^j\right)(z), \tag{4.38}$$

$$\tilde{a}^i(z) = \partial_{\tilde{b}^i}(z) := \left(\frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{b^j}\right)(z) + \left(\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}\Phi^r \partial_{\Phi^k}\right)(z), \tag{4.39}$$

$$\tilde{\Psi}^i(z) = \partial_{\tilde{\Phi}^i}(z) := \left(\frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{\Phi^j}\right). \tag{4.40}$$

The authors of [49] showed that these transformed fields again satisfy the OPEs of the $(bc-\beta\gamma)$-system. The transformation laws for derivatives and normal ordered products are given as follows: Let $c, d \in \{a, b, \Phi, \Psi\}$, then

$$\widetilde{c_n^i}(z) := \partial_z^n \tilde{c}^i(z), \tag{4.41}$$

$$(\widetilde{:c^i d^j:})(z) :=: \tilde{c}^i(z)\tilde{d}^j(z):. \tag{4.42}$$

The transformation laws determine a morphism of vertex algebras

$$\tilde{g} : \widehat{\Omega}_D \longrightarrow \widehat{\Omega}_D. \tag{4.43}$$

Theorem 4.2.1 ([49], 3.8)**.** *The assignment $g \mapsto \tilde{g}$ defines a group homomorphism $\mathbb{C}\{b^1,\dots b^D\} \to Aut(\hat{\Omega}_D)$, i.e. for two coordinate transformations g_1, g_2 we have $\widetilde{g_2 g_1} = \tilde{g}_2\tilde{g}_1$.*

This allows to glue the local sections of the chiral de Rham complex into a sheaf of vertex algebras on X and we arrive at:

Definition 4.2.2. The sheaf of vertex algebras constructed as described before is called the *(holomorphic) chiral de Rham complex* Ω_X^{ch} of X.

Because of the transformation laws, the fields $b^i(z)$, $a^i(z)$, $\Phi^i(z)$ and $\Psi^i(z)$ will in general not be globally defined. However, the fields $T^{top}(z)$, $J(z)$, $Q(z)$ and $G(z)$ generating the $N=2$ topological superconformal algebra will be for Calabi-Yau manifolds:

Theorem 4.2.3 ([49], 4.5). *If X is a compact D-dimensional Calabi-Yau manifold the fields $L(z), J(z), Q(z)$ and $G(z)$ are globally defined and Ω_X^{ch} is a sheaf of topological $N=2$ superconformal vertex algebras at central charge $c=3D$.*

In fact, the Calabi-Yau condition is only needed for $J(z)$ and $Q(z)$ to be globally defined. Since L_0^{top} and J_0 are always globally defined the bigrading always survives the gluing process and Ω_X^{ch} is a direct sum

$$\Omega_X^{ch} = \oplus_{l\in\mathbb{N}}\Omega_{X,l}^{ch} \quad \text{with} \quad \Omega_{X,l}^{ch} := \left\{ \omega \in \Omega_X^{ch} \mid L_0^{top}\omega = l\omega \right\}, \tag{4.44}$$

$$\Omega_X^{ch} = \oplus_{p\in\mathbb{Z}}\Omega_X^{ch,p} \quad \text{with} \quad \Omega_X^{ch,p} := \left\{ \omega \in \Omega_X^{ch} \mid J_0\omega = p\omega \right\}. \tag{4.45}$$

Since L_0^{top} and J_0 commute we can also simultaneously diagonalize Ω_X^{ch}

$$\Omega_X^{ch} = \oplus_{l\in\mathbb{N},p\in\mathbb{Z}}\Omega_{X,l}^{ch,p} \quad \text{with} \quad \Omega_{X,l}^{ch,p} := \left\{ \omega \in \Omega_X^{ch} \mid L_0^{top}\omega = l\omega, \quad J_0\omega = p\omega \right\}. \tag{4.46}$$

Furthermore the local embedding of the de Rham complex extends to the sheaf:

Theorem 4.2.4 ([49], 4.4). *Let X be a compact complex manifold. The sheaf complex embedding of the holomorphic de Rham complex into the chiral de Rham complex*

$$(\Omega_X, d_{DR}) \hookrightarrow (\Omega_X^{ch}, d) \tag{4.47}$$

is a quasi-isomorphism.

Proof. This is a local calculation and follows directly from 4.1.3. □

This implies that hypercohomology of the two complexes is isomorphic, while their sheaf cohomologies are quite distinct. We will call the sheaf cohomology $\mathrm{H}^*(X,\Omega_X^{ch})$ of Ω_X^{ch} the *chiral de Rham cohomology.*

It is important to remark that the chiral de Rham complex is not a quasi-coherent sheaf. We supplement the literature with the following consideration and example in order to illustrate this: Notice that while the chiral de Rham complex carries an $\mathcal{O}_X$-action, this action is not associative. We will investigate this now. Let U_1, U_2 be open sets in the open cover of X with transition function g on their intersection $U_{12} = U_1 \cap U_2$. An action of $\mathcal{O}_X$ on Ω_X^{ch} has to be compatible with the vertex algebra transition functions, i.e. the following diagram has to commute:

$$\begin{array}{ccccc} \mathcal{O}_X(U_1)\mid_{U_{12}} & \times & \Omega_X^{ch}(U_1)\mid_{U_{12}} & \longrightarrow & \Omega_X^{ch}(U_1)\mid_{U_{12}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_X(U_2)\mid_{U_{12}} & \times & \Omega_X^{ch}(U_2)\mid_{U_{12}} & \longrightarrow & \Omega_X^{ch}(U_2)\mid_{U_{12}} \end{array}. \tag{4.48}$$

Example 4.2.5. Let $X = \mathbb{CP}^1$. Even in this relatively simple case we see that a naive $\mathcal{O}_X$-action will not be compatible with the gluing data. Consider the coordinate transformation $g(b_0) = \frac{1}{b_0}$ in one complex dimension. The field $\frac{1}{b_0}(z)$ can be found using the geometric series ([49], 3.4):

$$\frac{1}{b_0}(z) = \left(b_0 + b_1 z + b_{-1} z^{-1} \ldots\right)^{-1} = b_0^{-1}\left(1 + b_0^{-2}(2b_1 b_{-1} + 2b_2 b_{-2} + \ldots) + \ldots\right). \tag{4.49}$$

Consider $(b_0, a_1) \in \mathcal{O}_X(U_1) \times \Omega_X^{ch}(U_1)$. Naively defining the action as multiplication of functions gives $b_0 a_1 \in \Omega_X^{ch}(U_1)$. Under coordinate transformations (b_0, a_1) maps to $(\frac{1}{b_0}, -b_0 b_0 a_1 - 2b_0 \Phi_0 \Psi_1) \in \mathcal{O}_X(U_2) \times \Omega_X^{ch}(U_2)$. Again, naively multiplying gives $-b_0 a_1 - 2\Phi_0\Psi_1 \in \Omega_X^{ch}(U_2)$. On the other hand, the image of $b_0 a_1$ under coordinate change is

$$:[\tilde{g}(b_0(z))]_0[\tilde{g}(a_1(z))]_1 : \mathbb{1} =: \left[\frac{1}{b_0}(z)\right]_0 [\tilde{g}(a_1(z))]_1 : \mathbb{1} = -b_0 b_0 a_1 - 2b_0\Phi_0\Psi_1 - 2b_0^{-1} b_1, \tag{4.50}$$

since $\left[\frac{1}{b_0}(z)\right]_0$ is not an annihilator and thus does not get ordered to the right. The point is that of course $\frac{1}{b_0} \neq \left[\frac{1}{b_0}(z)\right]_0$. Thus Diagram 4.48 does not commute and the action cannot be extended to all of X. Note that the extra terms appearing because of this mismatch are difficult to predict!

One fix, at the cost of loosing associativity, is to define the action as $(f, \varphi) \mapsto [f(z)]_0\varphi$ for $f \in \mathcal{O}_X(U_i)$ and $\varphi \in \Omega_X^{ch}(U_i)$. This yields a commuting diagram. We thus have

$$\begin{array}{ccccc} \mathcal{O}_X(U_i) & \times & \Omega_X^{ch}(U_i) & \longrightarrow & \Omega_X^{ch}(U_i) \\ f & & \varphi & \longmapsto & [f(z)]_0 \cdot \varphi \end{array}, \tag{4.51}$$

which is not associative. This is compatible with the bigrading given by L_0^{top} and J_0. On the conformal weight zero part, i.e. the classical holomorphic de Rham complex it reduces to the usual $\mathcal{O}_X$-action.

As the chiral de Rham complex Ω_X^{ch} is not even an $\mathcal{O}_X$-module there is no point in talking about (quasi-)coherence. The appropriate structure here is what Borisov defines as *(quasi-)loop-coherent* ([6]) in the algebraic setting: There is a filtration of the chiral de Rham complex compatible with the bigrading of L_0^{top} and J_0 whose graded objects are coherent. This filtration was introduced in 3.10 of [49] (also cf. [25], 3.5): $\Omega_X^{ch}(U_i)$ is a free $\mathcal{O}_X$-module with a base consisting of monomials in letters $b_n^i, \Phi_n^i, a_m^i, \Psi_m^i$ for $i = 1, \ldots, N$ and $n \geq 0, m > 0$. Define a partial order on the base by

1. $a > \Phi, a > \Psi, a > b, \Psi > \Phi, \Psi > b, \Phi > b; x_n^i > x_m^i$ if $n > m$ and $x \in \{a, b, \Phi, \Psi\}$,
2. extending this order to the whole set of monomials lexicographically such that the b_0^i are equivalent to the empty word, i.e. all monomials differing by products of b_0^i are equivalent.

This partial order determines an exhaustive increasing filtration

$$F^0\Omega_X^{ch}(U_i) \subset F^1\Omega_X^{ch}(U_i) \subset F^2\Omega_X^{ch}(U_i) \ldots \tag{4.52}$$

by using the two rules where $Gr_F\Omega_X^{ch}(U_i)$ is the associated graded object and $\pi : \Omega_X^{ch}(U_i) \to Gr_F\Omega_X^{ch}(U_i)$ is the projection map:

1. If M_1 and M_2 are two monomials and $M_1 < M_2$, then $\pi(M_1)$ lies in a strictly lower component in $Gr_F \Omega_X^{ch}(U_i)$ than $\pi(M_2)$.
2. If neither $M_1 < M_2$ nor $M_2 < M_1$, then $\pi(M_1)$ and $\pi(M_2)$ lie in the same graded component.

Because of the transformation properties (4.37) to (4.37) additional terms coming from coordinate changes $g^i \in \mathbb{C}\{b_0^1, \dots, b_0^D\}$ and stemming from the fact that $g \neq [g(z)]_0$ lie in lower graded components (cf. Example 4.2.5) and the filtration is defined on all of Ω_X^{ch}. Furthermore it is compatible with L_0 weights and J_0 charges by definition

$$F^0 \Omega_{X,k}^{ch,p}(U_i) \subset F^1 \Omega_{X,k}^{ch,p}(U_i) \subset F^2 \Omega_{X,k}^{ch,p}(U_i) \dots . \tag{4.53}$$

For example we have $F^0 \Omega_{X,0}^{ch,0} = \mathcal{O}_X$ and $F^0 \Omega_{X,1}^{ch,0} = \sum_{i=1}^N \mathcal{O}_X \, b_1^i$. The additional terms that prevent associativity of action (4.51) lie in lower parts of the filtration and thus the graded objects become $\mathcal{O}_X$-modules.

The filtration allows to deduce many results for quasi-loop-coherent sheaves from the world of quasi-coherent sheaves, for example we can prove the following using an idea of [6]:

Lemma 4.2.6. *Let X be a compact complex n-dimensional manifold. Then the sheaf cohomology of the chiral de Rham complex Ω_X^{ch} vanishes in degrees higher than the manifold dimension:*

$$\mathrm{H}^q(X, \Omega_X^{ch}) = 0 \quad \textit{for } q > n. \tag{4.54}$$

Proof. The chiral de Rham complex is graded by conformal weight

$$\Omega_X^{ch} = \bigoplus_l \Omega_{X,l}^{ch} \tag{4.55}$$

with each conformal weight space being of finite dimension. Each of these weight spaces in turn has a finite exhausting filtration

$$F^0 \subset F^1 \subset \dots \subset F^m = \Omega_{X,l}^{ch} \tag{4.56}$$

whose graded objects F^{i+1}/F^i are coherent sheaves. We proceed by induction on i: The short exact sequence

$$0 \longrightarrow F^i \longrightarrow F^{i+1} \longrightarrow F^{i+1}/F^i \longrightarrow 0 \tag{4.57}$$

gives rise to a long exact sequence in cohomology where higher vanishing of cohomology of F^i and F^{i+1}/F^i implies vanishing for F^{i+1}. Since the only infinity we are dealing with appears in the direct sum (4.55) the result carries to Ω_X^{ch}. □

Definition 4.2.7. We will call the sheaf cohomology $\mathrm{H}(X, \Omega_X^{ch})$ of the holomorphic chiral de Rham complex of a complex manifold X the *chiral de Rham cohomology of X*.

