CALABI-YAU MANIFOLDS AND THE BORCEA-VOISIN CONSTRUCTION

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Dedicated to Samuel Thacker

ABSTRACT. In this project we examine methods of constructing higher dimensional Calabi-Yau mirror manifolds using orbifolding procedures on one- and two-folds. We start with explanations of the geometry, topology and analysis needed, and proceed to the Borcea-Voisin construction of three-folds by involution and blow-up. We then consider possible extensions of the construction using different starting manifolds, and then using automorphisms of order four. We assume basic knowledge of manifold theory, tangent spaces, and topology.

1. MOTIVATIONS

An in-depth discussion of applications of Calabi-Yau manifolds to string theory is beyond the scope of this project; suffice it to say that uncovering tractable examples of Calabi-Yau 3-folds provides physicists with models for a higher-dimensional universe. Conceptually we imagine attaching such a manifold to each point in spacetime, but the manifolds are "small" enough to be modelled as points to a reasonable degree of accuracy by quantum mechanics. Calculations in string theory are often so complicated that they can only be solved by taking approximations; indeed the equations themselves must often be approximated. However, physicists have found that calculations on *mirror manifolds* yield the same results but are often much easier to perform. Thus classification of mirror pairs of Calabi-Yau 3-folds is of great interest to physicists. More about the applications of mirror manifolds can be found in [1].

2. COHOMOLOGY

Much of this section, including notation, is based on [2]. Let X denote a real manifold and consider the set

$$\Omega^1(X) = \{\text{1-forms on } X\}.$$

as a vector space over the smooth functions

$$C^\infty(X) = \{\text{infinitely differentiable functions on } X\}.$$

If X is a (real) n-manifold with local co-ordinates $x_1, ..., x_n$, then the space of 1-forms has a basis

$$\{dx_1, ..., dx_n\}.$$

We define an exterior product between elements of the space $\Omega^1(X)$, denoted by juxtaposition, starting with the usual tensor product and demanding the following relations hold:

- ${dx_i}^2 = 0$

- $dx_i dx_j = -dx_j dx_i$.

This allows us to locally define the vector space of p-forms by

$$\Omega^p(X) = span\{dx_{i^1}...dx_{i^p} | i_1, ..., i_p \in \{1, .., n\}\}$$

and thus we have the *algebra* of differential forms

$$\Omega^*(X) = \bigoplus_{p=1}^{\infty} \Omega^p(X).$$

We note that a typical p-form w can then be written (in local co-ordinates)

$$w = \sum_{1 \le i^1, ..., i^p \le n} f_{i^1 ... i^p} dx_{i^1} ... dx_{i^p}$$

where $f_{i^1...i^p}$ is a smooth function $X \to \mathbb{R}$. The uniqueness of this expression is guaranteed up to $dx_i dx_j = -dx_j dx_i$. Now we define the *exterior derivative* d as a map $\Omega^p(X) \to \Omega^{p+1}(X)$ as follows: if w is a p-form written as above, then

$$dw = \sum_{1 \le i^1, ..., i^{p+1} \le n} \frac{df_{i^1 ... i^p}}{dx_{i^{p+1}}} dx_{i^{p+1}} dx_{i^1} ... dx_{i^p}.$$

Proposition 2.1. $d^2 = 0$.

Proof : This follows immediately from the facts that mixed partials of the smooth function $f_{i^1...i^p}$ are equal, and that ${dx_i}^2 = 0$. We see that every term in the local expression of $d^2 w$ will contain either two copies of dx_a and will thus be zero, or is cancelled by a symmetrical term. □

Definition 2.2. If $w \in \Omega^p(X)$ then we say w is *closed* if $dw = 0$.
We say w is *exact* if $\exists u \in \Omega^{p-1}(X)$ s.t. $du = w$.

It clearly follows that w exact $\Rightarrow$ w closed, since $dw = d^2 u = 0$.

Definition 2.3. We define the *p-th DeRham cohomology* on X by

$$H^p(X) = \frac{\{w \in \Omega^p(X) | dw = 0\}}{\{dw | w \in \Omega^{p-1}(X)\}} = \frac{\{\text{closed } p\text{-forms on } X\}}{\{\text{exact } p\text{-forms on } X\}}.$$

$H^p(X)$ is called the *pth cohomology class of X*.

We note here that for $p = 0$, there are no $p-1$ forms, so we just have $H^0(X) = \{w \in \Omega^0(X) | dw = 0\}$ ={closed 0-forms on X}={locally constant real-valued functions on X}, so we see that when X is connected, $H^0(X) \cong \mathbb{R}$. We note also that $H^q(X)$ is considered as a vector space over $\mathbb{R}$ rather than over $C^\infty(X)$.

One important fact about cohomology structure is its invariance under diffeomorphism, which we prove with the following lemmas:

Lemma 2.4. *Let X and Y be real n-manifolds and $\phi : Y \to X$ a diffeomorphism. Then ϕ induces an isomorphism $\phi^p : \Omega^p(X) \to \Omega^p(Y)$.*

