An exploration of the (16,6) configuration with links to Kummer Surfaces

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Introduction

In 1864 E.E.Kummer wrote [2] about an interesting quadric surface which has 16 singularities. Now any such surface is called a Kummer surface, and with the singularities blown up it is often used as a tool to visualise K3 surfaces. It is even possible to build a model of a Kummer surface.

In [1] a correspondence was shown between Kummer surfaces embedded in $\mathbb{P}^3$ and certain non-degenerate $(16, 6)$ configurations. This is explained in section 2 where I state this theorem and outline a proof. But one point in the proof was the classification of non-degenerate $(16, 6)$ configurations (section 1), what intrigued me here were the two so called 'exotic' configurations, that couldn't live in $\mathbb{P}^3$.

Sections 3 and 4 both deal with the understanding of the symmetry group of all the non-degenerate $(16, 6)$ configurations.

Section 3 defines a symmetry of a $(16, 6)$ configuration and then looks at the symmetries that fix a given plane. The approach used here is through a view taken from the fixed plane.

Section 4 takes a different approach, using the matrix definition of the $(16, 6)$ configurations. This is another interesting way to look at symmetries. I use it to explore symmetries that fix no points or planes.

As a result I can argue that the exotic (16,6) configurations cannot correspond to Kummer surfaces at all.

1 Abstract (16,6) Configurations

My first aim in this section is to introduce the concept of $(16,6)$ configurations both abstractly and geometrically. Then I give a general construction to enable a $(16,6)$ configuration to be pictured as a collection of 10 cones over a chosen plane. The main part of this section presents a result from [1]:

Non-degenerate abstract $(16,6)$ configurations are classified into three types. Type (*) will be shown to correspond to Kummer surfaces in $\mathbb{P}^3$ (See section 2). Type A, and type B configurations will be called exotic.

Trying to understand the exotic cases has been the main motivation of my work.

1.1 Preliminaries

For the following definitions the main configuration to bear in mind is the $(16,6)$ configuration but I also use $(4,3)$, $(8,4)$ and other configurations. Hence I state the definitions in general.

Definition 1.1 *An abstract (a,b) configuration is an $a \times a$ matrix with elements in $\mathbb{F}_2 = \{0,1\}$ such that each row and each column has exactly b ones.*

However large matrices over $\mathbb{F}_2$ are not the most intuitive objects. So we can define a geometric (a,b) configuration as follows:

Definition 1.2 *A geometric (a,b) configuration is a collection of a points and a planes such that b points lie on each plane and b planes intersect at each point.*

We want to be be able to relate abstract and geometric configurations uniquely and for this we need to choose a convention. I define an incidence diagram as follows:

Definition 1.3 *A matrix* $(\alpha_{ij})_{1\leq i\leq n,\, 1\leq j\leq m}$ *with* $\alpha_{ij} \in \mathbb{F}_2$ *is considered an incidence diagram of* n *planes* H_i *and* m *points* P_i*: We associate a row to each plane and a column to each point, such that a point* P_j *lies on a plane* H_i *if and only if* $\alpha_{ij} = 1$.

This gives us immediately the desired result:

Remark 1.4 *Every abstract* (a, b) *configuration immediately defines a geometric* (a, b) *configuration, by considering the matrix to be an incidence diagram of abstract points and planes.*

For some special geometric (a, b) configurations we can associate a notion of non-degeneracy. Examples are $(4, 3)$ and $(16, 6)$ which allow non-degeneracy whereas $(8, 4)$ does not. In general (a, b) can be non-degenerate if $a = C_2^b + 1 = \frac{b(b-1)}{2} + 1$. The $(16, 6)$ configuration is particularly interesting since it is the smallest configuration having both non-degenerate and degenerate configurations.

Definition 1.5 *A geometrical* (a, b) *configuration is said non-degenerate if every pair of points or planes defines a unique line.*

1.2 The key tools

All pairs of planes intersect in a line containing exactly 2 points. Given a plane H the other 15 planes each define a unique line on H joining 2 of the 6 points lying on H, Figure 1.

Also any given point P not on H lies on 6 other planes and so defines a collection of 6 lines on H. Any given point on H and P both lie on exactly 2 planes and since P is not in H these two planes correspond to two lines on H. It follows that either the collection of 6 planes form a single cone(type 2) or they split into 2 cones of 3(type 1). Figure 2 illustrates an example of the intersection of the cones of type 1 and 2 with H.

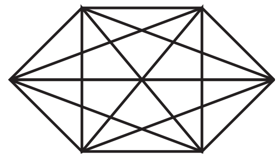


Figure 1: On any given plane H of a non-degenerate (16,6) configuration each of the other 15 planes corresponds to exactly one line of this picture.

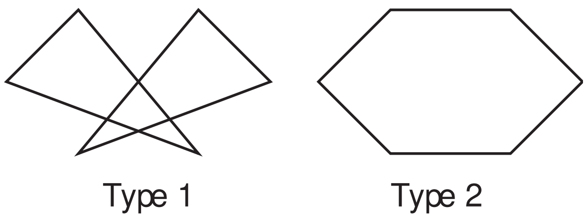


Figure 2: These diagrams illustrate how the cones of six planes, defined by a point not on H, could intersect H. The structure is not changed by interchanging points even though the picture could be.

Remark 1.6 *Given a point a on H there is a bijection between pairs of points on H and points not on H.*

Proof. a,b and a,c define lines on H which are planes in the (16,6) configuration. These planes intersect in a line which contains 2 points, one of which is a. Since the line does not lie in H neither does the second point d. Similarly given any point d outside H the line joining it to a comes from 2 planes which define 2 lines on H each containing a and one other point on H. Call these points b and c. These constructions are unique since no three planes intersect in a line and no three points lie on a line. $_{Q.E.D}$

Remark 1.7 *There is a bijection between pairs of points not in H and pairs of lines in H.*

Proof. This corresponds exactly to the non-degeneracy condition since any two points lie on the intersection of 2 planes. Since the points are not in H neither plane can be H so they both intersect H in a line. Similarly 2 lines define 2 planes which defines 2 points. $_{Q.E.D}$

1.3 Classification

Theorem 1.8 *There are exactly 3 non-isomorphic non-degenerate abstract* $(16, 6)$ *configurations.*

I approach the proof in two parts. Choose a special plane H. In part one of the proof I show that there are exactly 3 non-isomorphic configurations of the other 10 points and 15 planes which are $(16, 6)$ non-degenerate. To complete the proof all that is needed is to show that the configuration is independent of the choice of initial plane H. I only give the main ideas here.

Proof of part one. A configuration (called of type (*)) consisting solely of type 1 cones satisfies remarks 1.6 and 1.7. This can be checked immediately from figure 3.

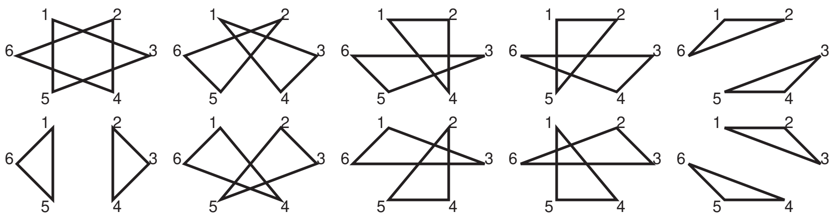


Figure 3: A configuration containing only triangles.

Now assume that a configuration has at least one cone of type 2. Without loss of generality consider point 7 to have the cone (1 2 3 4 5 6) where this means it contains the lines (12), (23), (34), (45), (56), (16). Now from remark 1.6, choosing base point 1, there is a bijection between pairs of points on H not equal to 1 and points not on H. We know that $(2, 6)$ maps to 7 so let $(2, 3)$ map to 8, $(2, 4)$ to 9 and $(2, 5)$ to 10. This puts us in the position shown in figure 4.

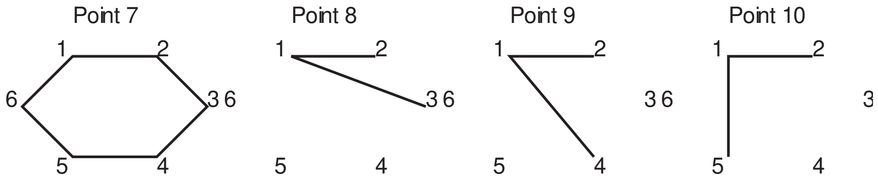


Figure 4: Step one in the building of a configuration with at least 1 cone of type 2.