4.3. Čech cohomology vertex algebra

In [6] Borisov equips the Čech cohomology of the chiral de Rham complex with the structure of a vertex algebra by using a cup product ([6], Proposition 3.7). He then proceeds to construct a vertex algebra for Calabi-Yau hypersurfaces in toric varieties via toric geometry which he shows to be equal to Čech cohomology of the chiral de Rham complex ([6], Proposition 7.14).
We will review the Čech cohomology vertex algebra structure now and the results for Calabi-Yau hypersurfaces in toric varieties in Section 4.5. Let $\mathcal{U} = \{U_i\}$ be a good open cover of X. Let $x \in \check{C}^p(\Omega_X^{ch})$ and $y \in \check{C}^q(\Omega_X^{ch})$ be two Čech-cocycles and $n \in \mathbb{Z}$. The product $x_n y$ is an element of $\check{C}^{p+q}(\Omega_X^{ch})$ given by

$$(x_n y)\left(U_{i_0,\dots,i_{p+q}}\right) \coloneqq (x_{i_0,\dots,i_p})_n y_{i_p,\dots,i_{p+q}}, \tag{4.58}$$

where the sections on the right are both restricted to $U_{i_0,\dots,i_{p+q}}$.

Theorem 4.3.1 ([6], Proposition 3.7, Definition 4.1, [7], Theorem 5.2)**.** *Let X be a Calabi-Yau manifold of dimension D. Then Čech cohomology $\check{\mathrm{H}}^*(X,\Omega_X^{ch})$ of the chiral de Rham complex with $\mathbb{Z}_2$-grading on $\check{\mathrm{H}}^q(X,\Omega_X^{ch})$ given by $(-1)^{J_0+q}$ carries the structure of a topological $N=2$ superconformal vertex algebra. If X is a Calabi-Yau hypersurface in a toric variety, the supertrace $y^{-\frac{D}{2}} y^{J_0} q^{L_0^{top}}$ yields the elliptic genus of X.*

In the next chapter Remark 5.1.8 will allow us to understand the shift in fermionic charge $(-1)^{F+q}$ using a Dolbeault style resolution of the chiral de Rham complex.

Since L_0^{top} is a globally defined operator the chiral de Rham complex is a direct sum of sheaves

$$\Omega_X^{ch} = \bigoplus_n \Omega_{X,n}^{ch}, \tag{4.59}$$

with $\Omega_{X,0}^{ch} = \Omega_X$, the holomorphic de Rham complex. For cohomology follows

$$\check{\mathrm{H}}^*(X,\Omega_X^{ch}) = \bigoplus_n \check{\mathrm{H}}^*(X,\Omega_{X,n}^{ch}), \tag{4.60}$$

such that we have an inclusion of cohomology

$$\check{\mathrm{H}}^*(X,\Omega_X) \hookrightarrow \check{\mathrm{H}}^*(X,\Omega_X^{ch}). \tag{4.61}$$

4.4. Identification with SCFT

In order to compare the chiral de Rham complex to the SCFT torus orbifold we need to undo the topological twist $T(z) = T^{top}(z) - \frac{1}{2}\partial J(z)$, which relates the zero modes by $L_0 = L_0^{top} + \frac{1}{2}J_0$. The comparison of the CFT and chiral de Rham incarnations of the elliptic genus hints towards an identification of their spaces of states (cf. [58], p. 25)

$$\mathcal{E}(\tau,z) = \mathrm{tr}_{\mathbb{H}^R}\left((-1)^F y^{J_0} q^{L_0-\frac{c}{24}}\right) \tag{4.62}$$

$$= y^{-\frac{c}{6}}\,\mathrm{tr}_{\mathbb{H}^{NS}}\left((-1)^F y^{J_0} q^{-\frac{1}{2}J_0} q^{L_0}\right) = y^{-\frac{c}{6}}\,\mathrm{tr}_{\mathbb{H}^{NS}}\left((-1)^F y^{J_0} q^{L_0^{top}}\right), \tag{4.63}$$

where we have used spectral flow going from the first to the second line. The prefactor of $y^{-\frac{c}{6}}$ is the same that appears in the chiral de Rham complex elliptic genus found by Borisov. This suggests that the sheaf cohomology of the chiral de Rham complex might be related to some cohomology of the NS-part of the SCFT. We will investigate this for complex tori and its $\mathbb{Z}_2$-orbifold.
Since in contrast to the SCFT only the complex structure and no complexified Kähler structure is used to define the chiral de Rham complex we cannot expect to retrieve all of the chiral algebra.

4.5. Toric geometry

Borisov's result for hypersurfaces in toric varieties in [6] gives in particular a result for elliptic curves as the zero set of polynomials of degree 3 in $\mathbb{P}^2$. We will review their construction and collect the data for an elliptic curve in the example below. We will not review the construction of toric varieties, for details see for example [28].

Let M be a free abelian group of rank $\dim M$ and $N = \mathrm{Hom}(M,\mathbb{Z})$ its dual. The vector space $(M \oplus N) \otimes \mathbb{C}$ has dimension $2\dim M$ and is equipped with a natural bilinear form. A toric variety is described by a fan Σ, a collection of rational polyhedral cones in N. Each cone is associated with an affine variety, which are glued on intersecting faces.

The corresponding vertex algebra is now constructed in the following way: Denote by $\mathrm{Fock}_{M\oplus N}$ the lattice algebra associated to $M \oplus N$. For every $m \in M$ and $n \in N$ we have fields

$$m \cdot B(z) = \sum_{k\in\mathbb{Z}} m \cdot B_k z^{k-1}, \quad n \cdot A(z) = \sum_{k\in\mathbb{Z}} n \cdot A_k z^{k-1}, \tag{4.64}$$

$$m \cdot \mathbf{\Phi}(z) = \sum_{k\in\mathbb{Z}} m \cdot \mathbf{\Phi}_k z^{k}, \quad n \cdot \mathbf{\Psi}(z) = \sum_{k\in\mathbb{Z}} n \cdot \mathbf{\Psi}_k z^{k-1} \tag{4.65}$$

with non-regular OPEs

$$m \cdot B(z) n \cdot A(w) \sim \frac{(m\cdot n)}{(z-w)^2}, \quad m \cdot \mathbf{\Phi}(z) n \cdot \mathbf{\Psi}(w) \sim \frac{(m\cdot n)}{z-w}. \tag{4.66}$$

As a vector space $\mathrm{Fock}_{M\oplus N}$ is given by

$$\mathrm{Fock}_{M\oplus N} = \bigoplus_{(m,n)\in M\oplus N} \mathbb{C}\left[B_k, A_k, \mathbf{\Phi}_l, \mathbf{\Psi}_k\right]_{k>0,l\geq 0} |\, m,n\rangle, \tag{4.67}$$

where the fields associated to ground states $|\, m,n\rangle$ are vertex operators

$$: \mathrm{e}^{\int (m\cdot B(z) + n\cdot A(z))} : . \tag{4.68}$$

In case of a one-dimensional M, the $(bc-\beta\gamma)$-system we know from the local chiral de Rham complex appears here as the following fields:

$$b(z) = e^{\int B(z)}, \qquad a(z) =: A(z) e^{-\int B(z)} : + : \mathbf{\Phi}(z)\mathbf{\Psi}(z) e^{-\int B(z)} :, \tag{4.69}$$

$$\Phi(z) = \mathbf{\Phi}(z) e^{\int B(z)}, \qquad \Psi(z) = \mathbf{\Psi}(z) e^{-\int B(z)}. \tag{4.70}$$

Let $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ be the degeneration of the vertex algebra structure on $\mathrm{Fock}_{M\oplus N}$ where the action of the field $\mathrm{e}^{\int(m\cdot B(z)+n\cdot A(z))}$ on the ground state $\mid m_1, n_1\rangle$ is set to zero when there is no cone in Σ that contains both n and n_1. This is the data coming from the toric variety. A hypersurface in a toric variety is specified by a map $f : \Delta \to \mathbb{C}$, where Δ is the polytope associated to the toric variety. The main result of [6] now is:

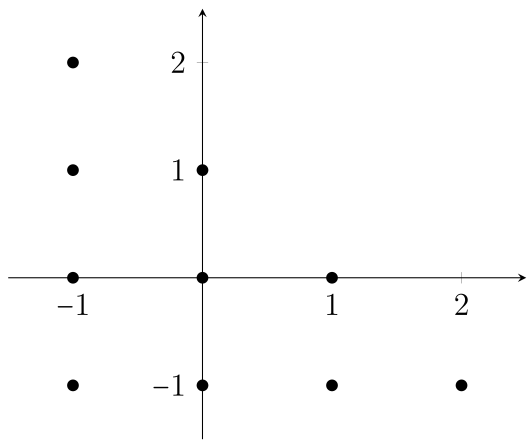
Theorem 4.5.1 ([6], Theorem 8.3)**.** *Let X be a Calabi-Yau hypersurface in a smooth toric nef-Fano variety, given by $f : \Delta \to \mathbb{C}$ and a fan Σ. Then cohomology of chiral de Rham complex of X equals the BRST cohomology of* $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ *by operator*

$$\mathcal{BRST}_{f,g} = \int \left(\sum_{m\in\Delta} f_m (m\cdot\Phi)(z)\,\mathrm{e}^{\int m\cdot B(z)} + \sum_{n\in\Delta^*} g_n (n\cdot\Psi)(z)\,\mathrm{e}^{\int n\cdot A(z)} \right) dz \tag{4.71}$$

with any choice of non-zero complex numbers $g_n \in \mathbb{C}$.

In the following example we apply this theorem to an elliptic curve which we interpret as the zero set of a polynomial in $\mathbb{P}^2$ resulting in an explicit example not found in the literature.

Example 4.5.2 (Elliptic curve as the zero set of a polynomial in $\mathbb{P}^2$)**.** In toric geometry projective space $\mathbb{P}^2$ is described by the fan $\sigma \subset N = \mathbb{Z}^2$ consisting of the three cones spanned by vectors $\{(1,0),(0,1),(-1,-1)\}$, these vectors themselves and 0. We want to regard an elliptic curve as the zero set of a section of the negative canonical line bundle $\mathcal{O}(-K_{\mathbb{P}^2}) \simeq \mathcal{O}_{\mathbb{P}^2}(3)$ of $\mathbb{P}^2$. The fan for this bundle is generated by the vectors $\{(1,0,1),(0,1,1),(-1,-1,1)\}$ ([38], p. 137). From now on this will be the fan Σ we use in the construction of [6] to give $\mathrm{Fock}^{\Sigma}_{M\oplus N}$. The associated polytope Δ is the triangle with vertices $(-1,2),(-1,-1)$ and $(2,-1)$:



Let x_1, x_2, x_3 be coordinates of $\mathbb{P}^2$ and $(t_1,t_2) = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$ the coordinates of $(\mathbb{C}^*)^2 \subset \mathbb{P}^2$. A basis of the sections $\Gamma(\mathcal{O}_{\mathbb{P}^2}(3))$ is given by the ten monomials of degree 3 in x_1, x_2, x_3. Each map $f : \Delta \to \mathbb{C}$ gives a section of $\mathcal{O}_{\mathbb{P}^2}(3)$ in the following way: In the coordinates of $(\mathbb{C}^*)^2$ we have

$$\begin{aligned} f_{(-1,2)} t_1^{-1} t_2^2 + f_{(-1,1)} t_1^{-1} t_2 + f_{(-1,0)} t_1^{-1} + f_{(-1,-1)} t_1^{-1} t_2^{-1} + f_{(0,1)} t_2 + f_{(0,0)} + f_{(0,-1)} t_2^{-1} \\ + f_{(1,0)} t_1 + f_{(1,-1)} t_1 t_2^{-1} + f_{(2,-1)} t_1^2 t_2^{-1} = 0, \end{aligned} \tag{4.72}$$

which extends to $\mathbb{P}^2$ as

$$\begin{aligned}f_{(-1,2)}x_3^3 + f_{(-1,1)}x_1x_3^2 + f_{(-1,0)}x_1^2x_3 + f_{(-1,-1)}x_1^3 + f_{(0,1)}x_2x_3^2 + f_{(0,0)}x_1x_2x_3 \\ + f_{(0,-1)}x_1^2x_2 + f_{(1,0)}x_2^2x_3 + f_{(1,-1)}x_1x_2^2 + f_{(2,-1)}x_2^3 = 0.\end{aligned} \tag{4.73}$$

This is the input data we need from toric geometry to use Theorem 4.5.1. Calculating this BRST cohomology is still very complicated and does not seem to be feasible if one wants to explicitly calculate the chiral de Rham cohomology of an elliptic curve.