Proof: Using local co-ordinates, let

$$w = \sum_{1 \le i^1, ..., i^p \le n} f_{i^1 ... i^p} dx_{i^1} ... dx_{i^p} \in \Omega^p(X)$$

and define

$$\phi^p(w) = \sum_{1 \le j^1, ..., j^p \le n} (f_{j^1 ... j^p} \circ \phi) dy_{j^1} ... dy_{j^p} \in \Omega^p(Y)$$

where each x_i is the component $y_i \circ \phi^{-1}$.
Aside : We note that we can combine these maps ϕ^p together into a map

$$\phi^* : \Omega^*(X) \to \Omega^*(Y)$$

by decomposing a form w

$$w = w_0 + ... + w_n$$

where w_i is an i-form, and defining

$$\phi^*(w) = \phi^0(w_1) + ... + \phi^n(w_n).$$

So because each of the ϕ^is an isomorphism, ϕ^* will be one too. It remains to show that ϕ^p is an isomorphism. Let $g, h \in C^\infty(X)$. Then if we let

$$w' = \sum_{1 \leq i^1, \ldots, i^p \leq n} f'_{i^1 \ldots i^p} dx_{i^1} \ldots dx_{i^p} \in \Omega^p(X)$$

then we have

$$\begin{aligned}
\phi^p(gw + hw') &= \phi^p \left(\sum_{1 \leq j^1, \ldots, j^p \leq n} (gf_{j^1 \ldots j^p} + hf'_{j^1 \ldots j^p}) dx_{j^1} \ldots dx_{j^p} \right) \\
&= \sum_{1 \leq j^1, \ldots, j^p \leq n} ((gf_{j^1 \ldots j^p} + hf'_{j^1 \ldots j^p}) \circ \phi) dy_{j^1} \ldots dy_{j^p} \\
&= \sum_{1 \leq j^1, \ldots, j^p \leq n} ((gf_{j^1 \ldots j^p} \circ \phi) + (hf'_{j^1 \ldots j^p} \circ \phi)) dy_{j^1} \ldots dy_{j^p} \\
&= g\phi^p(w) + h\phi^p(w').
\end{aligned}$$

Now we show that ϕ^p is an injection. If w and w' are as above,

$$\phi^p(w) = \phi^p(w')$$

$$\Rightarrow \sum_{1 \leq j^1, \ldots, j^p \leq n} (f_{j^1 \ldots j^p} \circ \phi) dy_{j^1} \ldots dy_{j^p} = \sum_{1 \leq j^1, \ldots, j^p \leq n} (f'_{j^1 \ldots j^p} \circ \phi) dy_{j^1} \ldots dy_{j^p}$$

$$\Rightarrow f_{j^1 \ldots j^p} = f'_{j^1 \ldots j^p}$$

$$\Rightarrow w = w'$$

thus ϕ^p is an injection. Now we show that ϕ^p is a surjection. If we let

$$z = \sum_{1 \leq j^1, \ldots, j^p \leq n} g_{j^1 \ldots j^p} dy_{j^1} \ldots dy_{j^p} \in \Omega^p(Y)$$

and let

$$w = \sum_{1 \leq j^1, \ldots, j^p \leq n} (g_{j^1 \ldots j^p} \circ \phi^{-1}) dx_{j^1} \ldots dx_{j^p} \in \Omega^p(X)$$

then clearly $\phi^p(z) = w$, so we have that ϕ^p is surjective. □

Lemma 2.5. *Under the above conditions for X, Y and ϕ, if $w \in \Omega^p(X)$ then for all $0 \leq p < n$*

$$\phi^{p+1}(dw) = d(\phi^p(w)).$$

In other words, the following diagram commutes :

$$\begin{array}{ccc} & \phi^p & \\ \Omega^p(X) & \longrightarrow & \Omega^p(Y) \\ d\downarrow & & \downarrow d \\ \Omega^{p+1}(X) & \longrightarrow & \Omega^{p+1}(Y) \\ & \phi^{p+1} & \end{array}$$

Proof: Again we use local co-ordinates, and the usual expression of w to get

$$\begin{aligned} \phi^{p+1}(dw) &= \phi^{p+1}\left(\sum_{1\leq i^1,\dots,i^{p+1}\leq n} \frac{df_{i^1\dots i^p}}{dx_{i^{p+1}}} dx_{i^p+1}dx_{i^1}\dots dx_{i^p}\right) \\ &= \sum_{1\leq i^1,\dots,i^{p+1}\leq n} \left(\frac{df_{i^1\dots i^p}}{dx_{i^{p+1}}}\circ\phi\right) dy_{i^{p+1}}dy_{i^1}\dots dy_{i^p} \\ &= d\left(\sum_{1\leq i^1,\dots,i^p\leq n} (f_{i^1\dots i^p}\circ\phi)dy_{i^1}\dots dy_{i^p}\right) \\ &= d(\phi^p(w)) \qquad \square. \end{aligned}$$

This leads us to conclude that cohomologies are invariant under diffeomorphism:

Theorem 2.6. *Under the above conditions for X, Y and ϕ, $H^p(X) \cong H^p(Y)$ for all $0 \leq p \leq n$ (as vector spaces).*

Proof: We note that the above lemmas lead us to conclude that

- w closed $\iff$ $\phi^p(w)$ closed, and
- w exact $\iff$ $\phi^p(w)$ exact.

So ϕ^p induces isomorphisms between

- $\{w \in \Omega^p(X)|dw = 0\}$ and $\{z \in \Omega^p(Y)|dz = 0\}$
- $\{dw|w \in \Omega^{p-1}(X)\}$ and $\{dz|z \in \Omega^{p-1}(Y)\}$

which proves the result. $\square$

We note also that non-diffeomorphic functions can also induce maps of cohomology. It should be noted that even when ϕ is not a bijection, the change of co-ordinates described in the definition of ϕ^* can still be performed. We apply the Inverse Function Theorem to the smooth function ϕ to get a local inverse, which gives us a local change of co-ordinates.

Example 2.7. If A, B are manifolds then we can introduce a smooth manifold structure on $A \times B$. Then the inclusion map

$$\pi : A \to A \times B$$

induces maps

$$\pi^* : \Omega^*(A \times B) \to \Omega^*(A)$$

and

$$\pi^p : H^p(A \times B) \to H^p(A)$$

as described above. (Because π is an inclusion,

We now calculate the cohomologies for a simple example space.

Example 2.8. Let X be $\mathbb{R}^2$. Then we observe that:

- because X is connected, $H^0(X) \cong \mathbb{R}$.