Already some restrictions become apparent. Point 8 has to have a cone of type 2 since if it was of type 1 it would have 4 lines and hence planes in common with point 7. This contradicts non-degeneracy (see remark 1.7).

If point 9 has a cone of type 1 it must be (1 2 4)(3 5 6) and if it were a cone of type 2, we can eliminate all possibilities other than (1 2 6 3 5 4). Simply checking the other cycles of the form (1 2 * * * 4) shows they all contradict remark 1.7.

Similarly if point 10 has a cone of type 1 it must be (1 2 5)(3 4 6) and if it were a cone of type 2 it is (1 2 6 4 3 5).

This gives us 4 cases as shown in figure 5. I have completed the cone for point 8 in cases 1,2 and 3 since it has become uniquely determined.

Case 4 corresponds to all 4 points being cones of type 2. This is invalid[1] since the cones in points 9 and 10 intersect in 3 lines.

Cases 2 and 3 are equivalent since we can get from one to the other by applying the following permutation (1 2)(3 6)(4 5)(8 9 10).

It is now possible to complete the proof directly by checking the remaining points directly and remarking that up to some permutation of points there are only two different configurations with a case 2 cone. [1] does this leaving some of the routine checking to the reader.

[1]A configuration is invalid if it contradicts either the $(16, 6)$ configuration or the non-degeneracy assumption. Invalidity also follows directly if 1.6 or 1.7 don't hold and these are often the easiest things to look for.

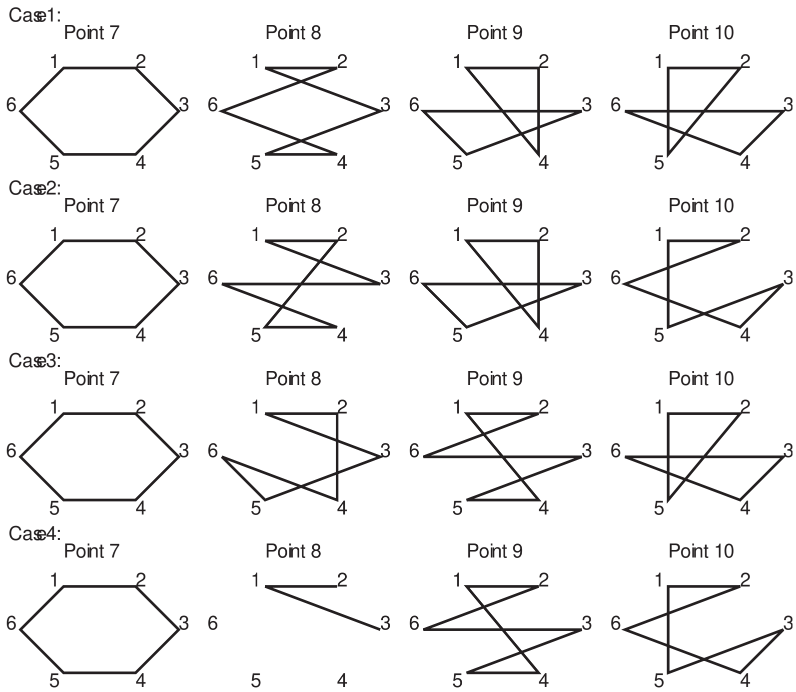


Figure 5: Step two: A configuration with at least 1 cone of type 2 has been shown to be one of these four cases. It is easy to see that case 4 is invalid and case 2 is equivalent to case 3.

Although that does complete the proof, a more useful exercise is to complete the proof by elimination after repeating the same argument.

The three points 8,9, and 10 were chosen so that they all lie on the plane $H_{1,2}$ intersecting H in the line 12. Since 7 also intersects H in 1 and 2 the plane $H_{1,2}$ contains the 6 points 1,2,7,8,9,10.

What we showed earlier was that there are exactly 2 ways to extend the points 1,2,3,4,5,6,7 by 3 new points without contradiction to the $(16, 6)$ configuration. It is useful to distinguish between cases 2 and 3 and think of them as being related.

In all cases the points 8,9,10 do not lie on the plane $H_{2,3}$. So three new points 11,12,13 can be defined in exactly the same way simply by rotating the diagrams by $\frac{2\pi}{6}$.

Recall we had the 3 valid cases for the 10 points 2 of which were equivalent. We now have 9 cases (since order does not matter) for 13 points and need to check validity and equivalence. I will not go into the details here since they are easy to do using the above methods.

- Case 1×1 is valid.
- Cases 1×2, 2×1, 1×3 and 3×1 are all valid and equivalent.
- Cases 2×2 and 3×3 are invalid, the easiest way to see this is to look in figure 5 at the points 6, 9, and 11[2] all three lie on both $H_{3,6}$ and $H_{5,6}$ for case 2×2. Consider the points 6, 9, and 13 to invalidate case 3×3.
- Cases 2×3 and 3×2 are valid and equivalent.

Each of the above valid cases have a unique completion. And they split into two types A and B (to use [1]'s notation). Type A is the first case in the list (figure 6) and type B encompasses the other 6 valid cases which are all equivalent (figure 7). *Q.E.D*

[2]the point 11 has cone defined by the cone of 8 rotated in the diagram by $\frac{2\pi}{6}$

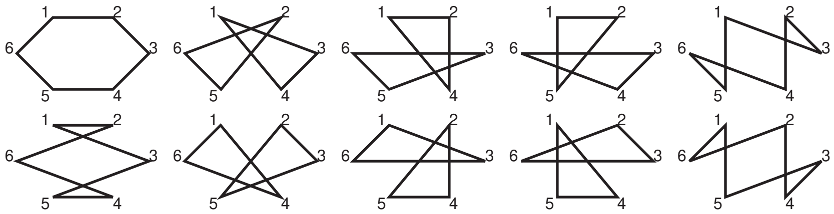


Figure 6: A configuration of type A.

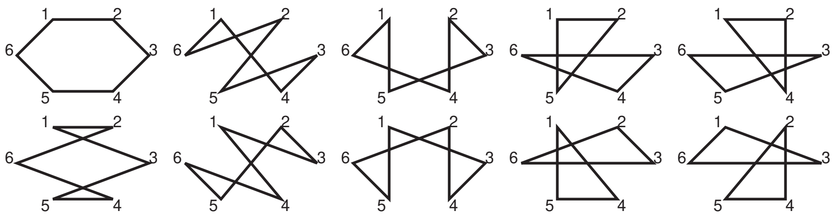


Figure 7: A configuration of type B.

Before I comment on part two of the proof I will state the following definition.

Definition 1.9 *A Rosenhain tetrahedron in a* $(16, 6)$ *configuration is a* $(4, 3)$ *sub-configuration.*

Idea and key steps for the proof of part 2. The idea of the proof is to show that the number of Rosenhain tetrahedra in a $(16, 6)$ configuration is 8n where n is the number of points having cones of type 1 with respect to a chosen plane H. This then means that n is independent of the choice of H. This would complete the proof.

The first key step is to show that every Rosenhain tetrahedron intersects a plane in either a point or a face (i.e. not an edge or an empty intersection).

It is obvious that H intersects 2n Rosenhain tetrahedra in a face.

All that's left is to show that H intersects 6n Rosenhain tetrahedra in a point. This comes directly from a very interesting bijection between Rosenhain tetrahedra and each pair of a point α on H and a point β not on H that defines a cone of type 1.

I find this bijection visually very interesting since given the pair of points we get a plane H' which corresponds to the line in the triangle containing α with $\alpha \notin H'$. H' defines 3 points which are not in H and not β, these points and α define a Rosenhain tetrahedron. All this is saying is that the cone defined by H' and α is of type 1 which is obvious. But I find it visually very nice. $_{Q.E.D}$

2 (16,6) Configurations in $\mathbb{P}^3$

In this section I outline the main results related to the embedding of (16,6) configurations into $\mathbb{P}^3$ from [1]. I do not go through the proofs but present the key ideas in a manner which will prepare the next section.

$(8,4)$ configurations are mentioned in 2.2 since they are the key tool in proving that type (*) configurations embed into $\mathbb{P}^3$ and also that the exotic configurations don't. They are also a key concept used in section 4.

The study of symmetries starts in subsection 2.3. Subsection 2.4 is an aside to the rest of the project but I like it, so I wanted to mention it.

2.1 Statement of Theorems

I follow the convention used in [1] to define a Kummer surface as a singular surface, as opposed to considering the surface after the singularities have been blown up, which would make it a K3 surface. This choice is justified since the $(16,6)$ configuration can be considered as a model for the singularities of a Kummer surface.