In the next chapter we will be able to give more explicit results for elliptic curves and more generally complex tori using a chiral Dolbeault resolution.

Chapter 5
Chiral de Rham complex of tori and orbifold

This chapter contains the main results of this thesis. Following an idea by Kapustin ([41]) we will define a resolution of the chiral de Rham complex. This will allow us to calculate the chiral de Rham cohomology for any complex torus and compare it with the vertex algebra part of the toroidal superconformal field theory. We will apply the CFT orbifolding ideas of Chapter 3 to this sheaf cohomology vertex algebra. The general orbifold procedure for the chiral de Rham cohomology was first given by Frenkel and Szczesny in [25]. Borisov and Libgober in [8] and Frenkel and Szczesny [25] showed compatibility of the elliptic genus with the orbifold procedure. This can be seen as evidence for compatibility of the chiral de Rham cohomology with the orbifold procedure, which was conjectured in [25]. We will further investigate this conjecture in the case of Kummer surfaces.

5.1. Dolbeault type resolution

Let X be a compact complex manifold of complex dimension N. An elegant way of computing sheaf cohomology of the holomorphic de Rham complex Ω_X of X is by utilizing the Dolbeault resolution. In this section we want to find an analogue for the holomorphic chiral de Rham complex. That means we want to find a resolution of sheaves

$$\Omega_X^{ch,p} \hookrightarrow \mathcal{A}_X^{ch,(p,0)} \longrightarrow \mathcal{A}_X^{ch,(p,1)} \longrightarrow \dots, \tag{5.1}$$

whose conformal weight zero component is the classical resolution

$$\Omega_X^{p} \hookrightarrow \mathcal{A}_X^{p,0} \longrightarrow \mathcal{A}_X^{p,1} \longrightarrow \dots, \tag{5.2}$$

where $\mathcal{A}_X^{p,q}$ is the sheaf of smooth (p,q)-forms on X. Since $\mathcal{A}_X^{0,0} = \mathcal{C}_X^\infty$ a good starting point is to consider the vertex algebra of the smooth ($\mathcal{C}_X^\infty$) de Rham complex Ω_X^∞. This structure was described in detail by Lian and Linshaw in [48] and we will denote it by Ω_X^{LL}.

Let U be a contractible open subset of X. If we apply the construction of [48] to a complex manifold where we choose local holomorphic and antiholomorphic coordinates $b^1,\dots,b^N,\bar b^1,\dots,\bar b^N$ on U and denote one-forms $\Phi^i = db^i, \bar\Phi^i = d\bar b^i$ and dual vectors $a^i = \frac{\partial}{\partial b^i}, \bar a^i = \frac{\partial}{\partial \bar b^i}$ and $\Psi^i = \frac{\partial}{\partial \Phi^i}, \bar\Psi^i = \frac{\partial}{\partial \bar\Phi^i}$ the local sections of Ω_X^{LL} on U are constructed as follows:

Let $C_0 = \mathbb{C}\{a^i, \bar{a}^i \mid 1 \le i \le N\}$ and $C_1 = \mathbb{C}\left\{\Psi^i, \bar{\Psi}^i \mid 1 \le i \le N\right\}$ and $C = C_0 \oplus C_1$. The abelian Lie super algebra C acts on the smooth forms $\Omega_X^\infty(U)$ and we can form the semi-direct product Lie algebra $\Lambda(U) := C \rhd \Omega_X^\infty(U)$ with Lie bracket

$$[(x_C, x_\Omega), (y_C, y_\Omega)] = (0, x_C(y_\Omega) - y_C(x_\Omega)). \tag{5.3}$$

Let $\Lambda(U)[t, t^{-1}]$ be the loop algebra of $\Lambda(U)$ with bracket

$$[ut^n, vt^m] = [u, v]t^{n+m}. \tag{5.4}$$

Let $\mathcal{V}(U)$ be the vertex algebra generated by $f_i = ft^i$ for $f \in \Omega_X^\infty(U), i \ge 0$ and $\beta_j = \beta t^j$ for $\beta \in C, j > 0$. For a function $f \in \mathcal{C}_X^\infty$ and a vector field $\beta \in C_0$ we have

$$\beta(z)f(w) \sim \frac{\beta(f)(w)}{z-w}, \tag{5.5}$$

which clarifies the vertex algebra structure.

The vertex algebra $\mathcal{V}(U)$ is too large, we need to quotion by the ideal $\mathcal{I}(U)$ defined by the relations

$$\begin{aligned}\frac{d}{dz}f(z) = \sum_{i=1}^n \frac{d}{dz}b^i(z)\frac{\partial f}{\partial b^i}(z) + \sum_{i=1}^n \frac{d}{dz}\bar{b}^i(z)\frac{\partial f}{\partial \bar{b}^i}(z)\\ + \sum_{i=1}^n \frac{d}{dz}\Phi^i(z)\frac{\partial f}{\partial \Phi^i}(z) + \sum_{i=1}^n \frac{d}{dz}\bar{\Phi}^i(z)\frac{\partial f}{\partial \bar{\Phi}^i}(z),\end{aligned} \tag{5.6}$$

$$(fg)(z) = f(z)g(z), \tag{5.7}$$

$$1(z) = \mathrm{id}, \tag{5.8}$$

for $f, g \in \Omega_X^\infty(U)$ (cf. [48], 2.5). We define

$$\Omega_X^{LL}(U) := \mathcal{V}(U)/\mathcal{I}(U). \tag{5.9}$$

The first relation (5.6) of $\mathcal{I}(U)$ ensures that the Taylor formula (4.28) for $f(z)$ remains true and that any f_k in $\Omega_X^{LL}(U)$ is in fact a product of a polynomial in $b_1^i, \dots, b_k^i, \bar{b}_1^i, \dots, \bar{b}_k^i$ for $i = 1, \dots, N$ and a $\mathcal{C}_X^\infty$ function on U. This in turn will allow us to again find an appropriate filtration to deduce results from the world of $\mathcal{C}_X^\infty$-modules.

The transformation laws follow from the geometric transformations just as for the chiral de Rham complex.

Theorem 5.1.1 ([48], Theorem 2.31). *For every compact smooth manifold X, we have a sheaf Ω_X^{LL} of vertex algebras which contains the smooth de Rham complex Ω_X^∞ as a subsheaf of vector spaces.*

The sheaf Ω_X^{LL} has two copies of the $N = 2$ topological superconformal algebra as global sections, one on the holomorphic and one on the antiholomorphic side. The holomorphic side is defined via formulas (4.9) to (4.12), the antiholomorphic one accordingly. We define

$$\Omega_X^{LL,(p,q)} := \left\{\omega \in \Omega_X^{LL} \mid J_0\omega = p\omega, \bar{J}_0\omega = q\omega\right\}. \tag{5.10}$$

The following consideration may be clear to the experts but we could not find it in the literature: We can define the analogue of a $\bar{\partial}$-operator as $\bar{\partial}^{ch} = \bar{Q}_0 = \sum_{i=1}^{N} \sum_{m \in \mathbb{Z}} \bar{a}_m \bar{\Phi}^i_{-m}$. This is a differential, since it is an odd operator and all its modes anti-commute. Also, since it commutes with everything in Ω^{ch}_X, we have $\Omega^{ch}_X \subseteq \ker \bar{\partial}^{ch}$. However, there is a problem with this choice: For $\bar{a}_1 \in \Omega^{LL,(0,0)}_X$ we have $\bar{\partial}^{ch} \bar{a}_1 = 0$ so that $\Omega^{ch}_X \subsetneq \ker \bar{\partial}^{ch}$! In [41] Kapustin suggests a resolution where we associate constant fields to the antiholomorphic coodinates and differential forms, while maintaining the known fields for the holomorphic side. But a recipe like this only works in the realm of real analytic differential forms, which is in general not sufficient for the sheaves to have favorable properties. For example, the sheafs $\mathcal{A}^{w,(p,q)}_X$ of real-analytic forms are not fine as we do not have partition of unity for real analytic functions. In fact, they are in general not even acyclic! It does, however, work for complex tori since here all $\mathcal{A}^{w,(p,q)}_X$ are at least acyclic.

Our solution to this conundrum is to define a sheaf that sits between the two constructions. Specifically, we modify the construction of Lian and Linshaw in such a way, that we do not include the fields for the antiholomorphic dual vectors $\bar{a}^i$ and $\bar{\Psi}^i$. In other words, we replace C by $C' := \mathbb{C}\{ a^i, \Psi^i \mid 1 \le i \le N \}$. Since we only allow holomorphic transition functions this is consistent and allows to define:

Definition 5.1.2. We denote the sheaf constructed above by $\mathcal{A}^{ch}_X$ and call it the *chiral Dolbeault sheaf.*

Remark 5.1.3. Note that the thus constructed sheaf is a subsheaf of the smooth chiral de Rham complex: $\mathcal{A}^{ch}_X \subset \Omega^{LL}_X$.

Notice that on $\mathcal{A}^{ch}_X$ the holomorphic $N = 2$ topological superconformal algebra is still present, while the antiholomorphic side cannot be defined as fields any more. However, while the grading operator for the antiholomorphic side $\bar{L}^{top}_0 = \sum_i \sum_n n : \bar{a}^i_{-n} \bar{b}^i_n :$ and $\bar{J}_0 = \sum_i \sum_n n : \bar{\Psi}^i_{-n} \bar{\Phi}^i_n :$ and the differential $\bar{\partial}^{ch}$ do not come from fields any more, they are well-defined operators on the sheaf $\mathcal{A}^{ch}_X$. The classical Dolbeault complex is the zero eigenspace of the operator $L^{top}_0 + \bar{L}^{top}_0$. We define the following subsheaves:

$$\mathcal{A}^{ch}_{X,k} := \left\{ \omega \in \mathcal{A}^{ch}_X \mid L^{top}_0 \omega = k\omega \right\}, \tag{5.11}$$

$$\mathcal{A}^{ch,(p,q)}_X := \left\{ \omega \in \mathcal{A}^{ch}_X \mid J_0 \omega = p\omega, \bar{J}_0 \omega = q\omega \right\}. \tag{5.12}$$

Remark 5.1.4. While in the toroidal CFT/SCFT we use the formal variable $\bar{z}$ for the antiholomorphic geometric objects, in the smooth chiral de Rham complex Ω^{LL}_X of Lian and Linshaw the formal variable z is used both on the holomorphic and on the antiholomorphic side.

Let U be an open subset of X. Similarly to the holomorphic chiral de Rham case we have a $\mathcal{C}^\infty_X$-action on $\mathcal{A}^{ch}_X$

$$\begin{array}{ccccc} \mathcal{C}^\infty_X(U) & \times & \mathcal{A}^{ch}_X(U) & \longrightarrow & \mathcal{A}^{ch}_X(U) \\ f & & \varphi & \longmapsto & [f(z)]_0 \cdot \varphi \end{array}, \tag{5.13}$$

which is not associative and differs from usual multiplication of functions by additional terms. So just as Ω^{ch}_X is not an $\mathcal{O}_X$-module, the sheaf $\mathcal{A}^{ch}_X$ is not a $\mathcal{C}^\infty_X$-module. However, there is an increasing exhaustive filtration F^i just as the one we defined in Section 4.2 (4.52) for Ω_X^{ch}, where we treat b_n^j and $\bar{b}_n^j$ as well as Φ_n^j and $\bar{\Phi}_n^j$ as being of same order: $\mathcal{A}_X^{ch}(U)$ is a free $\mathcal{C}_X^\infty$-module with a base consisting of monomials in letters $b_n^i, \bar{b}_n^i, \Phi_n^i, \bar{\Phi}_n^i, a_m^i, \Psi_m^i$ for $i = 1, \dots, N$ and $n \geq 0, m > 0$. Define a partial ordering on this base by

$$a > \left\{ \Phi, \bar{\Phi} \right\}, \quad a > \Psi, \quad a > \left\{ b, \bar{b} \right\}, \quad \Psi > \left\{ \Phi, \bar{\Phi} \right\}, \quad \Psi > \left\{ b, \bar{b} \right\}, \quad \left\{ \Phi, \bar{\Phi} \right\} > \left\{ b, \bar{b} \right\} \tag{5.14}$$

and

$$x_n^i > x_m^j \quad \text{if} \quad n > m \quad \text{where} \quad x \quad \text{is} \quad a, b, \bar{b}, \Phi, \bar{\Phi} \quad \text{or} \quad \Psi \tag{5.15}$$

and extend this order lexicographically to the whole set. This determines an increasing exhaustive filtration on the sheaves of fixed conformal weight k and $(J_0, \bar{J}_0)$-charge (p,q):

$$F^0\mathcal{A}_{X,k}^{ch,(p,q)} \subset F^1\mathcal{A}_{X,k}^{ch,(p,q)} \subset F^2\mathcal{A}_{X,k}^{ch,(p,q)} \subset \dots. \tag{5.16}$$

For example, $F^0\mathcal{A}_{X,0}^{ch,(0,0)} = \mathcal{C}_X^\infty$ and $F^0\mathcal{A}_{X,1}^{ch,(0,0)} = \bigoplus_{i=1}^N \mathcal{C}_X^\infty\, b_1^i \oplus \bigoplus_{i=1}^N \mathcal{C}_X^\infty\, \bar{b}_1^i$. The graded objects of this filtration F^i are $\mathcal{C}_X^\infty$-modules since additional terms that prevent associativity lie in lower parts of the filtration (see Section 4.2).