- If we take an arbitrary 1-form w given by
$$w = f_1 dx_1 + f_2 dx_2$$
then by defining
$$f(x_1, x_2) := \int_0^{x_1} f_1(u, x_2)du + \int_0^{x_2} f_2(x_1, u)du,$$
it is obvious that $w = df$, thus every 1-form is exact, so $H^1(X) \cong 0$.
- If we take an arbitrary 2-form w given by
$$w = f dx_1 dx_2$$
then by defining
$$g(x_1, x_2) := \int_0^{x_1} f(u, x_2)du$$
we see that
$$d(g dx_2) = \frac{\partial g}{\partial x_1} dx_1 dx_2 = f dx_1 dx_2 = w$$
thus w is exact, so $H^2(X) \cong 0$.
- For $p > 2$, all p-forms are 0, because of the relation $dx_i dx_i = 0$. So $H^p(X) = 0$.

This example can be generalised to $\mathbb{R}^n$ as in the Poincare lemma :

Theorem 2.9.
$$H^p(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} \textit{ when } p = 0 \\ 0 \textit{ otherwise .} \end{cases}$$

This is proved by induction on n. For a full proof, see [2].

3. The Mayer-Vietoris Sequence

This sections is intended to be a brief commentary on one of the common tools used in examining cohomology. Readers looking for more information should consult [2].

Definition 3.1. A set of vector spaces V_i together with homomorphisms f_i between them
$$\ldots \xrightarrow{f_{n-1}} V_{n-1} \xrightarrow{f_n} V_n \xrightarrow{f_{n+1}} V_{n+1} \xrightarrow{f_{n+2}} \ldots$$
is called an *exact sequence* if $ker(f_i) = image(f_{i-1})$ for all i. Note that exact sequences can be infinite or finite in both directions.

Definition 3.2. If an n-manifold X is the union of two open sets U and V, then we have inclusions
$$U \cup V \to X, \quad U \to U \uplus V, \quad V \to U \uplus V$$
where $\uplus$ indicates the disjoint union. These maps induce
$$\Omega^*(U \cup V) \to \Omega^*(U) \oplus \Omega^*(V), \quad \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V)$$
which gives us an exact sequence
$$0 \to \Omega^*(U \cup V) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0.$$

We call this sequence the *Mayer-Vietoris sequence* This in turn induces a long exact sequence of cohomology

$$H^q(U \cup V) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \to H^{q+1}(U \cup V)$$
$$\to H^{q+1}(U) \oplus H^{q+1}(V) \to H^{q+1}(U \cap V)$$

Theorem 3.3 (The Künneth formula)**.** *If M and N are two manifolds, then for all $n \geq 0$,*

$$H^n(M \times N) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(N)$$

where $\otimes$ is the tensor product.

Proof : The inclusion maps

$$\pi : M \to M \times N$$

$$\pi : N \to M \times N$$

induce a map in cohomology

$$\psi : H^p(M) \otimes H^q(N) \to H^{p+q}(M \times N)$$

Using the Mayer-Vietoris sequence we can check gives us the required property (see [2] for full details). □

Readers should note that we are also implicitly assuming that M and N have finite good covers (i.e. finite covers where finite intersections of the open sets are diffeomorphic to $\mathbb{R}^n$). This is not a problem as the manifolds we are dealing with will all have good covers.

4. Complex manifolds

The results in the previous section are all based on the cohomology of real manifolds (i.e. manifolds locally homeomorphic to $\mathbb{R}^n$), but we shall now extend this notion to cover complex manifolds.

Definition 4.1. A holomorphic bijection whose inverse is holomorphic is called a *biholomorphism*.

Definition 4.2. A *complex n-manifold X* is a topological space which is

- Hausdorff
- second-countable
- locally homeomorphic to $\mathbb{C}^n$

with an atlas such that the transition functions $\phi \circ \psi^{-1}\big|_{U \cap V}$ are holomorphic for all intersecting charts $(U, \phi), (V, \psi)$.

This allows us to define a holomorphic function $f : X \to \mathbb{C}$ as one where $f \circ \phi^{-1}$ is holomorphic for all choices of (U, ϕ) - the condition on transition functions assures us that this property is independent of our choice. This extends logically to maps between two complex manifolds.

As for non-complex manifolds we can have a notion of local co-ordinates, and in general we will choose n holomorphic and n anti-holomorphic co-ordinates, denoted

$z_1, ..., z_n$ and $\bar{z}_1, ..., \bar{z}_n$. We can also justify expressing the tangent space in local co-ordinates at $p \in X$ for a complex manifold

$$\begin{aligned} T_pX &= span_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_n}\right\} \\ &= T_p^{(1,0)}X \oplus T_p^{(0,1)}X \end{aligned}$$

where

$$T_p^{(1,0)}X = span_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\right\}$$

and

$$T_p^{(0,1)}X = span_{\mathbb{C}}\left\{\frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_1}\right\}.$$

and we apply this decomposition of co-ordinates to the dual space

$$T_p^*X = span_{\mathbb{C}}\{dz_1, ..., dz_n, d\bar{z}_1, ..., d\bar{z}_n\} = \Omega_p^{1,0}(X) \oplus \Omega_p^{0,1}(X)$$

where

$$\Omega_p^{1,0}(X) = span_{\mathbb{C}}\{dz_1, ..., dz_n\}$$
$$\Omega_p^{0,1}(X) = span_{\mathbb{C}}\{d\bar{z}_1, ..., d\bar{z}_n\}$$