Definition 2.1 *A Kummer surface in $\mathbb{P}^3$ is a reduced, irreducible surface of degree 4 having 16 nodes and no other singularities.*

The aim of this section is to prove the following result:

Theorem 2.2 *The non-degenerate $(16,6)$ configurations that embed into $\mathbb{P}^3$ are of type (*), form a 3-dimensional family and all correspond to a Kummer surface which is embedded in $\mathbb{P}^3$. Similarly every Kummer surface embedded in $\mathbb{P}^3$ has an associated type (*) configuration.*

The following steps are used to prove this result:

- A standard $(8,4)$ configuration embedded into $\mathbb{P}^3$ plus 2 special points uniquely generates a non-degenerate $(16,6)$ configuration of type (*) in $\mathbb{P}^3$.

- Every non-degenerate $(16,6)$ configuration contains a sub-structure equivalent to a standard $(8,4)$ configuration plus 2 special points. So when embedded into $\mathbb{P}^3$ we get a contradiction if the configuration is not of type (*).

- The points of a non-degenerate $(16,6)$ configuration of type (*) in $\mathbb{P}^3$ can be generated from a single point (a,b,c,d) in general position[3] under the action of a group which I call $F_0 \cong \mathbb{Z}_2^4$.

- The moduli space of non-degenerate $(16,6)$ configurations of type (*) in $\mathbb{P}^3$ is the quotient of an open subset of $\mathbb{P}^3$ by a finite group.

- Kummer surfaces in $\mathbb{P}^3$ define a unique non-degenerate $(16,6)$ configuration of type (*) and vice versa.

The main ideas concerning $(8,4)$ configurations and the group F_0 are introduced below since they relate both to some steps of the proof and to other work presented later.

2.2 $(8,4)$ Configurations

Recall definitions 1.1 and 1.2 of an (a,b) configuration.

Definition 2.3 *A standard abstract $(8,4)$ configuration is an abstract $(8,4)$ configuration that can be written, using row and column permutations, as the*

[3]The list of conditions stated at the end of 2.4 also correspond to the conditions on the point (a,b,c,d) that guaranty that F_0 generates 16 distinct points from it.

following matrix:

$$\begin{pmatrix}0&1&1&1&1&0&0&0\\1&0&1&1&0&1&0&0\\1&1&0&1&0&0&1&0\\1&1&1&0&0&0&0&1\\1&0&0&0&0&1&1&1\\0&1&0&0&1&0&1&1\\0&0&1&0&1&1&0&1\\0&0&0&1&1&1&1&0\end{pmatrix}$$

Remark 2.4 *A nice way to understand this definition of a standard (8,4) configuration is that we are simply attaching 2 Rosenhain tetrahedra together in a standard symmetrical way. Drawing the picture is tricky since this lives naturally in* $\mathbb{P}^3$ *but numbering the vertices and faces of 2 tetrahedra and seeing how they could fit together in a projective space is a good exercise in visualisation.*

Or alternatively consider 2 planes to be 'parallel' if they don't intersect in at least one point. Then a standard $(8,4)$ *configuration is made up of 4 pairs of 'parallel' planes and can be visualised as a pair of 'parallel' planes each containing 4 points.*

Remark 2.5 *Every non-degenerate* $(16,6)$ *configuration contains a standard* $(8,4)$ *sub-configuration. And moreover it can always be split into two standard* $(8,4)$ *configurations glued together by two* $(8,2)$ *configurations.*

These ideas will be developed further and used in section 4. I will mention the proof of this remark later. It is obvious by inspection of the three matrices presented in section 4.

2.3 Symmetries in $\mathbb{P}^3$

Let $F_0 \subseteq PGL(4)$ be the group of symmetries in $\mathbb{P}^3$ defined by changing signs of any 2 coordinates or interchanging any two pairs of coordinates. This group of symmetries is enough to generate a type (*) $(16, 6)$ configuration from a single point.

For readers familiar with the concept of abelian surfaces: It is known that every Kummer surface allows a 2:1 cover by an abelian surface, with branch locus given by the 16 nodes on the Kummer surface. The F_0 action is induced by half-period shifts on the abelian surface.

In section 3 and 4 I will explore the entire symmetry groups of the three non-degenerate configurations.

2.4 General equation of a Kummer surface in $\mathbb{P}^3$

This is a complete aside to the aims of my project but I mention it briefly because of its history. In [2] Kummer first wrote down an equation of what is now know as a Kummer surface, and the following equation was first known by [3]. Through this defining equation, using nothing but long calculations, [1] showed that any type (*) configuration is the associated $(16, 6)$ configuration for some Kummer surface.

The defining equation of a Kummer surface mapped to itself under the F_0 action defined in 2.3 can be written as:

$$f = x^4+y^4+z^4+t^4+2Dxyzt+A(x^2t^2+y^2z^2)+B(y^2t^2+x^2z^2)+C(x^2y^2+t^2z^2)$$

for appropriate choices of A, B, C, D that depend only on the coordinates $(a, b, c, d) \in \mathbb{P}^3$ of one node such that the coordinates satisfy the following conditions.

$$\begin{aligned} ad &\neq \pm bc, \\ ac &\neq \pm bd, \\ ab &\neq \pm cd, \\ a^2 + d^2 &\neq b^2 + c^2, \\ a^2 + c^2 &\neq b^2 + d^2, \\ a^2 + b^2 &\neq c^2 + d^2, \\ a^2 + b^2 + c^2 + d^2 &\neq 0. \end{aligned}$$

3 Symmetries of a (16,6) configuration

In this section I explore the abstract symmetries of a $(16,6)$ configuration that fix a given plane. The symmetry group of the type (*) configuration embedded in $\mathbb{P}^3$ was described in [1]. From the main result in the previous section the symmetry group of a configuration is independent of its embedding into $\mathbb{P}^3$.

I set up a method of finding abstract symmetries that fix a given plane. Then I apply it to all three configurations determining their symmetry groups that fix a plane.

In section 4 I will remove the condition that a given plane should be fixed, and, using a matrix argument, discuss the whole symmetry groups.

3.1 Preliminaries

I introduce the concept of an abstract symmetry of a $(16,6)$ configuration as follows:

Concept Let C be a non-degenerate abstract $(16,6)$ configuration, and define $I := \{1, 2, \ldots, 16\}$. Then consider C as a non-degenerate geometric $(16,6)$ configuration with points P_i $\forall i \in I$, and the associated planes H_n $\forall n \in I$ can be considered as sets of 6 points i.e. $H_n := \{P_i | c_{ni} = 1\}$.

We can define the 'Symmetry Group':

$$S_{16} \supset \mathcal{H}_C := \{\sigma \in S_{16} | \forall\ i \in I\ \exists j \in I\ s.t.\ H_j = \{\sigma(P_k) | P_k \in H_i\}\}$$

Note From the definition of a symmetry we automatically have that Rosenhain tetrahedra are mapped to Rosenhain tetrahedra.

Proof. For a non-degenerate $(16,6)$ configuration, a Rosenhain tetrahedron is uniquely determined by 4 points since 3 points uniquely determine a plane.

Hence the result follows directly from the definition of the 'Symmetry Group'.
Q.E.D

There is a natural splitting of the symmetries into those preserving a fixed plane and those that don't. This is a sensible splitting since all planes have the same structure (see the second part of the proof of theorem 1.8). This section explores the symmetries that fix a plane and the next section explores those that do not.

I now repeat the construction used in chapter one in this new context.

Assume that under a symmetry σ a plane H is fixed and it contains the points $P_1, \ldots, P_6$ (numbered 1 to 6 on the diagrams). Each other plane intersects H in one line defined by 2 points so we can denote the 15 other planes $H_{i,j}$ with i and j in $\{1, \ldots, 6\}$. Any point not on H defines 6 planes and from the non-degeneracy we get a cone which intersects H in either a hexagon or 2 triangles with vertices $P_1, \ldots, P_6$.

Any symmetry fixing H will take the set of points $\{P_1, \ldots, P_6\}$ to itself since they lie on and define H. It follows trivially from non-degeneracy that the symmetry group fixing H is a subgroup of S_6. I will determine it in all three specific cases.

3.2 Symmetries of type (*) Configurations

A type (*) configuration can be represented by figure 8, and using the above mentioned construction we can use this figure to see the abstract symmetries, fixing H. What I mean by this is that any permutation of the sixteen points that is a symmetry of the $(16, 6)$ configuration preserving H will preserve the diagram. And similarly any permutation that preserves the diagram is a symmetry of the $(16, 6)$ configuration preserving H.