We will now show that our construction does in fact give the desired resolution of Ω_X^{ch}!

Proposition 5.1.5. *Let X be a compact complex manifold. For all subsheaves $\mathcal{A}_{X,k}^{ch,(p,q)}$ of the chiral Dolbeault sheaf $\mathcal{A}_X^{ch}$ the higher cohomology groups vanish:*

$$\mathrm{H}^i\left(X, \mathcal{A}_{X,k}^{ch,(p,q)}\right) = 0 \quad \textit{for} \quad i > 0. \tag{5.17}$$

Therefore the sheaves $\mathcal{A}_{X,k}^{ch,(p,q)}$ are acyclic.

Proof. Fix an eigenvalue k of L_0^{top}. We want to use partition of unity of $\mathcal{C}_X^\infty$ to show the vanishing of higher cohomology groups. While $\mathcal{A}_X^{ch,(p,q)}$ is not a $\mathcal{C}_X^\infty$-module, it carries a non-associative action of $\mathcal{C}_X^\infty$. We cannot use this action to extend the partition of unity, since apart from usual function multiplication it involves additional terms. However, we can use the graded objects $Gr_n = F^n\mathcal{A}_{X,k}^{ch,(p,q)}/F^{n-1}\mathcal{A}_{X,k}^{ch,(p,q)}$ which are $\mathcal{C}_X^\infty$-modules and thus fine. Their higher cohomology groups vanish. We do an induction on n. The short exact sequence

$$0 \longrightarrow F^n\mathcal{A}_{X,k}^{ch,(p,q)} \longrightarrow F^{n+1}\mathcal{A}_{X,k}^{ch,(p,q)} \longrightarrow Gr_n \longrightarrow 0 \tag{5.18}$$

leads to a long exact sequence of cohomology groups where $\mathrm{H}^i\left(X, F^n\mathcal{A}_{X,k}^{ch,(p,q)}\right) = 0$ and $\mathrm{H}^i\left(X, Gr_n\right) = 0$ imply vanishing of $\mathrm{H}^i\left(X, F^{n+1}\mathcal{A}_{X,k}^{ch,(p,q)}\right)$ for $i > 0$. Since the filtration is exhaustive this completes the proof. □

Again, this fits into the general concept of *R-loop-coherent sheaves* introduced by Borisov in [6], where by using filtrations whose graded objects are coherent many statements can be deduced from the coherent world.

Proposition 5.1.6. *For any J_0-charge p of the holomorphic chiral de Rham complex Ω_X^{ch} the complex of sheaves*

$$\Omega_X^{ch,p} \hookrightarrow \mathcal{A}_X^{ch,(p,0)} \xrightarrow{\bar\partial^{ch}} \mathcal{A}_X^{ch,(p,1)} \xrightarrow{\bar\partial^{ch}} \dots \tag{5.19}$$

is exact.

Proof. We only need to check this locally for contractible $U \subset X$:

$$\Omega_X^{ch,p}(U) \hookrightarrow \mathcal{A}_X^{ch,(p,0)}(U) \longrightarrow \mathcal{A}_X^{ch,(p,1)}(U) \longrightarrow \dots.$$

All these subsheaves are also graded by conformal weight, i.e. the eigenvalues of operator $L_0^{top} + \bar L_0^{top}$. In weight zero the above resolution reduces to the classical case where by the $\bar\partial$-Lemma we get exactness. We proceed in two steps:

1. We consider the first term in the sequence. Write an element of $\mathcal{A}_X^{ch,(p,0)}(U)$ as $f\cdot\lambda(\bar b_m^i)\cdot\mu(b_m^i, a_m^i, \Phi_n^i, \Psi_m^i)\cdot 1$ with $f \in \mathcal{C}_X^\infty(U)$, λ and μ polynomials and $i = 1,\dots,N$, $n \geq 0$, $m > 0$. Without loss of generality assume that $f, \lambda, \mu \neq 0$. Then

$$\bar\partial^{ch}(f\cdot\lambda\cdot\mu\cdot 1) = [\bar\partial^{ch}, f]\cdot\lambda\cdot\mu\cdot 1 + f[\bar\partial^{ch},\lambda]\cdot\mu\cdot 1 \tag{5.20}$$

$$= \sum_i [\bar a_0^i, f]\cdot\bar\Phi_0^i\cdot\lambda\cdot\mu\cdot 1 + f\cdot\sum_i\sum_{n>0}[\bar a_{-n}^i,\lambda]\cdot\bar\Phi_n^i\cdot\mu\cdot 1. \tag{5.21}$$

   Since $\bar\Phi_n^i$ are basis elements this vanishes if and only if $[\bar a_0^i, f] = 0$ and $[\bar a_{-n}^i, \lambda] = \frac{\partial}{\partial \bar b_n^i}\lambda = 0$ for all i and n. This implies that f is holomorphic and λ is constant and thus $f\cdot\lambda\cdot\mu\cdot 1 \in \Omega_X^{ch,d}(U)$.

2. We define the operator $\bar G_0 := \sum_i\sum_n \bar\Psi_{-n}^i n\bar b_n^i$ just as in the holomorphic case. While it does not come from a field on $\mathcal{A}_X^{ch}$, it is globally defined just as $\bar L_0^{top}$ and we have $[\bar G_0, \bar\partial^{ch}] = \bar L_0^{top}$. Now let $v \in \mathcal{A}_X^{ch,(p,q)}(U)$ with $L_0^{top}v = k\cdot v \neq 0$ and $\bar\partial^{ch}v = 0$. Then because of

$$\bar L_0^{top}v = \bar G_0\bar\partial^{ch}v + \bar\partial^{ch}\bar G_0 v \tag{5.22}$$

   we find an inverse image $\frac{1}{k}\bar G_0 v$ for v under $\bar\partial^{ch}$.
   If $\bar L_0^{top}v = 0$ then we can write $v = f\cdot\lambda(\bar\Phi_0^i)\cdot\mu(b_m^i, a_m^i, \Phi_n^i, \Psi_m^i)\cdot 1$ with $f \in \mathcal{C}_X^\infty(U)$, λ and μ polynomials and $i = 1,\dots,N$, $n \geq 0$, $m > 0$. The differential $\bar\partial^{ch}$ reduces to the classical $\bar\partial$ and commutes with everything but $f\cdot\lambda(\bar\Phi_0^i)$ and we are again in the realm of the classical $\bar\partial$-Lemma.

□

Using both these propositions we arrive at

Theorem 5.1.7. *The acyclic chiral Dolbeault sheaf $\mathcal{A}_X^{ch}$ is a resolution of the holomorphic chiral de Rham complex Ω_X^{ch}. The holomorphic chiral de Rham cohomology is*

$$\begin{aligned}&\mathrm{H}^q(X, \Omega_X^{ch,p}) \cong \\ &\quad \mathrm{H}_{ch}^{p,q} := \ker\Big(\mathcal{A}_X^{ch,(p,q)}(X) \to \mathcal{A}_X^{ch,(p,q+1)}(X)\Big)\Big/\operatorname{im}\Big(\mathcal{A}_X^{ch,(p,q-1)}(X) \to \mathcal{A}_X^{ch,(p,q)}(X)\Big).\end{aligned} \tag{5.23}$$

The total cohomology of the holomorphic chiral de Rham complex is

$$\mathrm{H}^*(X,\Omega^{ch}) = \bigoplus_{p,q,i} \mathrm{H}^q(X,\Omega_i^{ch,p}) \cong \bigoplus_{p,q,i} \mathrm{H}^{p,q}_{ch,i}\,. \tag{5.24}$$

The vertex algebra structure on $\mathcal{A}_X^{ch}$ induces a vertex algebra structure on the quotients $\mathrm{H}^{p,q}_{ch}$. The vertex algebra structure is compatible with this quotient and all these statements hold at the level of vertex algebras (cf. [6], Proposition 3.8).

Remark 5.1.8. Note that q indicates the antiholomorphic form degree, which sheds more light on the shift in fermionic charge $(-1)^{J_0+q}$ mentioned in Theorem 4.3.1. In this differential forms interpretation the shift in fermionic charge can be made clear. A vector $v \in \mathrm{H}^q(T,\Omega_X^{ch})$ has a contribution by an antiholomorphic q-form which clearly changes the $\mathbb{Z}_2$-grading.

The proof of Proposition 5.1.6 also implies:

Corollary 5.1.9. *Every element of the holomorphic chiral de Rham cohomology has a representative in $\mathcal{A}_X^{ch}(X)$ with $\bar{L}_0^{top}$-weight zero.*

Cheung also defined a resolution he calls chiral Dolbeault in [12] using the different approach of vertex algebroids ([9], [34]).

5.2. Torus

We now want to use our chiral Dolbeault resolution to determine the vertex algebra structure for the torus.

We begin with a complex one-dimensional torus $X = T^2 = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Let x be the coordinate of $\mathbb{C}$. Then the coordinate functions of the torus transform as $x \mapsto x+1$ resp. $x \mapsto x+\tau$, i.e. as translations by constants. This implies that all elements of the chiral Dolbeault sheaf except for b_0 and $\bar{b}_0$ transform trivially. So the global sections of the chiral Dolbeault sheaf are given by the ones of the classical Dolbeault sheaf and elements involving $b_m, a_m, \Phi_n, \Psi_m, \bar{\Phi}_0$ with $n \geq 0$, $m > 0$. For the torus we find

$$\mathrm{H}^{p,q}_{ch} = \mathrm{H}^{p,q} \otimes \mathbb{C}[b_m, a_m, \Phi_m, \Psi_m].$$

As a vertex algebra $\mathrm{H}^*(T^2,\Omega^{ch}_{T^2})$ is generated by the fields

$$\partial b(z), a(z), \Phi(z), \Psi(z), \bar{\Phi}(z).$$

In general and in more detail we find:

Theorem 5.2.1. *The holomorphic chiral de Rham cohomology vertex algebra $\mathrm{H}^*(T^{2N},\Omega^{ch}_{T^{2N}})$ of a complex N-dimensional torus T^{2N} is generated by the fields*

$$\partial b^i(z), \quad a^i(z), \quad \Phi^i(z), \quad \Psi^i(z), \quad \bar{\Phi}^i(z) = \bar{\Phi}^i_0$$

for $i = 1,\dots,N$. The non-trivial OPEs are

$$a^i(z)\partial b^j(w) \sim \frac{\delta_{i,j}}{(z-w)^2} \quad \textit{and} \quad \Psi^i(z)\Phi^j(w) \sim \frac{\delta_{i,j}}{z-w}. \tag{5.25}$$

Proof. The lattice only enters the transformation laws for functions in $\mathcal{C}^\infty_{T^{2N}}$. All other sections of $\mathcal{A}^{ch}_{T^{2N}}$ transform trivially and are globally defined. We have

$$\mathcal{A}^{ch}(T^{2N}) = \mathcal{A}(T^{2N}) \otimes \mathbb{C}[b^i_m, \bar{b}^i_m, a^i_m, \Phi^i_m, \Psi^i_m, \bar{\Phi}^i_m], \tag{5.26}$$

with $m > 0$. Write $\omega \in \mathcal{A}^{ch,(p,q)}(T^{2N})$ as $\omega = \omega' \mu(\bar{b}^i_m)\lambda(b^i_m, a^i_m, \Phi^i_m, \bar{\Phi}^i_m, \Psi^i_m)$ with $\omega' \in \mathcal{A}(T^{2N})$ and μ, λ are polynomials. Without loss of generality assume that $\omega \neq 0$. Then

$$\begin{aligned} \bar{\partial}^{ch}(\omega'\mu(\bar{b}^i_m)\lambda) &= [\bar{\partial}, \omega']\mu(\bar{b}^i_m)\lambda + [\bar{\partial}^{ch}, \mu(\bar{b}^i_m)]\omega'\lambda & (5.27)\\ &= \lambda([\bar{\partial}, \omega']\mu(\bar{b}^i_m) + [\bar{\partial}^{ch}, \mu(\bar{b}^i_m)]\omega'). & (5.28) \end{aligned}$$

This vanishes if and only if both summands vanish since $[\bar{\partial}^{ch}, \mu(\bar{b}^i_m)]$ creates the basis element $\bar{\Phi}^i_m$, which cannot appear in $[\bar{\partial}, \omega']\mu(\bar{b}^i_m)$. This happens for $\omega' \in \ker \mathcal{A}(T^{2N})$ and $\mu(\bar{b}^i_n) = 0$. Also, the second term in (5.27) vanishes if $[\bar{\partial}^{ch}, \mu(\bar{b}^i_m)]$ creates a $\bar{\Phi}^i_m$ that is already present in λ.