We define $\Omega_p^{r,s}(X)$ as the space of forms with r holomorphic and s anti-holomorphic co-ordinates:

$$\Omega^{r,s}(X) = span_{\mathbb{C}}\{dz_{i^1}...dz_{i^r}d\bar{z}_{j^1}...d\bar{z}_{j^s}|i^a, j^b \in \{1, ..., n\}\}$$

so an arbitrary r, s-form w will take the form

$$w = \sum_{\substack{(1\leq i^1,...,i^r\leq n) \\ (1\leq j^1,...,j^s\leq n)}} f_{i^1,...,i^r,j^1,...,j^s} dz_{i^1}...dz_{i^r}d\bar{z}_{j^1}...d\bar{z}_{j^s}.$$

(For ease of notation we will write $f_{...}$ instead of writing the full subscript). We can decompose the exterior derivative into two components:

$$dw = \partial w + \bar{\partial} w$$

where

$$\partial w = \sum \frac{\partial f_{...}}{\partial z_{i^{r+1}}} dz_{i^{r+1}} dz_{i^1}...dz_{i^r}d\bar{z}_{j^1}...d\bar{z}_{j^s}$$

and

$$\bar{\partial} w = \sum \frac{\partial f_{...}}{\partial \bar{z}_{j^{s+1}}} d\bar{z}_{j^{s+1}} dz_{i^1}...dz_{i^r}d\bar{z}_{j^1}...d\bar{z}_{j^s}.$$

It is easy to show that both $\partial^2 = 0$ and $\bar{\partial}^2 = 0$.

Definition 4.3. Let g be a non-degenerate hermitian metric on a complex manifold X, and define (locally)

$$g_{ij} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$$
$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$$

with the obvious definitions for $g_{\bar{i}j}$ and $g_{\bar{i}\bar{j}}$. We define also

$$g^{ij} = g^{-1}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$$

$$g^{i\bar{j}} = g^{-1}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$$

where g^{-1} is the bilinear form obtained by taking the inverse of the matrix of g. We then make the obvious definitions of $g^{\bar{i}j}$ and $g^{\bar{i}\bar{j}}$.
The *Kähler form* on X is defined by

$$K = \sum_{1\leq i,j\leq n} \sqrt{-1}g_{i\bar{j}}dz_i d\bar{z}_j$$

We say that X is *Kähler* if it admits a metric g for which the Kähler form K is closed (i.e. $dK = 0$). For more on Kähler geometry, see [2] or [4].

We now define the complex equivalent of the DeRham cohomology.

Definition 4.4. The *(r,s)th Dolbeault cohomology* of a complex manifold X is defined to be

$$H^{r,s}_{\bar{\partial}}(X) = \frac{\{w \in \Omega^{r,s}(X)|\bar{\partial}w = 0\}}{\{\bar{\partial}w|w \in \Omega^{r-1,s}(X)\}}.$$

$H^{r,s}_{\bar{\partial}}(X)$ is called the *(r,s)th cohomology class of X*. We note that on compact Kähler manifolds, it is equivalent to formulate this using ∂ instead of $\bar{\partial}$. In this case we just write $H^{r,s}(X)$. We also write

$$H^*_{\bar{\partial}}(X) = \bigoplus_{r,s} H^{r,s}_{\bar{\partial}}(X).$$

Proposition 4.5 (Hodge decomposition)**.** *On a compact Kähler manifold,*

$$H^p(X) = \bigoplus_{r+s=p} H^{r,s}(X)$$

as vector spaces over $\mathbb{R}$*, and* $H^{r,s}(X) = \overline{H^{s,r}(X)}$*.*

Proof : See [4].

Definition 4.6. The *(r,s)th Hodge number* of a complex manifold X is defined as

$$h^{r,s}_X = dim_{\mathbb{C}} H^{r,s}_{\bar{\partial}}(X).$$

This is often written as just $h^{r,s}$.

We usually write the Hodge numbers of a manifold in a diamond, as in this example of a complex 2-manifold:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

We call this the *Hodge diamond*.

Theorem 4.7.

$$\chi = \sum_{i=0}^{n}(-1)^i\left(\sum_{r+s=i} h^{r,s}\right),$$

where χ *is the Euler characteristic of the manifold.*

Proof : See [2].

Proposition 4.8 (Künneth formula for complex manifolds)**.** *If X, Y are complex manifolds, then*

$$H^{r,s}_{\bar\partial}(X \times Y) = \bigoplus_{\substack{r_1+r_2=r \\ s_1+s_2=s}} H^{r_1,s_1}_{\bar\partial}(X) \otimes H^{r_2,s_2}_{\bar\partial}(Y)$$

Proof : The proof is similar to the proof of the Künneth formula for real manifolds - we use the Mayer-Vietoris sequence. □

5. Connections, curvature and holonomy

We recall that a *vector field on a manifold* X attaches a tangent vector to each point in X, and we denote

$$\mathcal{X}(X) = \{\text{vector fields on X}\}.$$

Definition 5.1. A *connection* D on a manifold X is a function

$$D : \mathcal{X}(X) \times \mathcal{X}(X) \to \mathcal{X}(X)$$

(we write $D_V W$ for $D(V, W)$) s.t.

- $D_V W$ is $C^\infty(X)$-linear in V
- $D_V W$ is $\mathbb{C}$-linear in W
- $D_V(f(W)) = (Vf)W + f(D_V W)$ for $f \in C^\infty(X)$.

Definition 5.2. Given two vector fields V, W on a manifold X we define a new vector field $[V, W]$ by

$$[V, W]_p(f) = V_p(Wf) - W_p(Vf)$$

for all $f \in C^\infty(X)$.

Theorem 5.3. *If X is a manifold with a non-degenerate metric $<,>$, then there exists a unique connection D s.t. $\forall U, V, W \in \mathcal{X}(X)$:*

- $[V, W] = D_V W - D_W V$
- $U < V, W >=< D_U V, W > + < V, D_U W >$

This is called the Levi-Civita connection*, and it is characterised by the* Kozul formula*:*

$$\begin{aligned} 2 < D_V W, U >= V < W, U > +W < U, V > -U < V, W > - \\ < V, [W, U] > + < W, [U, V] > + < U, [V, W] > \end{aligned}$$

$\forall U, V, W \in \mathcal{X}(X)$.