I now go through one way of finding and checking an element of the symmetry group. Readers who wish to omit this detail, could skip to Statement 3.1.

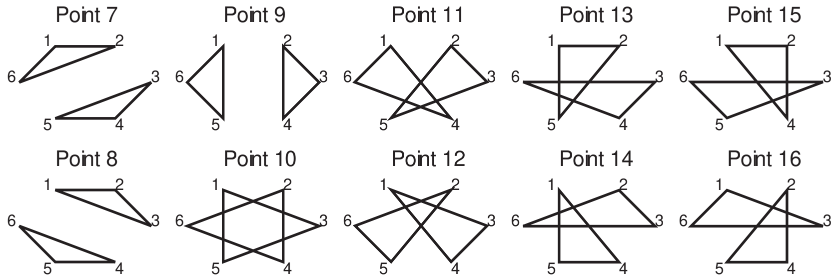


Figure 8: The cones of a type (*) configuration intersecting a fixed plane H.

Using the diagram it is trivial to see that the permutation that is the identity on $\{P_1, \ldots, P_6\}$ is the identity everywhere. This follows directly from the cones of the 10 other points being different, which is itself a direct consequence of non-degeneracy.

Given the transposition $(P_1\ P_2)$ in the group S_6 of the points $\{P_1, \ldots, P_6\}$, can it be extended to a symmetry σ of the $(16, 6)$ configuration? I follow this example through in detail to illustrate a method to find a valid symmetry of the $(16, 6)$ configuration preserving H.

It is useful to note that $\sigma(H_{i,j}) = H_{\sigma(i),\sigma(j)}$ when H is fixed under σ. This follows directly from the bijective correspondence between pairs of points on H and planes other than H. It follows that $\sigma(H_{1,i}) = H_{2,i}$ and $\sigma(H_{2,i}) = H_{1,i}$ for every i in $\{3, \ldots, 6\}$, and that every other plane is fixed under σ.

In this case it is now easy to check directly what $\sigma(P_i)$ is for all i by using figure 8 and what happens to each plane. However a more general method can be used when we have less information.

We can determine what the 6 points lying on $H_{1,3}$ are. P_1 and P_3 obviously lie on $H_{1,3}$. The other four can be found using figure 8 to identify the points in $\{P_7, \ldots, P_{16}\}$ whose cone contains the plane $H_{1,3}$. That is P_8, P_{10}, P_{12} and P_{16}.

If σ is a symmetry of the (16,6) configuration the image of the points of $H_{1,3}$ under σ must be the points of $\sigma(H_{1,3}) = H_{2,3}$. These are P_2, P_3, P_8, P_9, P_{11}, and P_{14}. We know that $\sigma(P_1) = P_2$ and $\sigma(P_3) = P_3$. So we get

$\sigma(\{P_8, P_{10}, P_{12}, P_{16}\}) = \{P_8, P_9, P_{11}, P_{14}\}$. Using $\sigma(H_{2,3}) = H_{1,3}$ in the same way we get $\sigma(\{P_8, P_9, P_{11}, P_{14}\}) = \{P_8, P_{10}, P_{12}, P_{16}\}$. From these two we know that $\sigma(P_8) = P_8$ otherwise we would get a contradiction.

Using the the same method for any plane which is not fixed we get:

- $\sigma(H_{1,4}) = H_{2,4}$ gives $\sigma(\{P_{11}, P_{12}, P_{14}, P_{15}\}) = \{P_9, P_{10}, P_{15}, P_{16}\}$
- $\sigma(H_{2,4}) = H_{1,4}$ gives $\sigma(\{P_9, P_{10}, P_{15}, P_{16}\}) = \{P_{11}, P_{12}, P_{14}, P_{15}\}$
- $\sigma(H_{1,5}) = H_{2,5}$ gives $\sigma(\{P_9, P_{10}, P_{13}, P_{14}\}) = \{P_{11}, P_{12}, P_{13}, P_{16}\}$
- $\sigma(H_{2,5}) = H_{1,5}$ gives $\sigma(\{P_{11}, P_{12}, P_{13}, P_{16}\}) = \{P_9, P_{10}, P_{13}, P_{14}\}$
- $\sigma(H_{1,6}) = H_{2,6}$ gives $\sigma(\{P_7, P_9, P_{11}, P_{16}\}) = \{P_7, P_{10}, P_{12}, P_{14}\}$
- $\sigma(H_{2,6}) = H_{1,6}$ gives $\sigma(\{P_7, P_{10}, P_{12}, P_{14}\}) = \{P_7, P_9, P_{11}, P_{16}\}$

Hence $\sigma(P_{10}) = P_{11}$ since any other value would produce a contradiction. Similarly we get: $\sigma(P_7) = P_7$, $\sigma(P_9) = P_{12}$, $\sigma(P_{11}) = P_{10}$, $\sigma(P_{12}) = P_9$, $\sigma(P_{13}) = P_{13}$, $\sigma(P_{14}) = P_{16}$, $\sigma(P_{15}) = P_{15}$, $\sigma(P_{16}) = P_{14}$. This completes the definition of σ.

To check that no contradictions occur here are the conditions imposed by the planes fixed by σ:

- $\sigma(H_{3,4}) = H_{3,4}$ gives $\sigma(\{P_7, P_9, P_{12}, P_{13}\}) = \{P_7, P_9, P_{12}, P_{13}\}$
- $\sigma(H_{3,5}) = H_{3,5}$ gives $\sigma(\{P_7, P_{10}, P_{11}, P_{15}\}) = \{P_7, P_{10}, P_{11}, P_{15}\}$
- $\sigma(H_{3,6}) = H_{3,6}$ gives $\sigma(\{P_{13}, P_{14}, P_{15}, P_{16}\}) = \{P_{13}, P_{14}, P_{15}, P_{16}\}$
- $\sigma(H_{4,5}) = H_{4,5}$ gives $\sigma(\{P_7, P_8, P_{14}, P_{16}\}) = \{P_7, P_8, P_{14}, P_{16}\}$
- $\sigma(H_{4,6}) = H_{4,6}$ gives $\sigma(\{P_8, P_{10}, P_{11}, P_{13}\}) = \{P_8, P_{10}, P_{11}, P_{13}\}$
- $\sigma(H_{5,6}) = H_{5,6}$ gives $\sigma(\{P_8, P_9, P_{12}, P_{15}\}) = \{P_8, P_9, P_{12}, P_{15}\}$

In disjoint cyclic notation $\sigma = (P_1\ P_2)(P_9\ P_{12})(P_{10}\ P_{11})(P_{14}\ P_{16})$ with respect to points and $\sigma = (H_{1,3}\ H_{2,3})(H_{1,4}\ H_{2,4})(H_{1,5}\ H_{2,5})(H_{1,6}\ H_{2,6})$ with respect to planes. From now on I just state the disjoint cyclic notation, validity can be checked from the figure 8.

One last comment on this method is that it can also be used when very little information about a symmetry is known. Hence it is a very useful exploratory tool. It was particularly useful when adapted to explore symmetries that do not preserve H, but the matrix manipulation presented in section 4 requires considerably less calculations, and gives more insight.

We can similarly check that the following transpositions in S_6 give the following valid symmetries of the (16,6) configuration fixing H. I see no gain for a reader to follow the previous method with these examples. However it is a good exercise in visualising the sixteen to check using figure 8 that these symmetries are valid.

- $(P_1\ P_3) \in S_6$ generates a symmetry which acts on points and planes by:
$$(P_1\ P_3)(P_7\ P_{14})(P_9\ P_{15})(P_{11}\ P_{13})(H_{1,2}\ H_{2,3})$$
$$\circ(H_{1,4}\ H_{3,4})(H_{1,5}\ H_{3,5})(H_{1,6}\ H_{3,6})$$
- $(P_1\ P_4) \in S_6$ generates a symmetry which acts on points and planes by:
$$(P_1\ P_4)(P_7\ P_{10})(P_8\ P_9)(P_{13}\ P_{16})(H_{1,2}\ H_{2,4})$$
$$\circ(H_{1,3}\ H_{3,4})(H_{1,5}\ H_{4,5})(H_{1,6}\ H_{4,6})$$
- $(P_1\ P_5) \in S_6$ generates a symmetry which acts on points and planes by:
$$(P_1\ P_5)(P_7\ P_{12})(P_8\ P_{11})(P_{15}\ P_{16})(H_{1,2}\ H_{2,5})$$
$$\circ(H_{1,3}\ H_{3,5})(H_{1,4}\ H_{4,5})(H_{1,6}\ H_{5,6})$$
- $(P_1\ P_6) \in S_6$ generates a symmetry which acts on points and planes by:
$$(P_1\ P_6)(P_8\ P_{14})(P_{10}\ P_{15})(P_{12}\ P_{13})(H_{1,2}\ H_{2,6})$$
$$\circ(H_{1,3}\ H_{3,6})(H_{1,4}\ H_{4,6})(H_{1,5}\ H_{5,6})$$

Since span($\{(P_1\ P_2), (P_1\ P_3), (P_1\ P_4), (P_1\ P_5), (P_1\ P_5)\}$) is isomorphic to S_6 and we noted above that the 'symmetry group' preserving H of every configuration is contained in S_6, we have proved the following statement.