We now have to quotient out the image of $\bar{\partial}^{ch}$. Any element involving $\bar{\Phi}^i_m$ is in the image because of $\bar{\partial}^{ch}\bar{b}^i_m = \bar{\Phi}^i_m$ and gets quotiented out. Note that this is not true for $\bar{\Phi}^i_0$, since $\bar{b}^i_0$ is not a global section. Furthermore we have the image of the classical $\bar{\partial}$. This implies that there are no $\bar{b}^i_n$ and $\bar{\Phi}^i_m$ in $\mathrm{H}^*(T^{2N}, \Omega^{ch}_{T^{2N}})$. We find that $\mathrm{H}^*(T^{2N}, \Omega^{ch}_{T^{2N}})$ consists of classical Dolbeault cohomology of the torus and sections b_m, a_m, Φ_m, Ψ_m. Since for $m > 0$ the sections $\bar{\Phi}^i_m$ are quotiented out, the field $\bar{\Phi}^i_0(z) = \bar{\Phi}^i_0$ is constant. □

Corollary 5.2.2. *The sheaf cohomology vertex algebra* $\mathrm{H}^*(T^{2N}, \Omega^{ch}_{T^{2N}})$ *is independent of the choice of complex structure on* T^{2N}.

Proof. The complex structure of the torus $T^{2N} = \mathbb{C}^N/\Lambda$ is encoded in the lattice Λ. The only sections of $\mathcal{A}^{ch}_{T^{2N}}$ whose transformation laws depend on the specific lattice are b^i_0. Since these sections do not appear in $\mathrm{H}^*(T^{2N}, \Omega^{ch}_{T^{2N}})$ it is independent of the choice of lattice, i.e. choice of complex structure. □

This shows that we really get the same vertex algebra for all complex tori of a fixed dimension. This is in contrast to the holomorphic chiral algebra (Definition 1.3.4) of the toroidal CFT which changes depending on the complex structure and Kähler structure. At special points in the moduli space the chiral algebra is enlarged (see Subsection 1.3.1). Since the holomorphic chiral de Rham complex only depends on the complex structure and an enlarged chiral algebra depends also on a condition on the Kähler structure as well, this may not come as a surprise. We do, however, find part of the generic chiral algebra that all toroidal theories share.

Lemma 5.2.3. *The chiral de Rahm complex elliptic genus of a complex torus vanishes.*

Proof. The elliptic genus is given by

$$\begin{aligned} \mathcal{E}(\tau, z) &:= \mathrm{tr}_{\mathrm{H}^*(T^{2N}, \Omega^{ch}_{T^{2N}})}\, y^{-\frac{c}{6}}(-1)^{J_0 + \bar{J}_0} q^{L_0^{top}} y^{J_0} & (5.29)\\ &= y^{-\frac{c}{6}} \cdot 0 \cdot \left(\prod_{k=1}^{\infty} \frac{1}{1-q^k}\right)^{2N} \prod_{k=0}^{\infty}\left(1 - q^k y^1\right)^N \prod_{k=1}^{\infty}\left(1 - q^k y^{-1}\right)^N = 0 & (5.30) \end{aligned}$$

due to the zero contribution of the constant fields $\bar{\Phi}^i(z)$. □

This is in accordance with Theorem 4.3.1 and a well-known result for the torus.

We have seen that the cohomology of the holomorphic chiral de Rham complex of complex tori does not depend on the specific complex structure. If we want to see the complex structure and thus properties like complex multiplication we need to include additional data. Since $\mathrm{H}^*(T^{2N}, \Omega_{T^{2N}}) \subset \mathrm{H}^*(T^{2N}, \Omega^{ch}_{T^{2N}})$ and the complex structure of a torus can be encoded by the relative position of $\mathrm{H}^*(T^{2N}, \mathbb{Z}) \subset \mathrm{H}^*(T^{2N}, \Omega_{T^{2N}})$ a potential integral structure for $\mathrm{H}^*(T^{2N}, \Omega^{ch}_{T^{2N}})$ seems like a natural candidate.

A different point of view is taken by Ben-Zvi, Heluani and Szczesny in [5] where they start with all of Ω^{LL}_X instead of the holomorphic chiral de Rham complex and then use a Riemannian metric to define sections of Ω^{LL} that generate a $N = 1$ superconformal algebra. If the metric is Ricci-flat and Kähler this algebra is extended to $N = 2$.

5.2.1. Čech cohomology

In complex dimension one we want to compare the vertex algebra structure coming from the chiral Dolbeault resolution to that of the Čech cohomology specified by Borisov.

Let $T^2 = \mathbb{C}/\Lambda$ be a complex one-dimensional torus. We can cover this torus by dividing the fundamental domain into three by three open sets U_i, $i = 1, \dots, 9$. These nine open sets give a good cover of the torus. Transition functions are translations by elements of the lattice Λ.

Because of $\mathrm{H}^*(T^2, \Omega_{T^2}) \subset \mathrm{H}^*(T^2, \Omega^{ch}_{T^2})$ we can investigate the vertex operators associated to the classical forms dz, $d\bar{z}$ and $dz \wedge d\bar{z}$. The Hodge diamond of T^2 is

$$\begin{array}{ccc} & \mathrm{H}^0(T^2, \Omega^0_{T^2}) & \\ \mathrm{H}^0(T^2, \Omega^1_{T^2}) & & \mathrm{H}^1(T^2, \Omega^0_{T^2}). \\ & \mathrm{H}^1(T^2, \Omega^1_{T^2}) & \end{array} \tag{5.31}$$

In order to find the field associated to $d\bar{z} \in \mathrm{H}^1(T^2, \Omega^0_{T^2})$ we need its representation in Čech cohomology. For this consider the Čech-Dolbeault double complex:

$$\begin{array}{ccccccccc} & & \check{C}^0(T^2, \Omega^0_{T^2}) & \to & \check{C}^1(T^2, \Omega^0_{T^2}) & \to & \check{C}^2(T^2, \Omega^0_{T^2}) & \to & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{A}^{0,0}_{T^2}(T^2) & \to & \check{C}^0(T^2, \mathcal{A}^{0,0}_{T^2}) & \to & \check{C}^1(T^2, \mathcal{A}^{0,0}_{T^2}) & \to & \check{C}^2(T^2, \mathcal{A}^{0,0}_{T^2}) & \to & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & . \\ \mathcal{A}^{0,1}_{T^2}(T^2) & \to & \check{C}^0(T^2, \mathcal{A}^{0,1}_{T^2}) & \to & \check{C}^1(T^2, \mathcal{A}^{0,1}_{T^2}) & \to & \check{C}^2(T^2, \mathcal{A}^{0,1}_{T^2}) & \to & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & & \dots & & \dots & & \dots & & \end{array} \tag{5.32}$$

We show the explicit form of the vertex operator associated to $d\bar{z}$, which cannot be found in the literature:

Proposition 5.2.4. *In Čech cohomology the field associated to $d\bar{z} = \bar{\Phi}_0$ is given by*

$$\left(\left(\widehat{d\bar{z}}\right)_n x\right)\left(U_{i_0,\dots,i_{1+q}}\right) = (\bar{z}_{i_1} - \bar{z}_{i_0})_n x_{i_1,\dots,i_{1+q}} = \begin{cases} (\bar{z}_{i_1} - \bar{z}_{i_0}) x_{i_1,\dots,i_{1+q}} & n = 0 \\ 0 & n \neq 0 \end{cases}. \tag{5.33}$$

Here x is any element of the cohomology vertex algebra and we have specified the field associated to $d\bar{z}$ by giving the n-product with x and $\bar{z}_{i_1} - \bar{z}_{i_0} \in \Gamma$.

Proof. We find the image of $d\bar{z}$ in Čech cohomology by following the diagram from $\mathcal{A}^{0,1}_{T^2}(T^2)$ to $\check{C}^1_{T^2}(T^2,\Omega^0)$, where we denote the corresponding element by $\widehat{d\bar{z}}$. On intersections U_{ij} its value is $\widehat{d\bar{z}}(U_{ij}) = \bar{z}_j - \bar{z}_i \in \Lambda$. Let $x \in \check{\mathrm{H}}^q(T^2,\Omega^{ch}_{T^2})$ and $n \in \mathbb{Z}$. We want to describe the vertex operator associated to $\widehat{d\bar{z}}$ by determining $\big(\widehat{d\bar{z}}\big)_n x \in \check{\mathrm{H}}^{1+q}(T^2,\Omega^{ch}_{T^2})$. By definition (Formula (4.58)) this is

$$\Big(\big(\widehat{d\bar{z}}\big)_n x\Big)\big(U_{i_0,\dots,i_{1+q}}\big) = (\bar{z}_{i_1} - \bar{z}_{i_0})_n x_{i_1,\dots,i_{1+q}}. \tag{5.34}$$

Since $\bar{z}_{i_1} - \bar{z}_{i_0} \in \Lambda$ is constant this is only nonzero for $n = 0$ and we deduce that the vertex operator of $\widehat{d\bar{z}}$ is a constant field. Similarly, we find for $dz \wedge d\bar{z}$:

$$\Big(\big(\widehat{dz \wedge d\bar{z}}\big)_n x\Big)\big(U_{i_0,\dots,i_{1+q}}\big) = ((\bar{z}_{i_1} - \bar{z}_{i_0})dz_{i_0})_n x_{i_1,\dots,i_{1+q}}. \tag{5.35}$$

So this is the product of the field associated to dz and a constant field. □

Since in the case of a complex torus T the transition functions are simply lattice translations we find that each conformal weight component $\Omega^{ch}_{T,n}$ is a free $\mathcal{O}_T$-module. We arrive at the same vertex algebra as in the previous subsection. Yet the way the fields are collected using the Dolbeault style resolution is maybe more suggestive.

Note that while the lattice structure enters here as the value of the Čech-cocycle $\widehat{d\bar{z}}$ the whole structure is still independent of the specific complex torus just as $\check{\mathrm{H}}^*(T,\Omega_T)$ is independent of the specific choice.

5.2.2. Identification with SCFT

We give a specific identification of fields in the holomorphic chiral de Rham complex and in the SCFT for the torus. Note that the fields differ due to taking cohomology. For example, the antiholomorphic fermion $\bar{\Phi}_0$ appears as a constant field on the holomorphic chiral de Rham complex side, as $\bar{\Phi}(z) = \sum_n \bar{\Phi}_n z^n$ on the chiral Dolbeault resolution and as $\bar{\Psi}^+(\bar{z}) = \sum_n \bar{\Psi}^+_n \bar{z}^n$ on the SCFT side.

Proposition 5.2.5. *The generating fields of* $\mathrm{H}^*(T^{2N},\Omega^{ch}_{T^{2N}})$ *can be identified with those of the SCFT in the following way:*

L_0^{top}	$\bar{L}_0^{top}$	J_0	$\bar{J}_0$	$\mathrm{H}^*(T^2,\Omega^{ch}_{T^2})$	$\mathcal{A}^{ch}_{T^2}(T^2)$	$\mathbb{H}_{NS}(T^2)$
0	0	0	0	$\mathbb{1}(z)$	$\mathbb{1}(z)$	$\mathbb{1}(z)$
1	0	0	0	$\partial b(z), a(z)$	$\partial b(z), a(z)$	$j_-(z), j_+(z)$
0	0	1	0	$\Phi(z)$	$\Phi(z)$	$\Psi^+(z)$
1	0	−1	0	$\Psi(z)$	$\Psi(z)$	$\Psi^-(z)$
0	0	0	1	$\bar{\Phi}(z) = \bar{\Phi}_0$	$\bar{\Phi}(z)$	$\bar{\Psi}^+(\bar{z})$

Notice that we have only identified the generating vectors. The corresponding fields differ since we took cohomology.