Proof : Existence: This follows from the fact that

$$V \mapsto V^*, \text{ where } V^*(U) =< V, U >$$

is a $C^\infty(X)$-linear isomorphism $\mathcal{X}(X) \to \Omega^{1,0}(X)$
Uniqueness: Suppose D satisfies the conditions above. Then for all $U, V, W \in \mathcal{X}(X)$.

$$\begin{aligned} V < W, U > &= < D_V W, U > + < W, D_V U > \\ W < U, V > &= < D_W U, V > + < U, D_W V > \\ U < V, W > &= < D_U V, W > + < V, D_U W > \end{aligned}$$

and

$$\begin{aligned} < V, [W,U] > &= < V, D_W U - D_U W > = < V, D_W U > - < V, D_U W > \\ < W, [U,V] > &= < W, D_U V - D_V U > = < W, D_U V > - < W, D_V U > \\ < U, [V,W] > &= < U, D_V W - D_W V > = < U, D_V W > - < U, D_W V > \end{aligned}$$

so we can verify the Kozul formula. The uniqueness of D follows from the fact that if $< D_V W, U > = < D'_V W, U > \ \forall U, V, W \in \mathcal{X}(X)$, $D_V W = D'_V W$ and thus $D = D'$. □

Theorem 5.4. *If X is a manifold with a non-degenerate metric g, $I \subset \mathbb{R}$, and $\alpha : I \to X$ is a curve, then we can define a notion of the derivative of a vector field. There is a unique function taking every vector field Z on α to a vector field Z' on α, called the* induced covariant derivative*, s.t.*

- $(aZ_1 + bZ_2)' = aZ'_1 + bZ'_2$ *where* $a, b \in \mathbb{C}$
- $(hZ)' = (dh/dt)Z + hZ'$ *where* $h \in C^\infty(I)$
- $(V\alpha)'(t) = D_{\alpha'(t)}(V)$ *where* $t \in I, V \in \mathcal{X}(X)$
- $(d/dt) < Z_1, Z_2 > = < Z'_1, Z_2 > + < Z_1, Z'_2 >$

Proof : See [3].

Definition 5.5. A vector field is called *parallel* if $Z' = 0$.

Proposition 5.6. *Let X be a manifold with a non-degenerate metric. Let $\alpha : (a, b) \to X$ be a smooth curve on X, and $z \in T_{\alpha(a)}X$. Then there exists a unique parallel vector field Z defined on all of $\alpha(a, b)$ s.t. $Z(a) = z$.*

Proof : This is a consequence of the properties of the induced covariant derivative. For more detail, see [3]. □

Definition 5.7. Under the above conditions, we define the *parallel transport of z along α* to be $Z(b)$ where Z is the unique parallel vector field s.t. $Z(a) = z$. The *holonomy* of X is defined to be

$$\left\{ \text{linear maps } A : T_{\alpha(a)}X \to T_{\alpha(a)}X \,\middle|\, \begin{array}{l} A(v) \text{ is the parallel transport of } v \text{ along } \alpha \\ \alpha : (a, b) \to X \text{ smooth closed curve} \end{array} \right\}.$$

In other words, the holonomy of a manifold is the set of all linear maps that describe the action of parallel transport around some closed curve.

6. Calabi-Yau Manifolds

The following definitions and results will help us to define Calabi-Yau manifolds.

Theorem 6.1. *If X is a complex, Kähler manifold with Euler characteristic $\chi \neq 0$ and $c_1(X)$, then $h^{1,0} = 0$.*

Proof : This is a consequence of the Poincaré-Hopf theorem, which states that χ counts zeroes of vector fields on X (with multiplicity). Let $V \in H^{1,0}(X)$, so V is a holomorphic one form. By the Weitzenböck formula, V is therefore constant. V is constant and it has a zero, so V is zero everywhere. □

Definition 6.2. On a manifold X with a non-degenerate metric g we define the *Christoffel symbols*

$$\Gamma^i_{jk} = \sum_l \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial z_j} + \frac{\partial g_{lj}}{\partial z_k} - \frac{\partial g_{jk}}{\partial z_l} \right),$$

where we allow replacement of i by $\bar{\imath}$, j by $\bar{\jmath}$, k by $\bar{k}$ and l by $\bar{l}$.

We note that on a Kähler manifold, we have

$$\frac{\partial g_{i\bar{\jmath}}}{\partial z_l} = \frac{\partial g_{l\bar{\jmath}}}{\partial z_i}$$

so the only non-zero Christoffel symbols are

$$\Gamma^l_{jk} = \sum_s g^{l\bar{s}} \frac{\partial g_{k\bar{s}}}{\partial z_j}$$

and

$$\Gamma^{\bar{l}}_{\bar{\jmath}\bar{k}} = \sum_s g^{\bar{l}s} \frac{\partial g_{\bar{k}s}}{\partial z_{\bar{\jmath}}}.$$

Definition 6.3. The *Riemann curvature tensor* is defined by

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial z^l} - \frac{\partial \Gamma^i_{jl}}{\partial z^k} + \sum_s \left(\Gamma^i_{sl}\Gamma^s_{jk} - \Gamma^i_{sk}\Gamma^s_{jl} \right)$$

where we allow replacement of i by $\bar{\imath}$, j by $\bar{\jmath}$, k by $\bar{k}$ and l by $\bar{l}$. The *Ricci curvature tensor* is defined by

$$R_{j\bar{k}} = \sum_i R^i_{ij\bar{k}}.$$

Under the Kähler conditions above the Ricci tensor simplifies to

$$R_{\bar{\imath}j} = -\sum_{\bar{k}} \frac{\partial \Gamma^{\bar{k}}_{\bar{\imath}\bar{k}}}{\partial z_j}$$

We say that a metric is *Ricci-flat* if the induced Ricci curvature tensor is identically zero.