Statement 3.1 *The group of symmetries that preserve a given plane H of a non-degenerate (16,6) configuration of type (*) is isomorphic to* S_6.

This result was found by [1] but the symmetries were not found explicitly. They were simply counted. Using the same methods as for the type (*) configuration we can now look at both exotic configurations.

3.3 Symmetries of type A Configurations

The same reasoning holds to show that the group of symmetries that fix H is isomorphic to a subgroup of S_6. To see this we need to check that no two points have the same cone in figure 9; we know this to be the case from non-degeneracy. I now present a set of elements that generate the relevant subgroup of S_6. Then I state what the subgroup is and give a separate explanation of its origin.

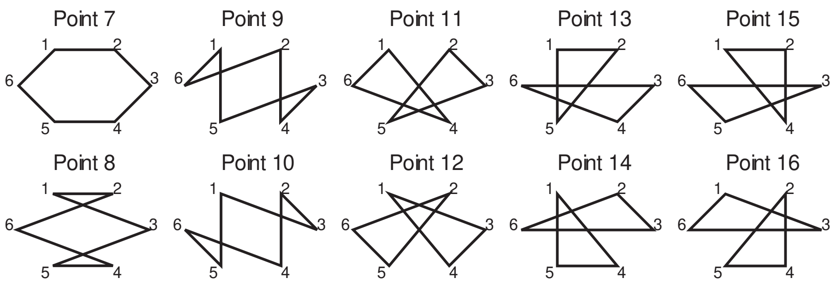


Figure 9: The cones of a type A configuration intersecting a fixed plane H.

I do not go through all the details of the exploration. To see that the set of symmetries is strictly smaller than S_6, note that a symmetry swapping P_1 with P_2 and fixing the points P_3, P_4, P_5, and P_6 takes the cone of the point 7 to a cone that does not belong to the configuration. Hence this is invalid.

It becomes valid when composed with the symmetry swapping P_4 with P_5. In this way we get the following symmetries in disjoint cyclic notation acting on points, the action on planes follows trivially:

- $(P_1\ P_4)(P_7\ P_{10})(P_8\ P_9)(P_{13}\ P_{16})$
- $(P_2\ P_5)(P_7\ P_9)(P_8\ P_{10})(P_{14}\ P_{15})$
- $(P_3\ P_6)(P_7\ P_8)(P_9\ P_{10})(P_{11}\ P_{12})$
- $(P_1\ P_2)(P_4\ P_5)(P_7\ P_8)(P_{11}\ P_{12})(P_{13}\ P_{15})(P_{14}\ P_{16})$
- $(P_1\ P_5)(P_2\ P_4)(P_9\ P_{10})(P_{11}\ P_{12})(P_{13}\ P_{14})(P_{15}\ P_{16})$
- $(P_1\ P_3)(P_4\ P_6)(P_8\ P_{10})(P_{11}\ P_{13})(P_{12}\ P_{16})(P_{14}\ P_{15})$
- $(P_1\ P_6)(P_3\ P_4)(P_7\ P_9)(P_{11}\ P_{16})(P_{12}\ P_{13})(P_{14}\ P_{15})$
- $(P_2\ P_3)(P_5\ P_6)(P_7\ P_{10})(P_{11}\ P_{14})(P_{12}\ P_{15})(P_{13}\ P_{16})$
- $(P_2\ P_6)(P_3\ P_5)(P_8\ P_9)(P_{11}\ P_{15})(P_{12}\ P_{14})(P_{13}\ P_{16})$

It is easy to check that any symmetry not generated by these is invalid, using the figure 9.

There is a $\mathbb{Z}_2$ subgroup which is defined by the involution σ which is the composition of the first three symmetries above;

$$\sigma := (P_1\ P_4)(P_2\ P_5)(P_3\ P_6)(P_{11}\ P_{12})(P_{14}\ P_{15})(P_{13}\ P_{16})$$

Moreover, there is an $\mathbb{S}_4$ subgroup generated by the last six symmetries above. This can be seen as an $\mathbb{S}_4$ action on the points $\{P_7, P_8, P_9, P_{10}\}$. One checks that these $\mathbb{Z}_2$ and S_4 type subgroups commute, implying the following statement:

Statement 3.2 *The group of symmetries that fix H of a non-degenerate $(16,6)$ configuration of type A is isomorphic to $\mathbb{Z}_2 \times S_4$.*

3.4 Symmetries of type B Configurations

This case could be understood in the same way as the previous two by direct calculation but the following description gives more insight into the nature of this configuration. We note from figure 10 that, as in the previous cases, we are looking at a subgroup of S_6.

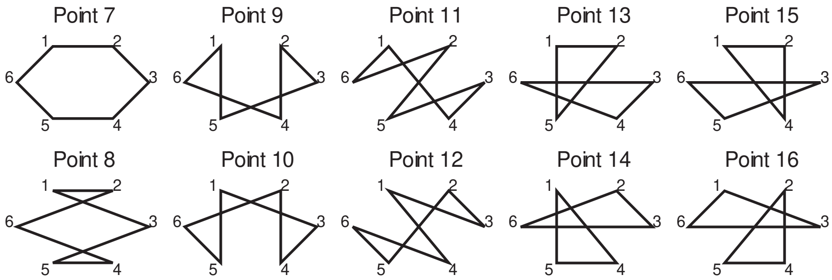


Figure 10: The cones of a type B configuration intersecting a fixed plane H.

Using the fact that tetrahedra get mapped onto tetrahedra by symmetries of a (16,6) configuration we get that for a symmetry fixing H the set of points $\{P_{13}, P_{14}, P_{15}, P_{16}\}$ must be mapped to itself, and they all lie on $H_{3,6}$ so this plane must also be preserved by the action. We know that the 'symmetry group' is a subgroup of S_6 on the points $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ and it follows from the invariance of $H_{3,6}$ that we are now looking for a subgroup of $\mathbb{Z}_2 \times S_4$. Namely the two distinct actions: $\mathbb{Z}_2$ corresponds to the permutations of the points $\{P_3, P_6\}$, and S_4 gives the permutations of $\{P_1, P_2, P_4, P_5\}$. In the S_4 action we are looking at the symmetries of a standard $(8, 4)$ configuration fixing a plane. I will mention this again below and explain it in section 4.

Actually we don't have the full S_4 action on the points $\{P_1, P_2, P_4, P_5\}$, we have the action of even permutations A_4. This restriction comes from the way the second standard $(8, 4)$ configuration interacts with the first. I will go into much more detail in section 4.

The other way to read off the symmetry group is to check directly. I will write down the generators for this case to help with the overall picture, these can be used to check that the description of the group is accurate.