Proof. This follows directly from comparing eigenspaces of the four operators $L_0^{top}, \bar{L}_0^{top}, J_0, \bar{J}_0$ and using $L_0 = L_0^{top} + \frac{1}{2}J_0$. □

Remark 5.2.6. Because of the topological twist we are working with L_0^{top} rather than $L_0 = L_0^{top} + \frac{1}{2}J_0$. Thus the holomorphic chiral de Rham cohomology vertex algebra chooses other odd fields than the holomorphic chiral algebra of CFT (Definition 1.3.4).

5.3. $\mathbb{Z}_2$-orbifold of tori

Let $T^{2N} = \mathbb{C}^N/\Lambda$ be a complex torus. We define a $\mathbb{Z}_2$-action on $\mathbb{C}^N$

$$\begin{array}{cccc} \kappa \ : & \mathbb{C}^N & \longrightarrow & \mathbb{C}^N \\ & (x^1,\dots,x^N) & \longmapsto & (-x^1,\dots,-x^N). \end{array} \tag{5.36}$$

Because of $\kappa(\Lambda) \subset \Lambda$ this action descends to any torus. The group generated by κ is isomorphic to $\mathbb{Z}_2$. One can quotient out the action of $\mathbb{Z}_2$ on T^{2N}, which for $N > 1$ yields a singular space $T^{2N}/\mathbb{Z}_2$. Blowing up the singularities we arrive at a manifold again. For $N = 2$ this is a Kummer surface, a special K3-surface. We are interested in the sheaf cohomology of the holomorphic chiral de Rham complex of this Kummer surface. Instead of directly calculating it from the sheaf on the Kummer surface, we will apply the ideas of orbifolding CFTs described in Chapter 3. Here the role of trace over the different sectors will be played by the elliptic genus. In [25] Frenkel and Szczesny first described the orbifold procedure of the holomorphic chiral de Rham complex. Before treating their general results, we will calculate the $\mathbb{Z}_2$-orbifold of the torus holomorphic chiral de Rham complex.

We change the orbifold notation slightly in order to match the chiral de Rham complex setting. Let G be a group acting on a N-dimensional complex manifold X and $g \in G$. Define

$$g\underset{\mathbb{1}}{\square}(\tau,z) := \mathrm{tr}_{\mathrm{H}^*(X,\Omega_X^{ch})}\, g y^{-\frac{N}{2}} (-1)^{J_0+\bar{J}_0} q^{L_0^{top}} y^{J_0}. \tag{5.37}$$

From now on we will restrict to $X = T^4$ and $G = \mathbb{Z}_2$. The first step in the orbifold procedure is projecting to the invariant states via the projector $\frac{1}{2}(\mathbb{1} + \kappa)$. The identity sector is nothing but the elliptic genus of the torus:

$$\mathbb{1}\underset{\mathbb{1}}{\square}(\tau,z) = \mathcal{E}_{T^4}(\tau,z) = 0,$$

by Lemma 5.2.3.

We have to determine the action of κ on the trace. Since all monomials flip their sign under κ we find

$$\kappa\underset{\mathbb{1}}{\square}(\tau,z) = y^{-1}\cdot 4\cdot\left(\prod_{k=1}^{\infty}\frac{1}{1+q^k}\right)^4\cdot\left(\prod_{k=0}^{\infty}\left(1+q^k y^1\right)\right)^2\cdot\left(\prod_{k=1}^{\infty}\left(1+q^k y^{-1}\right)\right)^2 \tag{5.38}$$

$$= y^{-1}\cdot 4\cdot 4\cdot\frac{\eta(\tau)^2}{\vartheta_2(\tau)^2}\cdot q^{\frac{1}{6}}\cdot\frac{\vartheta_2(\tau,z)^2}{\eta(\tau)^2}\cdot q^{-\frac{1}{6}}\cdot y \tag{5.39}$$

$$= 16\cdot\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau)^2}. \tag{5.40}$$

The elliptic genus is a weak Jacobi form, as should be the orbifold elliptic genus. However, a trace only involving the invariant states does not lead to a weak Jacobi form just as in a CFT we did not find a modular invariant partition function. We also have to take the twists in the space direction of the world sheet into account:

$$\mathcal{E}_{T^4}^{\mathbb{Z}_2}(\tau,z) = \frac{1}{2}\cdot\left(\mathbb{1}\underset{\mathbb{1}}{\square}(\tau,z) + \kappa\underset{\mathbb{1}}{\square}(\tau,z)\right) + \frac{1}{2}\cdot\left(\mathbb{1}\underset{\kappa}{\square}(\tau,z) + \kappa\underset{\kappa}{\square}(\tau,z)\right). \tag{5.41}$$

From the modular transformation properties of the elliptic genus as well as the geometric world-sheet interpretation of the orbifold boxes we can deduce the traces of the twisted sector:

$$\mathbb{1}\underset{\kappa}{\square}(\tau,z) = \mathrm{e}^{-2\pi i\frac{z^2}{\tau}}\cdot\kappa\underset{\mathbb{1}}{\square}(-\frac{1}{\tau},\frac{z}{\tau}) \tag{5.42}$$

$$= \mathrm{e}^{-2\pi i\frac{z^2}{\tau}}\cdot\left(16\cdot\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau)^2}\right)(-\frac{1}{\tau},\frac{z}{\tau}) \tag{5.43}$$

$$= 16\cdot\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2}, \tag{5.44}$$

$$\kappa\underset{\kappa}{\square}(\tau,z) = \mathbb{1}\underset{\kappa}{\square}(\tau+1,z) \tag{5.45}$$

$$= \left(16\cdot\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2}\right)(\tau+1,z) \tag{5.46}$$

$$= 16\cdot\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau)^2}. \tag{5.47}$$

We arrive at:

Corollary 5.3.1. *Using the orbifold procedure on the holomorphic chiral de Rham cohomology of T^4 we retrieve the known K3-elliptic genus:*

$$\mathcal{E}_{T^4}^{\mathbb{Z}_2}(\tau,z) := 8\cdot\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau)^2}+\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau)^2}+\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2}\right) = \mathcal{E}_{K3}(\tau,z). \tag{5.48}$$

5.3.1. Twisted sector space of states

We can use the trace in the twisted sector to deduce the form of its space of states $\mathbb{H}_\kappa$.

$$\mathbb{1}\underset{\kappa}{\square}(\tau,z) = 16\cdot\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2} \tag{5.49}$$

$$= y^{-1}\cdot y\cdot 16\cdot\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2} \tag{5.50}$$

$$= y^{-1}\cdot 16\cdot y\cdot\left(\prod_{k=1}^{\infty}\frac{1}{1-q^{k-\frac{1}{2}}}\right)^4\cdot\left(\prod_{k=1}^{\infty}\left(1-q^{k-\frac{1}{2}}y^1\right)\right)^2\cdot\left(\prod_{k=1}^{\infty}\left(1-q^{k-\frac{1}{2}}y^{-1}\right)\right)^2 \tag{5.51}$$

$$= y^{-1}\cdot\mathrm{tr}_{\mathrm{H}_\kappa}\,(-1)^{J_0+\bar{J}_0}\,q^{L_0^{top}}y^{J_0}, \tag{5.52}$$

suggesting that the twisted sector is build from 16 isomorphic vector spaces build by the action of the twisted fields

$$a^{\kappa,i}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} a_n^i z^{n-1}, \tag{5.53}$$

$$(\partial b)^{\kappa,i}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} n b_n^i z^{n-1}, \tag{5.54}$$

$$\Phi^{\kappa,i}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \Phi_n^i z^{n}, \tag{5.55}$$

$$\Psi^{\kappa,i}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \Psi_n^i z^{n-1} \tag{5.56}$$

on a lowest weight vector with L_0^{top}-conformal weight 0 and J_0-charge 1, where $i = 1,2$ account for the two complex dimensions. This 16 corresponds to the 16 singularities of the geometric orbifold. The modes satisfy the same commutator relations as in the untwisted sector:

$$[a_n^i, b_m^j] = \delta_{i,j}\delta_{n,-m}, \tag{5.57}$$

$$\{\Phi_n^i, \Psi_m^j\} = \delta_{i,j}\delta_{n,-m}. \tag{5.58}$$

Formally we can calculate their OPEs

$$a^{\kappa,i}(z)(\partial b)^{\kappa,j}(w) \sim \frac{\frac{1}{2}\delta_{i,j}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right)}{(z-w)^2}, \tag{5.59}$$

$$\Phi^{\kappa,i}(z)\Psi^{\kappa,j}(w) \sim \frac{\frac{1}{2}\delta_{i,j}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right)}{z-w}, \tag{5.60}$$

implementing the twist by κ when going around the origin $t \mapsto \mathrm{e}^{2\pi i t}\cdot z$ for $t \in [0,1)$. While these OPEs show that the twisted fields are not local to each other, the normal ordered products appearing when restricting to the $\mathbb{Z}_2$-invariant fields, in fact, will be. This is in accordance with the results of Frenkel and Szczesny in [25].

We yet have to find the field corresponding to the twisted ground state and its OPEs. For this we first turn to the corresponding SCFT orbifold and in the next section to the holomorphic chiral de Rham complex of the resolution, i.e. the Kummer surface.

5.3.2. Identification with the SCFT orbifold

As explained in Section 4.4 we turn to the NS-part of the toroidal SCFT $\mathbb{Z}_2$-orbifold. Since the holomorphic chiral de Rham complex seems to pick a part which is independent of the moduli space, we restrict to a point of the moduli space which is especially well behaved, the point of intersection of the torus and the $\mathbb{Z}_2$-orbifold model.

Consider an elliptic curve, i.e. complex dimension $N = 1$ and central charge $c = 3$:

$$\begin{aligned}\mathcal{E}_{T^2}^{\mathbb{Z}_2}(\tau,z) &= \frac{1}{2}\cdot\left({}_{\mathbb{1}}\underset{\mathbb{1}}{\boxed{}}(\tau,z) + {}_{\kappa}\underset{\mathbb{1}}{\boxed{}}(\tau,z)\right) + \frac{1}{2}\cdot\left({}_{\mathbb{1}}\underset{\kappa}{\boxed{}}(\tau,z) + {}_{\kappa}\underset{\kappa}{\boxed{}}(\tau,z)\right)\\ &= y^{\frac{1}{2}}\cdot 2\cdot\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau)} + \frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau)} + \frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau)}\right) = \mathrm{Str}_{\mathbb{H}^{NS}}\, y^{J_0} q^{L_0^{top}}.\end{aligned} \tag{5.61}$$

Let $\mathbb{H}_\kappa$ denote the space of states of the κ-twisted sector. We are interested in the lowest weight vectors of the twisted sector.

$$\operatorname{Str}_{\mathbb{H}_\kappa} y^{J_0} q^{L_0^{top}} = \mathbb{1}\underset{\kappa}{\boxed{}}(\tau, z) = 4 \cdot y^{\frac{1}{2}} \cdot \frac{\vartheta_4(\tau, z)}{\vartheta_4(\tau)} \tag{5.62}$$

$$= 4 \cdot y^{\frac{1}{2}} \cdot \left(\prod_{k=1}^{\infty} \frac{1}{1 - q^{k-\frac{1}{2}}} \right)^2 \cdot \left(\prod_{k=1}^{\infty} \left(1 - q^{k-\frac{1}{2}} y^1\right) \right) \cdot \left(\prod_{k=1}^{\infty} \left(1 - q^{k-\frac{1}{2}} y^{-1}\right) \right). \tag{5.63}$$

Thus the four twist field ground states have L_0^{top}-weight 0 and J_0-charge $\frac{1}{2}$ and are of bosonic nature. We want to identify the holomorphic chiral de Rham cohomology with the NS-part of the twisted sector of the toroidal SCFT. In the realm of the chiral de Rham complex all modes of fields are with respect to the topological L_0^{top}. So this is the moding we should also use for the toroidal SCFT. Since we take cohomology of $\mathcal{A}^{ch}_{T^2}(T^2)$ with respect to the operator ∂^{ch} there is some choice when looking for a representative in the SCFT. By Corollary 5.1.9 every element of the holomorphic chiral de Rham cohomology has a representative with $\bar{L}_0^{top}$-weight 0. This seems like a natural choice since we expect a vertex algebra without any appearance of the formal variable $\bar{z}$. The NS-part of the twisted sector is given by

$$\mathbb{1}\underset{\kappa}{\boxed{}}(\tau) A \underset{P}{\boxed{}}(\tau, z). \tag{5.64}$$