We are now ready to define a Calabi-Yau manifold.

Definition 6.4. A complex n-manifold X is *Calabi-Yau* if

- X is compact
- X is Kähler
- X has a holonomy that is a subgroup of $SU(n)$

where $SU(n)$ is the group of special unitary $n \times n$ matrices.

Proposition 6.5. *If a complex compact manifold X admits a Ricci-flat metric then the holonomy of X is a subgroup of $SU(n)$ with respect to that metric.*

Proof : See [5].
This leaves us with the unenviable task of trying to find explicit Ricci-flat metrics on manifolds. The next section gives us an alternate approach.

7. Bundles

Definition 7.1. Let G be a topological group which acts effectively on a space F on the left. A *fiber bundle* is a surjection $\pi : E \to B$ between topological spaces s.t. B has an open cover $\{U_\alpha\}$ and homeomorphisms

$$\phi_\alpha : E\big|_{U_\alpha} \to U_\alpha \times F$$

and the transition functions are continuous functions with values in G. In this case we call F the *fiber* and G the *structure group.*

Note that the set E is often referred to as the "bundle".

Definition 7.2. A *complex vector bundle of rank* n is a fiber bundle with fiber $\mathbb{C}^n$ and structure group $GL(n;\mathbb{C})$. If $n = 1$ then we call it a *complex line bundle.*

Note that by removing the complex structure from a complex line bundle we get a real vector bundle of rank 2. We call this the *underlying real bundle.*

Definition 7.3. The *1st Chern class* $c_1(E)$ of a complex line bundle E is the Euler class of its underlying real bundle.

Definition 7.4. If X is a complex n-manifold, the *canonical bundle* of X is the $n - fold$ exterior product (denoted by $\bigwedge$)

$$K_X = \bigwedge^n T^*X$$

which is a complex line bundle of X.

Definition 7.5. The first Chern class $c_1(X)$ of a complex manifold X is defined to be the first Chern class of its canonical bundle.

Definition 7.6. A complex vector bundle is *trivial* if it takes the form

$$\pi : M \times \mathbb{C}^k \to M$$

where k is some positive integer.

Proposition 7.7. *A complex line bundle is trivial iff it has zero 1st Chern class.*

Proof : See [2]

Theorem 7.8 (Yau)**.** *A Kähler manifold* X *with* $c_1(X) = 0$ *admits a Ricci-flat metric.*

Proof : See [7]

This theorem frees us from the difficult task of explicitly finding Ricci-flat metrics on Kähler manifolds in order to prove that they are Calabi-Yau. Now all we have to do is check that their 1st Chern class is zero. It is worth noting that this theorem is not constructive; i.e. it does not provide an explicit Ricci-flat metric.

8. More Calabi-Yau manifolds

We now look at some common properties of Calabi-Yau manifolds.

Theorem 8.1 (Poincaré Duality)**.** *If* X *is a compact,* n*-dimensional manifold, then there exists a linear isomorphism*

$$\sigma : H^r(X) \to H^{n-r}(X).$$

Proof : See [6]

Proposition 8.2. *On a connected Calabi-Yau manifold* X*,*

(i) $h^{0,0} = 1$
(ii) $h^{r,s} = h^{n-r,n-s}$
(iii) $h^{r,s} = h^{s,r}$
(iv) $h^{n,0} = h^{0,n} = 1$

Proof : Property (i) follows directly from connectedness : the closed 0,0-forms must all be constant, so $H^{0,0}(X) \cong \mathbb{C}$.
Property (ii) is detailed in [2].
Property (iii) follows from the Hodge decomposition of real cohomology.
Property (iv) follows from the fact that holonomy is some subgroup of $SU(n)$. □
So we can already write the hodge diamond for a one-dimensional Calabi-Yau manifold:

$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array} = \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

These manifolds are in fact elliptic curves, i.e. quotient spaces $\frac{\mathbb{C}}{\Lambda}$ where Λ is a lattice generated by two linearly independent vectors in $\mathbb{C}$.
We now classify Calabi-Yau 2-folds. These fall into two categories: firstly, the complex 2-torus.

Definition 8.3. We define a *complex 2-torus* as a quotient space $\frac{\mathbb{C}^2}{\Lambda}$ where Λ is a lattice generated by four vectors in $\mathbb{C}^2$ which are linearly independent over $\mathbb{R}$.

Proposition 8.4. *A complex 2-torus T is a Calabi-Yau manifold.*

Proof: Compactness is easy to show; we need to show that $c_1(T) = 0$. This is easy to see if we write T as the product space of two elliptic curves E_1, E_2. The canonical bundle of T will be the product $K_{E_1} \times K_{E_2}$. E_1 and E_2 are Calabi-Yau manifolds, so K_T is the product of two trivial bundles and is therefore trivial. So $c_1(T) = 0$ by 7.7. □
We now have that the Hodge diamond for T will look like

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0} & & h^{0,1} & \\ 1 & & h^{1,1} & & 1 \\ & h^{2,1} & & h^{1,2} & \\ & & 1 & & \end{array}.$$

Examining the cohomology on the cover $\mathbb{C}^2$ we get that $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 2$, and $h^{1,1} = 4$. This confirms that the Euler characteristic is $\chi = 1 - (2+2) + (1+4+1) - (2+2) + 1 = 8 - 8 = 0$ (which is verifiable by triangulation).
Now we examine the cohomology of the case $\chi \neq 0$. By Theorem 6.1 $h^{1,0} = 0$, so we deduce $h^{0,1} = h^{2,1} = h^{1,2} = 0$. It remains only to calculate $h^{1,1}$. Results from index theory tell us that the Euler characteristic χ is 24, and thus by Theorem 4.7 :

$$24 = 1 - 0 + (1 + h^{1,1} + 1) - 0 + 1$$

and therefore $h^{1,1} = 20$. We call this a *K3 surface*. One can show that all Calabi-Yau 2-folds with $\chi \neq 0$ are diffeomorphic; see [13].