- $(P_3\ P_6)(P_7\ P_8)(P_9\ P_{10})(P_{11}\ P_{12})$
- $(P_1\ P_4)(P_2\ P_5)(P_7\ P_8)(P_{11}\ P_{12})(P_{13}\ P_{16})(P_{14}\ P_{15})$
- $(P_1\ P_2)(P_4\ P_5)(P_7\ P_8)(P_9\ P_{10})(P_{13}\ P_{15})(P_{14}\ P_{16})$
- $(P_1\ P_5)(P_2\ P_4)(P_9\ P_{10})(P_{11}\ P_{12})(P_{13}\ P_{14})(P_{15}\ P_{16})$
- $(P_1\ P_2\ P_4)(P_7\ P_{10}\ P_{12})(P_8\ P_9\ P_{11})(P_{13}\ P_{16}\ P_{14})$
- $(P_1\ P_4\ P_2)(P_7\ P_{12}\ P_{10})(P_8\ P_{11}\ P_9)(P_{13}\ P_{14}\ P_{16})$
- $(P_1\ P_2\ P_5)(P_7\ P_{11}\ P_{10})(P_8\ P_{12}\ P_9)(P_{14}\ P_{15}\ P_{16})$
- $(P_1\ P_5\ P_2)(P_7\ P_{10}\ P_{11})(P_8\ P_9\ P_{12})(P_{14}\ P_{16}\ P_{15})$
- $(P_1\ P_4\ P_5)(P_7\ P_9\ P_{12})(P_8\ P_{10}\ P_{11})(P_{13}\ P_{15}\ P_{16})$
- $(P_1\ P_5\ P_4)(P_7\ P_{12}\ P_9)(P_8\ P_{11}\ P_{10})(P_{13}\ P_{16}\ P_{15})$
- $(P_2\ P_4\ P_5)(P_7\ P_{11}\ P_9)(P_8\ P_{12}\ P_{10})(P_{13}\ P_{15}\ P_{14})$
- $(P_2\ P_5\ P_4)(P_7\ P_9\ P_{11})(P_8\ P_{10}\ P_{12})(P_{13}\ P_{14}\ P_{15})$

Given the necessary checks the following statement can be proved.

Statement 3.3 *The group of symmetries of a non-degenerate* $(16, 6)$ *configuration of type B that fix H is isomorphic to* $\mathbb{Z}_2 \times A_4$.

4 Matrix representation of a (16,6) configuration

In this section I present ideas on the matrix representation, then go back to exploring symmetries of the $(16, 6)$ configuration. Through the matrix description of the configurations I find a set of generators of the entire symmetry groups for each configuration.

In the end of this section I provide visual aids for each of the three types of configurations. By this I mean a geometrical representation as we have used previously and a corresponding matrix. The three matrices appear to be very similar. These are intended as tools to understanding the schematic proofs. To keep the flow of the argument I have left out the details of most proofs, which are not hard.

4.1 Preliminaries

We have so far considered the $(16, 6)$ configurations as geometrical objects, this is because generally the matrices are unwieldy objects. However they give insight in a number of ways. A lot of the proofs in this section follow from an understanding of general (a, b) configurations and in particular what the non-degeneracy condition means. I will not explain the details of these proofs since my main interest is the geometry of the configurations, and the proofs offer little insight.

As with any abstract idea it is important to find a meaningful representation. In this case putting the matrices in a form that displays as many independent[4] Rosenhain tetrahedra as possible gives the matrices more geometric meaning. In each of the three configurations it is always possible to extend a set of independent Rosenhain tetrahedra until it contains 4 elements. And moreover any pair of such Rosenhain tetrahedra form an $(8, 4)$ configuration. There are other interesting ways to represent the matrix. In particular to see the possible standard $(8, 4)$ configurations in type A the

[4]By independent I mean that they have no points or planes in common.

matrix can be split into 12 $(4,2)$ configurations and 4 (4,0) configurations, see figure 14.

Once we have a matrix representing a configuration, abstract symmetries of this configuration correspond to sets of row and column operations that leave the matrix representation invariant.

In each configuration it is possible given a standard $(8,4)$ configuration to find a second independent standard $(8,4)$ configuration. Given the form of the main matrices in section 4.3 the result is obvious given either the 'upper-left' or 'lower-right' $(8,4)$ configurations. I ask the reader to accept or check that these standard $(8,4)$ configurations are not special in any way. Remark 2.5 is a consequence.

As an aside, in each configuration it is not difficult to see the number of Rosenhain tetrahedra directly in the matrix, or to deduce it from the number of standard $(8,4)$ configurations containing a given point or plane. Everything comes down to standard $(8,4)$ configurations and how they fit together.

4.2 The Symmetries

A symmetry of a $(16,6)$ configuration acts on a matrix representing the $(16,6)$ configuration by taking it to itself. There are many ways to think of this. Above I said this could be done using sets of row and column operations. These can be seen as a pair of $(16,1)$ configurations which act by matrix multiplication one on the left and the other on the right. Then the symmetry group of an abstract $(16,6)$ configuration A is simply the set of pairs of $(16,1)$ configurations which leave A invariant.

The groups obtained in section 3 did not depend on the choice of fixing the plane H, if any other plane or point had been fixed the same groups would have been obtained[5]. Both type (*) and type A have some duality between planes and points, i.e. their matrices can be made symmetric. This

[5]This follows from the second part of the main proof in section 1. That shows that the diagrams did not depend on the choice of H.

makes the equivalence of the groups obvious. For type B configurations it is also visible in the matrices, or can be shown by direct calculation.

Finding symmetries that can take H to any other plane is the next step. For the type (*) configuration this is easy but I will use a method that can be generalised. I will not go through exactly how I find symmetries but the idea is simply to read off from the matrices a set of transpositions that leave the matrix invariant.

The first symmetry must take the first row to the second.

$$(P_2\ P_{16})(P_4\ P_5)(P_1\ P_{14})(P_{13}\ P_{15})(P_3\ P_7)(P_9\ P_{11})(P_6\ P_8)(P_{10}\ P_{12})$$
$$\circ(H\ H_{4,5})(H_{2,5}\ H_{2,4})(H_{3,6}\ H_{1,2})(H_{1,4}\ H_{1,5})(H_{1,6}\ H_{2,3})$$
$$\circ(H_{3,4}\ H_{3,5})(H_{1,3}\ H_{2,6})(H_{4,6}\ H_{5,6})$$

The second symmetry must take the first two rows to the second two rows.

$$(P_4\ P_{16})(P_2\ P_5)(P_1\ P_{13})(P_{14}\ P_{15})(P_3\ P_{11})(P_7\ P_9)(P_6\ P_{12})(P_8\ P_{10})$$
$$\circ(H\ H_{2,5})(H_{4,5}\ H_{2,4})(H_{3,6}\ H_{1,4})(H_{1,2}\ H_{1,5})(H_{1,6}\ H_{3,4})$$
$$\circ(H_{2,3}\ H_{3,5})(H_{1,3}\ H_{4,6})(H_{2,6}\ H_{5,6})$$

The third symmetry must take the first four rows to the second four rows.

$$(P_1\ P_{16})(P_2\ P_{14})(P_4\ P_{13})(P_5\ P_{15})(P_3\ P_6)(P_7\ P_8)(P_{11}\ P_{12})(P_9\ P_{10})$$
$$\circ(H\ H_{3,6})(H_{4,5}\ H_{1,2})(H_{2,5}\ H_{1,4})(H_{2,4}\ H_{1,5})(H_{1,6}\ H_{1,3})$$
$$\circ(H_{2,3}\ H_{2,6})(H_{3,4}\ H_{4,6})(H_{3,5}\ H_{5,6})$$

The fourth symmetry must take the first eight rows to the second eight rows.

$$(P_3\ P_{16})(P_2\ P_7)(P_4\ P_{11})(P_5\ P_9)(P_1\ P_6)(P_{14}\ P_8)(P_{13}\ P_{12})(P_{15}\ P_{10})$$
$$\circ(H\ H_{1,6})(H_{4,5}\ H_{2,3})(H_{2,5}\ H_{3,4})(H_{2,4}\ H_{3,5})(H_{3,6}\ H_{1,3})$$
$$\circ(H_{1,2}\ H_{2,6})(H_{1,4}\ H_{4,6})(H_{1,5}\ H_{5,6})$$

These four symmetries generate a symmetry group $G := \mathbb{Z}_2^4$ that takes the plane H to any other plane. It is easy to see that the full symmetry group G_* of a type (*) configuration can be obtained by taking the span of G and the group $G' \cong S_6$ of symmetries that fix H. In fact G_* is defined by the following exact sequence:

$$1 \longrightarrow G \longrightarrow G_* \longrightarrow G' \longrightarrow 1$$

This is the same description that was found in [1]. The method described above can also be used in the exotic cases to find generators that can take the plane H to any other plane.