In Section 3.3.2 we identified this with a tensor product of three bosons on a circle, the third of which was the fermionic part. Since this fermionic part is a Dirac fermion $\mathbb{H}_{Dirac} = \mathbb{H}_{circ}\left(\frac{1}{\sqrt{2}}\right)$ there is only one state with J_0-charge $\frac{1}{2}$ and L_0^{top}-weight zero, the one associated to the lattice vector $\left(\frac{1}{2}, \frac{1}{2}\right) \in \Gamma_{R=\frac{1}{\sqrt{2}}}$. This field has L_0-weight $\frac{1}{8}$. In order to reach weight $\frac{1}{4}$ we need another $\frac{1}{8}$ coming from the bosonic side. As seen in Section 3.3.2 there are four possible fields[1]:

$$e^{\frac{1}{2} i X^1(z) + \frac{1}{2} i X^3(z) + \frac{1}{2} i \bar{X}^1(\bar{z}) + \frac{1}{2} i \bar{X}^3(\bar{z})}, \tag{5.65}$$

$$e^{-\frac{1}{2} i X^1(z) + \frac{1}{2} i X^3(z) - \frac{1}{2} i \bar{X}^1(\bar{z}) + \frac{1}{2} i \bar{X}^3(\bar{z})}, \tag{5.66}$$

$$e^{\frac{1}{2} i X^2(z) + \frac{1}{2} i X^3(z) + \frac{1}{2} i \bar{X}^2(\bar{z}) + \frac{1}{2} i \bar{X}^3(\bar{z})}, \tag{5.67}$$

$$e^{-\frac{1}{2} i X^2(z) + \frac{1}{2} i X^3(z) - \frac{1}{2} i \bar{X}^2(\bar{z}) + \frac{1}{2} i \bar{X}^3(\bar{z})}, \tag{5.68}$$

where we identify $J(z) = i\partial X^3(z)$. The OPEs between these are all non-singular. Their conjugated fields have same L_0-weight $\frac{1}{4}$ and J_0-charge $-\frac{1}{2}$. Their fermionic part comes from the lattice vector $\left(-\frac{1}{2}, -\frac{1}{2}\right) \in \Gamma_{R=\frac{1}{\sqrt{2}}}$. Again, there are four possible fields:

$$e^{\frac{1}{2} i X^1(z) - \frac{1}{2} i X^3(z) + \frac{1}{2} i \bar{X}^1(\bar{z}) - \frac{1}{2} i \bar{X}^3(\bar{z})}, \tag{5.69}$$

$$e^{-\frac{1}{2} i X^1(z) - \frac{1}{2} i X^3(z) - \frac{1}{2} i \bar{X}^1(\bar{z}) - \frac{1}{2} i \bar{X}^3(\bar{z})}, \tag{5.70}$$

$$e^{\frac{1}{2} i X^2(z) - \frac{1}{2} i X^3(z) + \frac{1}{2} i \bar{X}^2(\bar{z}) - \frac{1}{2} i \bar{X}^3(\bar{z})}, \tag{5.71}$$

$$e^{-\frac{1}{2} i X^2(z) - \frac{1}{2} i X^3(z) - \frac{1}{2} i \bar{X}^2(\bar{z}) \cdot \frac{1}{2} i \bar{X}^3(\bar{z})}. \tag{5.72}$$

[1]We are omitting the cocycle factors here.

Some of the twist fields and their conjugated fields have singular OPE, e.g.:

$$e^{\frac{1}{2}iX^1(z)+\frac{1}{2}iX^3(z)+\frac{1}{2}i\bar{X}^1(\bar{z})+\frac{1}{2}i\bar{X}^3(\bar{z})}e^{-\frac{1}{2}iX^1(w)-\frac{1}{2}iX^3(w)-\frac{1}{2}i\bar{X}^1(\bar{w})-\frac{1}{2}i\bar{X}^3(\bar{w})} \sim (z-w)^{-\frac{1}{2}}(\bar{z}-\bar{w})^{-\frac{1}{2}}\left\{\mathbb{1}+\mathcal{O}(z-w)+\mathcal{O}(\bar{z}-\bar{w})\right\}, \tag{5.73}$$

up to phases that depend on the choice of cocycle factors.

5.4. Kummer surface

We have investigated the $\mathbb{Z}_2$-orbifold of the holomorphic chiral de Rham complex of a torus T^4. On the side of target spaces, i.e. the manifold on which we investigate the holomorphic chiral de Rham complex, the $\mathbb{Z}_2$-orbifold of T^4 results in a singular surface. The resolution of this surface is a Kummer surface. Taking $\mathbb{Z}_2$-invariants and adding twisted sectors to ensure modular (co)variance on the holomorphic chiral de Rham complex side should correspond to the geometric $\mathbb{Z}_2$-orbifold and resolving singularities.

In the classical Dolbeault cohomology of T^4 all odd forms vanish and we find 16 additional $(1,1)$-forms coming from the blowups. We want to use this to further our understanding of the twist field ground states of the previous section.

We expect the $\mathbb{Z}_2$-orbifold of the holomorphic chiral de Rham cohomology of the torus T^4 to be the holomorphic chiral de Rham cohomology of the Kummer surface. This has been formulated in more generality by Frenkel and Szczesny in their following conjecture:

Conjecture 5.4.1 ([25]). *Let X be a smooth complex variety which carries an action of a finite group G. Let $\widetilde{G}$ denote the set of conjugacy classes of G and $C(g)$ the centralizer of an element g of G. Then*

$$H = \bigoplus_{[g]\in\widetilde{G}} H^*(X,\Omega_X^{ch,g})^{C(g)} \tag{5.74}$$

is the conjectured space of states of chiral de Rham complex of the resolution of the orbifold X/G.

Some remarks are due considering this notation. For $g\in G$ let X^g denote the fixed-point set of g on X. Then $\Omega_X^{ch,g}$ is the twisted sheaf supported on X^g. In our setting the conjectured space of states reduces to

$$H = H^*(T^4,\Omega_{T^4}^{ch})^{\mathbb{Z}_2} \oplus H^*(T^4,\Omega_{T^4}^{ch,\kappa})^{\mathbb{Z}_2}, \tag{5.75}$$

where $\Omega_X^{ch,\kappa}$ is a direct sum of 16 sheaves, each supported on one of the singularities. As vector spaces they are generated by the actions of fields (5.53) to (5.56) on a twist field ground state.

In order to compare the orbifold holomorphic chiral de Rham complex with that of the resolution we investigate the local description of the Kummer surface resulting from resolving the 16 singularities of $T^4/\mathbb{Z}_2$. Locally these singularities are of the form $\mathbb{C}^2/\mathbb{Z}_2$.

Choosing coordinates[2] (z_1, z_2) on $\mathbb{C}^2$ we can identify $\mathbb{C}^2/\mathbb{Z}_2$ with $K := \{ u \in \mathbb{C}^3 \mid u_1 u_2 = u_3^2 \}$ via the $\mathbb{Z}_2$-invariant monomials $u_1 = z_1^2$, $u_2 = z_2^2$ and $u_3 = z_1 z_2$ (cf. [3]). Let $\varepsilon > 0$ and $K_\varepsilon := K \cap \overline{B_\varepsilon(0)} \setminus \{ 0 \}$. To resolve the singularity in the origin we consider

$$\check{V} := \left\{ ((u_1, u_2, u_3), (v_1 : v_2 : v_3)) \in K_\varepsilon \times \mathbb{CP}^2 \quad \middle| \quad u_i v_j = u_j v_i \quad \forall i, j \right\}. \tag{5.76}$$

Then $V' = \overset{\circ}{\overline{\check{V}}}$ with projection σ to $K \cap B_\varepsilon(0)$ is the resolution of the singularity and the exceptional divisor is $E := \sigma^{-1}(0)$. Outside the exceptional divisor σ is a biholomorphic map to $(K_\varepsilon) \setminus \{ 0 \}$. This is done for all 16 singularities. Denote by $X = \widetilde{T^4/\mathbb{Z}_2}$ the resulting Kummer surface.

Outside of the exceptional divisor we can choose the open sets $U_{13} := \{ u_1 \neq 0 \}$ with coordinates (u_1, u_3) and $U_{23} := \{ u_2 \neq 0 \}$ with coordinates (u_2, u_3). Throughout this section we will use the following charts and coordinates around the singularity:

$$U_1 = \{ v_1 \neq 0 \} \quad \text{with coordinates } (u_1, u) := (u_1, \frac{v_3}{v_1}), \tag{5.77}$$

$$U_2 = \{ v_2 \neq 0 \} \quad \text{with coordinates } (u_2, v) := (u_2, \frac{v_3}{v_2}). \tag{5.78}$$

On the intersection of U_1 and U_2 the coordinate change is given by $(u_2, v) = (u^2 u_1, \frac{1}{u})$. These kinds of coordinates have to be chosen around each of the 16 singularities, away from them they are patched together as on the torus via lattice translations.

On the classical level of Dolbeault cohomology the $\mathbb{Z}_2$-invariant part of the untwisted sector is spanned by representatives $1, dz_1 \wedge dz_2, dz_i \wedge d\bar{z}_j, d\bar{z}_1 \wedge d\bar{z}_2$ and $dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2$. That these are in fact global sections of the Dolbeault sheaf of the Kummer surface can be checked using the local coordinates above. An important feature for the well-definedness of these sections is the fermionic nature of one-forms, which is exemplified in the following coordinate transformation: On U_1 the two-form $dz_1 \wedge dz_2$ takes the form

$$\frac{1}{2} du_1 \wedge du \tag{5.79}$$

and on the intersection $U_1 \cap U_2$ it transforms as

$$\frac{1}{2} du_1 \wedge du = \frac{1}{2} \left(v^2 du_2 + 2 u_2 v dv \right) \wedge \left(-\frac{1}{v^2} dv \right) \tag{5.80}$$

$$= -\frac{1}{2} du_2 \wedge dv \tag{5.81}$$

due to fermionic cancellation. This enables one to consistently define a global section.

When considering the full orbifold holomorphic chiral de Rham complex we find global sections that do not exhibit this cancellation and cannot be easily identified with global sections on the Kummer surface.

[2]Contrary to our notation so far we will denote the coordinate index by a subscript in this section. Conformal weights will be specified by subscripts put in parenthesis. Furthermore, 1-forms and vector fields will not get their own letters. For example, a_n^i will now be denoted by $\left(\frac{\partial}{\partial z_i}\right)_{(n)}$. If no conformal weight is given this will always mean conformal weight zero, i.e. classical geometric objects.

Proposition 5.4.2. *The $\mathbb{Z}_2$-invariant section $(z_1)_{(1)}dz_1$ of the torus cannot be trivially extended to the resolution of the singularity.*

Proof. Consider $(z_1)_{(1)}dz_1$ or $(b^1)_{(1)}\Phi_0^1$ in the notation of [49]. On U_1 this becomes $\frac{1}{4}\frac{1}{u_1}(u_1)_{(1)}du_1$ which on $U_1 \cap U_2$ transforms to

$$\frac{1}{4}\left(v^2u_2\right)\left((v)_{(1)}2vu_2 + (u_2)_{(1)}v^2\right)\left(v^2du_2 + 2vu_2dv\right), \tag{5.82}$$

while going directly to U_2 gives

$$\left(\frac{1}{2}u_2^{-\frac{1}{2}}v(u_2)_{(1)} + u_2^{\frac{1}{2}}(v)_{(1)}\right)\left(\frac{1}{2}u_2^{-\frac{1}{2}}vdu_2 + u_2^{\frac{1}{2}}dv\right). \tag{5.83}$$

□

Remark 5.4.3. Note, however, that we do expect $(z_1)_{(1)}dz_1$ to appear in the Kummer chiral de Rham cohomology since it is accounted for in the elliptic genus which is, in fact, compatible with the orbifold procedure ([8], [25]). It may be more complicated to extend the section just as in the case of the 16 additional $(1,1)$-forms coming from the exceptional divisors and being extend to the Kummer surface which was shown thoroughly by R. Kobayashi in [45]. We also may have to consider higher cohomology: Song states in [53] that the only global sections of the Kummer holomorphic chiral de Rham complex are the ones generating the $N = 4$ superconformal algebra. If this is the case we may have to look for the other sections predicted by the orbifold in higher cohomology as it is not clear that the orbifold should respect cohomology degree.

We can, however, say more about the twist field ground states and shed light on the shift in J_0-charge that was introduced in [25] somewhat artificially.