9. Blow-up of singularities

The Borcea-Voisin construction involves a process called *orbifolding*, whereby we identify points on a Calabi-Yau manifold by means of an automorphism. However, if the automorphism has any fixed points, we will no longer have smooth structure on the result. These fixed points are called *singularities*. However, this section details methods by which we can restore the smooth structure by “glueing” copies of complex projective space $\mathbb{CP}^n$ along the fixed points.

Definition 9.1. $\mathbb{CP}^n := \{$1-dimensional subspaces of $\mathbb{C}^{n+1}\}$.

Definition 9.2. Let Δ be a disc about the origin in $\mathbb{C}^n$ with co-ordinates $z_1, ..., z_n$ and let $y_1, ..., y_n$ be homogeneous co-ordinates on $\mathbb{CP}^{n-1}$. Define

$$\tilde{\Delta} := \{(z, y) | z_i y_j = z_j y_i \forall i, j\},$$

which is a submanifold of $\Delta \times \mathbb{CP}^{n-1}$. Let $\pi : \Delta \times \mathbb{CP}^{n-1} \longrightarrow \Delta$ be the natural projection onto the first co-ordinate, and we call the restriction of π to $\tilde{\Delta}$ the *blow-up of* Δ *at 0*.

Proposition 9.3. *This blow-up is independent of the choice of co-ordinates on* $\mathbb{C}^n$ *and* $\mathbb{CP}^{n-1}$.

Proof : See [4].

Now that we can blow-up zero inside Δ, we can blow up any isolated point on an n-surface since we have a neighbourhood of that point that is locally biholomorphic to $\mathbb{C}^n$.

Definition 9.4. If p is an isolated point on a complex n-manifold X, let U be a neighbourhood of p biholomorphic to Δ by the map $\phi : \Delta \longrightarrow U$. Let $\tilde{U}$ be the image of the map $\tilde{\Delta} \longrightarrow U \times \mathbb{CP}^{n-1}$ mapping $(z, l) \mapsto (\phi(z), l)$. Then the *blow-up* $\tilde{X}$ *of* X *at* p is defined to be the toplogical space $(X - \{p\}) \cup_\pi \tilde{U}$ together with the natural projection $\pi : \tilde{X} \longrightarrow X$. (Often just $\tilde{X}$ will be referred to as the blow-up.) We call $\pi^{-1}(p)$ the *exceptional divisor* of the blow up.

Theorem 9.5. $\tilde{X}$ *can be given a smooth structure compatible with the structure of* X.

Proof : See [4].

Proposition 9.6. *If* D *is the exceptional divisor of the blow-up* $\tilde{X}$ *at a smooth point of an* n*-manifold* X*, and* $\pi : \tilde{X} \longrightarrow X$ *is the natural projection, then*

$$\pi^* K_X = K_{\tilde{X}} - (n-1)D$$

Proof : See [4].

Definition 9.7. A *complex orbifold* is a manifold in which each set in the atlas is biholomorphic not to $\mathbb{C}^n$ but to $\mathbb{C}^n/G$ where G is some discrete group of automorphisms fixing the origin of $\mathbb{C}^n$. (G isn't necessarily the same everywhere.)

So we now have a notion of a topological space that has smooth structure everywhere except on the fixed points of the group G. We can still talk about cohomology and the Euler characteristic of an orbifold, even though it doesn't have smooth structure everywhere.

We can also perform blow-up along any submanifold of an orbifold X.

Proposition 9.8. *If* X *is a complex* n*-orbifold with* Y *a singular* k*-submanifold, then by a finite number of blow-ups of* Y *on* X*, using copies of* $\mathbb{CP}^{n-k-1}$*, we can obtain a smooth manifold* $\tilde{X}$*. This is called the* resolution *of* X.

Proof : See [4].

10. The Borcea-Voisin construction

The Borcea-Voisin construction is a method for obtaining a Calabi-Yau 3-fold by taking involutions (i.e. automorphisms of order 2) on an elliptic curve and a K3 surface.

Lemma 10.1. *If S is a K3 surface with an involution j s.t. j induces a non-trivial automorphism on $H^{2,0}(S)$, then the fixed points of j (which will be the singularities of S/j) have several possibilities:*

(1) *no fixed points,*
(2) *a finite number of rational curves and at most one curve with genus > 0, or*
(3) *two elliptic curves.*

Proof : see [8].
Note that Borcea assures us in [11] that such involutions on K3 surfaces can be found; in fact he explicitly gives equations for them.

Theorem 10.2. *Let E be an elliptic curve $\mathbb{C}/\Lambda$ and i the involution induced by the involution on $\mathbb{C}$, $z \mapsto -z$.*
Let S be a K3 surface with an involution j inducing a non-trivial automorphism on $H^{2,0}(S)$, and let $k(e,s) := (i(e), j(s))$ be the product automorphism on $E \times S$. Then

$$X = \frac{\widetilde{E \times S}}{k},$$

the minimal resolution of the orbifold $(E \times S)/k$, is a Calabi-Yau manifold.