For the type A configuration:

The first symmetry must take the first row to the second but it is not possible to find a symmetry leaving nothing invariant satisfying this.

$$(P_1\ P_{16})(P_2\ P_{14})(P_3\ P_7)(P_{10}\ P_{11})(P_9\ P_{12})(P_6\ P_8)(H\ H_{4,5})$$

$$\circ(H_{2,5}\ H_{1,5})(H_{3,6}\ H_{1,2})(H_{1,4}\ H_{2,4})(H_{1,6}\ H_{1,3})(H_{3,5}\ H_{5,6})$$

The second symmetry must take the first two rows to the second two rows.[6]

$$(P_4\ P_{16})(P_2\ P_5)(P_1\ P_{13})(P_{14}\ P_{15})(P_3\ P_{11})(P_7\ P_9)(P_6\ P_{12})(P_8\ P_{10})$$

$$\circ(H\ H_{2,5})(H_{4,5}\ H_{2,4})(H_{3,6}\ H_{1,4})(H_{1,2}\ H_{1,5})(H_{1,6}\ H_{3,4})$$
$$\circ(H_{2,3}\ H_{3,5})(H_{1,3}\ H_{4,6})(H_{2,6}\ H_{5,6})$$

The third symmetry must take the first four rows to the second four rows.[7]

$$(P_1\ P_{16})(P_2\ P_{14})(P_4\ P_{13})(P_5\ P_{15})(P_3\ P_6)(P_7\ P_8)(P_{11}\ P_{12})(P_9\ P_{10})$$

$$\circ(H\ H_{3,6})(H_{4,5}\ H_{1,2})(H_{2,5}\ H_{1,4})(H_{2,4}\ H_{1,5})(H_{1,6}\ H_{1,3})$$

[6] This is the same as for the type (*) configuration.
[7] This is the same as for the type (*) configuration.

$$\circ(H_{2,3}\ H_{2,6})(H_{3,4}\ H_{4,6})(H_{3,5}\ H_{5,6})$$

The fourth symmetry must take the first eight rows to the second eight rows. No reflection satisfies this, but if we apply the fourth symmetry for the type (*) configuration we can then 'correct' the matrix by applying $(P_2\ P_5)(P_{14}\ P_{15})(H_{2,3}\ H_{3,5})(H_{2,6}\ H_{5,6})$ which gives:

$$(P_3\ P_{16})(P_2\ P_7\ P_5\ P_9)(P_4\ P_{11})(P_1\ P_6)(P_{14}\ P_8\ P_{15}\ P_{10})(P_{13}\ P_{12})$$

$$\circ(H\ H_{1,6})(H_{2,3}\ H_{4,5}\ H_{3,5}\ H_{2,4})(H_{2,5}\ H_{3,4})(H_{3,6}\ H_{1,3})$$

$$\circ(H_{2,6}\ H_{1,2}\ H_{5,6}\ H_{1,5})(H_{1,4}\ H_{4,6})$$

Again these 4 symmetries are enough to go from a plane H to any other plane, hence the entire symmetry group is the span of these symmetries and $\mathcal{A} \cong \mathbb{Z}_2 \times S_4$ the symmetries fixing H. Unlike in the previous case the intersection of the spans of both symmetry groups is not just the identity[8], so it can't be written as an easy exact sequence.

For the type B configuration:

The first symmetry must take the first row to the second row but it is not possible to find a symmetry leaving nothing invariant satisfying this.

$$(P_1\ P_{16})(P_2\ P_{14})(P_3\ P_7)(P_{10}\ P_{11})(P_9\ P_{12})(P_6\ P_8)(H\ H_{4,5})$$

$$\circ(H_{2,5}\ H_{1,5})(H_{3,6}\ H_{1,2})(H_{1,4}\ H_{2,4})(H_{1,6}\ H_{1,3})(H_{3,5}\ H_{5,6})$$

The second symmetry must take the first two rows to the second two rows, but it is not possible to find a symmetry leaving nothing invariant satisfying this.

$$(P_1\ P_{16})(P_4\ P_{13})(P_3\ P_{11})(P_7\ P_{10})(P_6\ P_{12})(P_8\ P_9)(H\ H_{2,5})$$

$$\circ(H_{4,5}\ H_{1,5})(H_{3,6}\ H_{1,4})(H_{1,2}\ H_{2,4})(H_{1,6}\ H_{1,3})(H_{2,3}\ H_{2,6})$$

[8]It contains the square of the fourth symmetry and an element which is the conjugation of the first by both the second and the third symmetries.

The third symmetry must take the first four rows to the second four rows.[9]

$$(P_1\ P_{16})(P_2\ P_{14})(P_4\ P_{13})(P_5\ P_{15})(P_3\ P_6)(P_7\ P_8)(P_{11}\ P_{12})(P_9\ P_{10})$$

$$\circ(H\ H_{3,6})(H_{4,5}\ H_{1,2})(H_{2,5}\ H_{1,4})(H_{2,4}\ H_{1,5})(H_{1,6}\ H_{1,3})$$

$$\circ(H_{2,3}\ H_{2,6})(H_{3,4}\ H_{4,6})(H_{3,5}\ H_{5,6})$$

The fourth symmetry must take the first eight rows to the second eight rows. No reflection satisfies this, but if we apply the fourth symmetry for the type (*) configuration we can then 'correct' the matrix by applying $(P_2\ P_4\ P_5)(P_{14}\ P_{13}\ P_{15})(H_{2,3}\ H_{3,4}\ H_{3,5})(H_{2,6}\ H_{4,6}\ H_{5,6})$ which gives:

$$(P_3\ P_{16})(P_2\ P_7\ P_4\ P_{11}\ P_5\ P_9)(P_1\ P_6)(P_{14}\ P_8\ P_{13}\ P_{12}\ P_{15}\ P_{10})$$

$$\circ(H\ H_{1,6})(H_{2,3}\ H_{4,5}\ H_{3,4}\ H_{2,5}\ H_{3,5}\ H_{2,4})(H_{3,6}\ H_{1,3})$$

$$\circ(H_{2,6}\ H_{1,2}\ H_{4,6}\ H_{1,4}\ H_{5,6}\ H_{1,5})$$

Again these 4 symmetries are enough to go from a plane H to any other plane, hence the entire symmetry group is the span of these symmetries and $\mathcal{B} \cong \mathbb{Z}_2 \times A_4$ the symmetries fixing H. Again the intersection of the spans of both symmetry groups is not just the identity, so it can't be written as an easy exact sequence.

We have found sets of symmetries for each of the exotic cases that generate their entire symmetry groups. But the groups that take H to any other point are not $\mathbb{Z}_2^4$, and I leave it to the reader to check that for both exotic cases no such $\mathbb{Z}_2^4$ could be found. This means that the exotic configurations do not correspond to Kummer surfaces.

However if we quotient[10] the groups out by the symmetries that fix the plane H, we get[11] $\mathbb{Z}_2 * \mathbb{Z}_2^3$ for type A and $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2^2$ for type B.

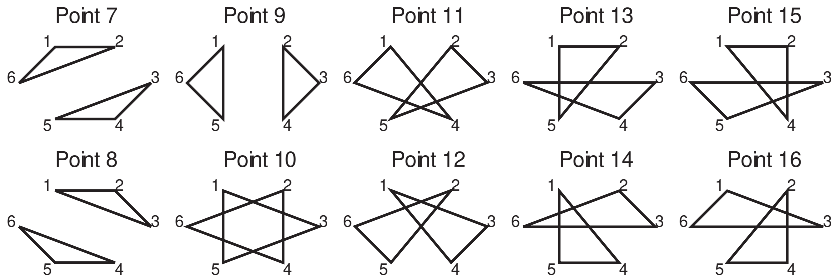
[9]This is the same as for the type (*) configuration.

[10]It is not clear that this is valid it is intended merely as an informal idea to give a bit more understanding of the interaction between the groups.

[11]I use $*$ to denote a product, that is probably not direct, but I have not had time to check.

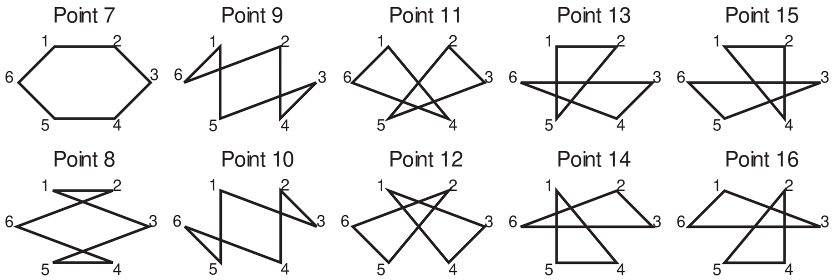
4.3 Extra visual aids

The matrix representations given here are chosen so that all three link to their corresponding geometrical representation in the same way. There are other matrices obtained from these by permutation of rows and columns such as figure 14. The difference in what is easy to see is amazing when the matrix is changed by permutations of rows and columns.