Proposition 5.4.4. *The 16 twist fields of the $\mathbb{Z}_2$-orbifold of the holomorphic chiral de Rham cohomology of the torus T^4 correspond to the 16 additional $(1,1)$-forms $\hat{\omega}^i, i = 1, \ldots 16$ coming from the exceptional divisors and have non-singular OPEs*

$$\hat{\omega}^i(z)\hat{\omega}^j(w) = -2\delta_{i,j}\cdot \mathrm{vol}(w) + \mathcal{O}(z-w), \tag{5.84}$$

where vol *is the top $(2,2)$-form of the Kummer surface, e.g. in torus coordinates*

$$\mathrm{vol} = dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2. \tag{5.85}$$

Proof. The 16 twist field ground states have L_0^{top}-weight 0 and J_0-charge 1, so they are part of the classical Dolbeault cohomology and, if the orbifold and the resolution are to be related, correspond to the 16 additional $(1,1)$-forms of the Kummer surface coming from the exceptional divisors. We can write this in coordinates in the following way:
We embed $\mathbb{P}^1$ into V' as

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & E \subset V' \\ (s_0 : s_1) & \longmapsto & ((0,0,0),(s_0^2 : s_1^2 : s_0s_1)) \end{array}. \tag{5.86}$$

By choice of our coordinates on V' the exceptional divisor E only has non-trivial intersection with U_1 and U_2. On each of these we can identify u respectively v with the coordinates $u := \frac{s_1}{s_0}$ respectively $v := \frac{s_0}{s_1}$ of $\mathbb{P}^1 = U \cup V$. Let $\{ \rho_U, \rho_V \}$ be the partition of unity for $\{ U, V \}$. In coordinates the $(1,1)$-form of $\mathbb{P}^1$ can be written as

$$\omega = \rho_U (\bar{\partial} \rho_V) \frac{du \wedge d\bar{u}}{u} + \rho_V (\bar{\partial} \rho_U) \frac{dv \wedge d\bar{v}}{v}. \tag{5.87}$$

This cannot be extended trivially to a closed $(1,1)$-form $\hat{\omega}$ on the Kummer surface with $\hat{\omega}\,|_E = \omega$: Let ρ be a cutoff function for a small tubular neighborhood of E. On U_1 such a form is given by

$$\hat{\omega}(U_1) = \rho\omega + \alpha du \wedge d\bar{u}_1 + \beta du_1 \wedge d\bar{u} + \gamma du_1 \wedge d\bar{u}_1 \tag{5.88}$$

with smooth functions $\alpha, \beta, \gamma \in \mathcal{C}^\infty \left(\widetilde{T^4/\mathbb{Z}_2} \right)$. Since we need $\bar{\partial}\hat{\omega} = 0$ not all α, β, γ can be trivial. R. Kobayashi describes this highly non-trivial procedure of gluing in the $(1,1)$-forms in [45]. We can choose charts U_1^i, U_2^i with coordinates $(u_1^i, u^i), (u_2^i, v^i)$ and a $(1,1)$-form $\hat{\omega}^i$ around each of the 16 exceptional divisors for $i = 1, \dots, 16$. The 16 $(1,1)$-forms are supported around each of the exceptional divisors.

We turn to the OPEs: As elements of $\mathcal{A}_X^{ch} \left(\widetilde{T^4/\mathbb{Z}_2} \right)$ we know how to construct the fields $\hat{\omega}^i(z)$ associated to $\hat{\omega}^i$. They only involve functions and differential forms as well as their higher modes. By construction of the $(bc - \beta\gamma)$-system this all commutes with each other and we arrive at the non-singular OPE

$$\hat{\omega}^i(z)\hat{\omega}^j(w) =: \hat{\omega}^i(w)\hat{\omega}^j(w) : + \mathcal{O}(z - w). \tag{5.89}$$

The state associated to the field $: \hat{\omega}^i(w)\hat{\omega}^j(w) :$ is obtained via the state-field-correspondence

$$: \hat{\omega}^i(w)\hat{\omega}^j(w) : \mathbb{1}\,|_{w=0} = (\hat{\omega}^i)_{(0)} (\hat{\omega}^j)_{(0)}. \tag{5.90}$$

Since the conformal weight zero subspace of $\mathcal{A}_X^{ch}$ is $\mathcal{A}_X$ and the vertex algebra 0-product reduces to the classical wedge product, $(\hat{\omega}^i)_{(0)}(\hat{\omega}^j)_{(0)}$ is the wedge product $\hat{\omega}^i \wedge \hat{\omega}^j$ of two of the 16 additional $(1,1)$-forms and thus equal to $-2\delta_{i,j} \cdot \mathrm{vol}$ in cohomology. Here vol is the volume form which, for example, on U_1 takes the form $du_1 \wedge du \wedge d\bar{u}_1 \wedge d\bar{u}$, i.e. the top $(2,2)$-form on the Kummer surface. □

Because of the relatively concrete nature of the representation of the twist field ground states in the previous proposition other OPEs can be calculated easily. For example, it is apparent that $\hat{\omega}^i$ has L_0^{top}-weight zero and J_0-charge 1 but is still a boson because of the antiholomorphic form $d\bar{u}, d\bar{u}_1$ (resp. $d\bar{v}$, $d\bar{u}_2$) contribution. Both the J_0-charge and the bosonic nature appear naturally in $\mathcal{A}^{ch}$ and shed light on the shift in J_0-charge in [25].

The 16 conjugated fields of $\hat{\omega}^i$ with L_0^{top}-weight 1 and J_0-charge -1 can be constructed from the $\hat{\omega}^i$ by application of the mode J_1^- of the $N = 4$ superconformal algebra:

$$J_1^- \hat{\omega}^i. \tag{5.91}$$

This is an element of $\mathcal{A}_X^{ch}\left(\widetilde{T^4/\mathbb{Z}_2}\right)$. On U_1^i we have

$$J_1^-(U_1^i) = \sum_{n+m=1} \left(\frac{\partial}{\partial du_1^i}\right)_{(n)} \left(\frac{\partial}{\partial du^i}\right)_{(m)}. \tag{5.92}$$

Applied to $\hat{\omega}^i$ it exchanges holomorphic 1-forms du, du_1 with holomorphic vector fields $\left(\frac{\partial}{\partial du_1^i}\right)_{(1)}$, $\left(\frac{\partial}{\partial du^i}\right)_{(1)}$ respectively:

$$J_1^-\hat{\omega}^i(U_1^i) =$$
$$\rho^i(\bar{\partial}\rho_V^i)\frac{1}{u^i}\left(\frac{\partial}{\partial du_1^i}\right)_{(1)} d\bar{u}^i + \alpha^i\left(\frac{\partial}{\partial du_1^i}\right)_{(1)} d\bar{u}_1^i + \beta^i\left(\frac{\partial}{\partial du^i}\right)_{(1)} d\bar{u}^i + \gamma^i\left(\frac{\partial}{\partial du^i}\right)_1 d\bar{u}_{(1)}^i \tag{5.93}$$

and analogously on U_2^i. Interestingly, the application of J_1^- replaces the $\mathbb{P}^1$-direction du^i with the perpendicular direction $\left(\frac{\partial}{\partial du_1^i}\right)_{(1)}$. Thus $J_1^-\hat{\omega}^i$ is unique to the Kummer surface. While the $\hat{\omega}^i$ come from the exceptional divisors, J_1^- comes from a global section of the torus and only in their combination $J_1^-\hat{\omega}^i$ arises.

Since we work with sheaf cohomology and not only $\mathcal{A}_X^{ch}\left(\widetilde{T^4/\mathbb{Z}_2}\right)$ we still have to show that the $J_1^-\hat{\omega}^i$ are cohomologically non-trivial:

Proposition 5.4.5. *The 16 states $J_1^-\hat{\omega}^i$ are non-trivial elements of the holomorphic chiral de Rham cohomology of the Kummer surface with L_0^{top}-weight 1 and J_0-charge -1.*

Proof. Assume that $J_1^-\hat{\omega}^i$ is cohomologically trivial, i.e. that there is an element $\chi \in \mathcal{A}_X^{ch}\left(\widetilde{T^4/\mathbb{Z}_2}\right)$ such that

$$\bar{\partial}^{ch}\chi = J_1^-\hat{\omega}^i. \tag{5.94}$$

For the modes J_{-1}^+ and J_1^- of the $N=4$ superconformal algebra we have the commutator relation

$$\left[J_{-1}^+, J_1^-\right] = 1 + 2J_0. \tag{5.95}$$

Applying these modes to (5.94) yields:

$$\bar{\partial}^{ch}\chi = J_1^-\hat{\omega}^i \tag{5.96}$$
$$\Longrightarrow \quad J_{-1}^+\bar{\partial}^{ch}\chi = J_{-1}^+J_1^-\hat{\omega}^i \tag{5.97}$$
$$\Longrightarrow \quad \bar{\partial}^{ch}J_{-1}^+\chi = \left[J_{-1}^+, J_1^-\right]\hat{\omega}^i + J_1^-J_{-1}^+\hat{\omega}^i \tag{5.98}$$
$$\Longrightarrow \quad \bar{\partial}^{ch}J_{-1}^+\chi = \hat{\omega}^i + 2J_0\hat{\omega}^i + J_1^-J_{-1}^+\hat{\omega}^i \tag{5.99}$$
$$\Longrightarrow \quad \bar{\partial}^{ch}J_{-1}^+\chi = 3\hat{\omega}^i + J_1^-J_{-1}^+\hat{\omega}^i. \tag{5.100}$$

The second summand $J_1^-J_{-1}^+\hat{\omega}^i$ is trivial since $\hat{\omega}^i$ has L_0^{top}-weight zero and J_{-1}^+ reduces weight by one. We arrive at

$$\bar{\partial}^{ch}J_{-1}^+\frac{1}{3}\chi = \hat{\omega}^i. \tag{5.101}$$

This, however, is a contradiction because $\hat{\omega}^i$ is cohomologically non-trivial and thus so is $J_1^-\hat{\omega}^i$. The 16 states $J_1^-\hat{\omega}^i$ are distinct since they are supported around the 16 exceptional divisors. □

Proposition 5.4.6. *The 16 states $J_1^-\hat{\omega}^i$ have non-singular OPEs among themselves. With their conjugated states $\hat{\omega}^j$ they have OPEs with a pole of order at most one in $z = w$.*

Proof. Consider the coordinate representation of $J_1^-\hat{\omega}^i$ on U_1^i:

$$J_1^-\hat{\omega}^i(U_1^i) = \rho^i(\bar{\partial}\rho_V^i)\frac{1}{u^i}\left(\frac{\partial}{\partial du_1^i}\right)_{(1)} d\bar{u}^i + \alpha^i\left(\frac{\partial}{\partial du_1^i}\right)_{(1)} d\bar{u}_1^i + \beta^i\left(\frac{\partial}{\partial du^i}\right)_{(1)} d\bar{u}^i + \gamma^i\left(\frac{\partial}{\partial du^i}\right)_1 d\bar{u}_{(1)}^i \quad (5.102)$$

Since all of these elements commute with each other $(J_1^-\hat{\omega}^i)(z)$ has non-singular OPE with itself. Their conjugated states $\hat{\omega}^j$ have coordinate representation

$$\hat{\omega}(U_1) = \rho\omega + \alpha du \wedge d\bar{u}_1 + \beta du_1 \wedge d\bar{u} + \gamma du_1 \wedge d\bar{u}_1. \quad (5.103)$$

Note that du, du_1 do not commute with $\left(\frac{\partial}{\partial du^i}\right)_{(1)}, \left(\frac{\partial}{\partial du_1^i}\right)_{(1)}$ respectively. All other modes appearing in $\hat{\omega}^i$ and $(J_1^-\hat{\omega}^i)$ commute. Since both are composed of free fields by the Wick Theorem we find a pole of order at most one.

On U_2^j the argument is the same. Because both $(J_1^-\hat{\omega}^i)(z)$ and $\hat{\omega}^j(w)$ are supported around the exceptional divisors this completes the proof. □

Interestingly the singular part of the OPEs between $J_1^-\hat{\omega}^i(z)$ and $\hat{\omega}^j(w)$ does not come from the $\mathbb{P}^1$ part but from the procedure of gluing it into the Kummer surface. If $\hat{\omega}^j$ only had a $du \wedge d\bar{u}$ contribution then its OPEs with $J_1^-\hat{\omega}^i(z)$ would be non-singular!

Remark 5.4.7. As in the case of orbifold toroidal SCFT, the twist fields of the chiral de Rham cohomology of the Kummer surface have singular OPEs with their conjugated fields with a pole of order one. So our results for the OPEs of the twist fields and their conjegates fits nicely with the results in the orbifold toroidal SCFT. This can be seen as further evidence for the orbifold conjecture. Note that in Subsection 5.3.2 we gave the OPE in the $c = 3$ case while the Kummer surface chiral de Rham complex is $c = 6$.

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