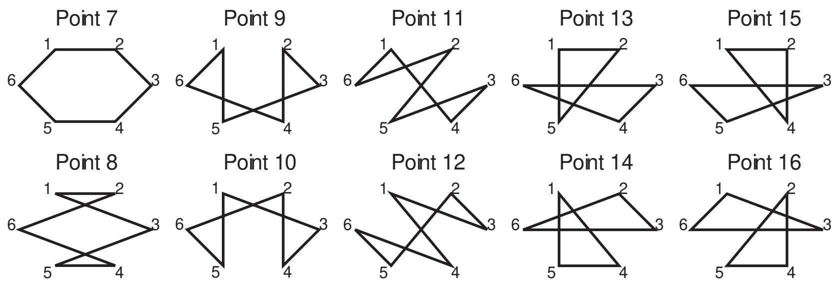
	P_{16}	P_2	P_4	P_5	P_1	P_{14}	P_{13}	P_{15}	P_3	P_7	P_{11}	P_9	P_6	P_8	P_{12}	P_{10}
H	0	1	1	1	1	0	0	0	1	0	0	0	1	0	0	0
H_{45}	1	0	1	1	0	1	0	0	0	1	0	0	0	1	0	0
H_{25}	1	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0
H_{24}	1	1	1	0	0	0	0	1	0	0	0	1	0	0	0	1
H_{36}	1	0	0	0	0	1	1	1	1	0	0	0	1	0	0	0
H_{12}	0	1	0	0	1	0	1	1	0	1	0	0	0	1	0	0
H_{14}	0	0	1	0	1	1	0	1	0	0	1	0	0	0	1	0
H_{15}	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	1
H_{16}	1	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0
H_{23}	0	1	0	0	0	1	0	0	1	0	1	1	0	1	0	0
H_{34}	0	0	1	0	0	0	1	0	1	1	0	1	0	0	1	0
H_{35}	0	0	0	1	0	0	0	1	1	1	1	0	0	0	0	1
H_{13}	1	0	0	0	1	0	0	0	1	0	0	0	0	1	1	1
H_{26}	0	1	0	0	0	1	0	0	0	1	0	0	1	0	1	1
H_{46}	0	0	1	0	0	0	1	0	0	0	1	0	1	1	0	1
H_{56}	0	0	0	1	0	0	0	1	0	0	0	1	1	1	1	0

Figure 11: A type (*) configuration.



	P_{16}	P_2	P_4	P_5	P_1	P_{14}	P_{13}	P_{15}	P_3	P_7	P_{11}	P_9	P_6	P_8	P_{12}	P_{10}
H	0	1	1	1	1	0	0	0	1	0	0	0	1	0	0	0
H_{45}	1	0	1	1	0	1	0	0	0	1	0	0	0	1	0	0
H_{25}	1	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0
H_{24}	1	1	1	0	0	0	0	1	0	0	0	1	0	0	0	1
H_{36}	1	0	0	0	0	1	1	1	1	0	0	0	1	0	0	0
H_{12}	0	1	0	0	1	0	1	1	0	1	0	0	0	1	0	0
H_{14}	0	0	1	0	1	1	0	1	0	0	1	0	0	0	1	0
H_{15}	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	1
H_{16}	1	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0
H_{23}	0	1	0	0	0	1	0	0	1	1	1	0	0	0	0	1
H_{34}	0	0	1	0	0	0	1	0	1	1	0	1	0	0	1	0
H_{35}	0	0	0	1	0	0	0	1	1	0	1	1	0	1	0	0
H_{13}	1	0	0	0	1	0	0	0	1	0	0	0	0	1	1	1
H_{26}	0	1	0	0	0	1	0	0	0	0	0	1	1	1	1	0
H_{46}	0	0	1	0	0	0	1	0	0	0	1	0	1	1	0	1
H_{56}	0	0	0	1	0	0	0	1	0	1	0	0	1	0	1	1

Figure 12: A type A configuration.



	P_{16}	P_2	P_4	P_5	P_1	P_{14}	P_{13}	P_{15}	P_3	P_7	P_{11}	P_9	P_6	P_8	P_{12}	P_{10}
H	0	1	1	1	1	0	0	0	1	0	0	0	1	0	0	0
H_{45}	1	0	1	1	0	1	0	0	0	1	0	0	0	1	0	0
H_{25}	1	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0
H_{24}	1	1	1	0	0	0	0	1	0	0	0	1	0	0	0	1
H_{36}	1	0	0	0	0	1	1	1	1	0	0	0	1	0	0	0
H_{12}	0	1	0	0	1	0	1	1	0	1	0	0	0	1	0	0
H_{14}	0	0	1	0	1	1	0	1	0	0	1	0	0	0	1	0
H_{15}	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	1
H_{16}	1	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0
H_{23}	0	1	0	0	0	1	0	0	1	1	0	1	0	0	1	0
H_{34}	0	0	1	0	0	0	1	0	1	1	1	0	0	0	0	1
H_{35}	0	0	0	1	0	0	0	1	1	0	1	1	0	1	0	0
H_{13}	1	0	0	0	1	0	0	0	1	0	0	0	0	1	1	1
H_{26}	0	1	0	0	0	1	0	0	0	0	1	0	1	1	0	1
H_{46}	0	0	1	0	0	0	1	0	0	0	0	1	1	1	1	0
H_{56}	0	0	0	1	0	0	0	1	0	1	0	0	1	0	1	1

Figure 13: A type B configuration.

	P_{16}	P_2	P_1	P_{14}	P_4	P_5	P_{13}	P_{15}	P_{11}	P_7	P_{12}	P_8	P_3	P_9	P_6	P_{10}
H_{25}	1	1	0	0	0	1	1	0	1	0	1	0	0	0	0	0
H_{45}	1	0	0	1	1	1	0	0	0	1	0	1	0	0	0	0
H_{14}	0	0	1	1	1	0	0	1	1	0	1	0	0	0	0	0
H_{12}	0	1	1	0	0	0	1	1	0	1	0	1	0	0	0	0
H	0	1	1	0	1	1	0	0	0	0	0	0	1	0	1	0
H_{24}	1	1	0	0	1	0	0	1	0	0	0	0	0	1	0	1
H_{36}	1	0	0	1	0	0	1	1	0	0	0	0	1	0	1	0
H_{15}	0	0	1	1	0	1	1	0	0	0	0	0	0	1	0	1
H_{16}	1	0	1	0	0	0	0	0	1	1	0	0	0	1	1	0
H_{23}	0	1	0	1	0	0	0	0	1	1	0	0	1	0	0	1
H_{13}	1	0	1	0	0	0	0	0	0	0	1	1	1	0	0	1
H_{26}	0	1	0	1	0	0	0	0	0	0	1	1	0	1	1	0
H_{34}	0	0	0	0	1	0	1	0	0	1	1	0	1	1	0	0
H_{35}	0	0	0	0	0	1	0	1	1	0	0	1	1	1	0	0
H_{46}	0	0	0	0	1	0	1	0	1	0	0	1	0	0	1	1
H_{56}	0	0	0	0	0	1	0	1	0	1	1	0	0	0	1	1

Figure 14: A different matrix of a type A configuration, that illustrates choices of $(8, 4)$ configurations.

Conclusion

My main aim was to investigate the exotic (16,6) configurations. In the first section I went through the classification of non-degenerate $(16, 6)$ configurations. This and the second section were both strongly based on [1]. The second section states the main result from her work, and sketches her proof.

The third section explored the symmetries using a geometric representation. This can be done for any symmetry, but it is particularly useful for symmetries that fix a given plane, since everything can then be read from the diagrams introduced there. In fact the diagrams directly give the fact that the symmetry group fixing a given plane of each configuration is a subgroup of S_6.

An interesting point is the viewing of the symmetries in terms of the subconfigurations, particularly the Rosenhain tetrahedra and the different $(8, 4)$ configurations. This is an idea that was taken further in section 4. There my main aims were to show the insight that can be gained from the matrices, and describe the entire symmetry groups.

It was disappointing to realise that the exotic configurations do not correspond to Kummer surfaces. But it would still be interesting to find out what space these configurations live in.

References

[1] Maria R. Gonzalez-Dorrego, $(16, 6)$ *configurations and geometry of kummer surfaces in* $\mathbb{P}^3$, Memoirs of the American Mathematical Society **107** (1994), no. 512, ISSN 0065-9266.

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[3] R.W.H.T.Hudson, *Kummer's quartic surface*, Cambridge University Press, 1905, ISBN 0 521 39790 1.