Proof: (summarized from [8]) : clearly X is compact, because every cover of X can be written as a product of covers of $\widetilde{E/i}$ and $\widetilde{S/j}$, each of which are compact (by the compactness of E and S, and the fact that the exceptional divisors are all compact).
We note that there are four fixed points of the involution i - namely, if Λ is generated by v_1, v_2, the fixed points will be $\{0, \frac{1}{2}v_1, \frac{1}{2}v_2, \frac{1}{2}v_1 + \frac{1}{2}v_2\}$. We denote these p_1, p_2, p_3, p_4. By Lemma 10.1, we note that j will have N fixed curves $C_1, ..., C_N$ on S. (N is possibly zero.)
To show that $c_1(X) = 0$ we show that K_X is trivial. First, let $\tau : \widetilde{E \times S} \to E \times S$ be the blow up of $E \times S$ along all the curves $p_r \times C_s$. Let $\phi : \widetilde{E \times S} \to X$ be the natural quotient map. Then we have $\phi^* K_X = K_{\widetilde{E \times S}} - D$ and $K_{\widetilde{E \times S}} = \tau^* K_{E \times S} + D$ (by Proposition 9.6) where D is the exceptional divisor. S, E Calabi-Yau implies that K_S, K_E are trivial, so $K_{E \times S}$ is trivial also, so $\phi^* K_X$ is trivial, so X is a Calabi-Yau 3-fold. □
We now calculate the cohomology of X. From the Theorem 10.2 we already know that $h^{0,0} = h^{3,0} = h^{0,3} = h^{3,3} = 1$, the rest of the Hodge numbers we calculate by means of the Künneth formula on the cohomologies of $\widetilde{E/i}$ and $\widetilde{S/j}$. We note that we can divide the Hodge diamond of E into the positive and negative eigenspaces of i. We do this by considering the induced action of i on a basis element of $H^{r,s}(E)$. Clearly i acts trivially on $H^{0,0}(E)$ and $H^{1,1}(E)$, $i^*(dz) = d(-z) = -dz$

and $i^*(d\bar{z}) = d(\overline{-z}) = -d\bar{z}$. This shows that

$$H^*(E^+): \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array} \qquad H^*(E^-): \begin{array}{ccc} & 0 & \\ 1 & & 1 \\ & 0 & \end{array}$$

and we can, with a little calculation, do the same for S. Clearly the global (0,0)-form 1 is invariant under j as is the space of (2,2)-forms. The Hodge diamond of S shows there are no (1,0)-forms, (0,1)-forms, (2,1)-forms, or (1,2)-forms to act on. We can calculate the Euler characteristic of S/j by

$$\begin{aligned} \chi(S/j) &= \chi(S/j - \bigcup_s C_s) + \sum_s \chi(C_s) \\ &= \frac{1}{2}\chi(S - \bigcup_s C_s) + \sum_s \chi(C_s) \end{aligned}$$

and if we let $N' = \sum_s g_s$ where g_s is the genus of C_s, we have $\chi(S/j) = 12+N-N'$. Also, because j induces a non-trivial action on $H^{2,0}(S)$, the positive eigenspace of the action of j on S is

$$H^*(S^+): \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & a & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and because we know that the Euler characteristic is $12 + N - N'$, we calculate that $a = 10 + N - N'$. The negative eigenspace is the complement (in the vector space sense), so we deduce that it is

$$H^*(S^-): \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & b & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

where $b = 10 - N + N'$. We now use the Künneth formula to combine these, noting that in order to get a valid differential form, we need to combine forms either from both positive eigenspaces or both negative. So combining the positive (E^+ and S^+) gives us

$$H^*(E^+ \times S^+): \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a+1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & a+1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

and the negative

$$H^*(E^- \times S^-): \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & b+1 & & b+1 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array},$$

and it remains to calculate the contribution from the exceptional divisor $D_{r,s}$ of the fixed curve $p_r \times C_s$. By Proposition 9.8 with n=3, k=1, we are blowing up using $\mathbb{CP}^1$, so the Hodge diamond of the exceptional divisor comes from

$$H^*(\mathbb{CP}^1)\ \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array} \qquad H^*(C_s)\ \begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}.$$

We calculate g by using the equation

$$\chi = \sum_{i=0}^{n} (-1)^i \left(\sum_{r+s=i} h^{r,s} \right),$$

so we have

$$2 - 2g_s = \chi = 2 - 2g.$$

Thus $g = g_s$. There are 4N such curves, 4 with each genus, so the total contribution will be

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 4N & & 0 & \\ 0 & & 4N' & & 4N' & & 0 \\ & 0 & & 4N & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

because we only get a single contribution from (0,0)-forms. This gives us the final Hodge diamond for $\widetilde{E \times S}/k$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \alpha & & 0 & \\ 1 & & \beta & & \beta & & 1 \\ & 0 & & \alpha & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

where $\alpha = 11 + 5N - N'$ and $\beta = 11 + 5N' - N$. The significance of this result is that Nikulin's classification ([9]) implies that if (S, j) is a K3 surface with an involution, and j has N fixed curves with total genus N', there exists a complementary pair (S', j') with N' fixed curves that have total genus N. This implies that using the

Borcea-Voisin construction on S', we get the Hodge diamond of $X' = (\widetilde{S' \times E})/k'$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \beta & & 0 & \\ 1 & & \alpha & & \alpha & & 1 \\ & 0 & & \beta & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} .$$

This makes (X, X') a *mirror pair* of manifolds. Manifolds constructed using this method always come in mirror pairs. For more on mirror manfolds, the reader should consult [10].

11. Extensions of the Borcea-Voisin construction

We consider in this section two possible modifications to the Borcea-Voisin constructions. The first is to replace the elliptic curve E with a second K3 surface with another involution. The result will be Calabi-Yau fourfolds which occur in mirror pairs, as we see below.

Theorem 11.1. *Let S, T be two K3 surfaces. Let i be an involution on S that induces a non-trivial action on $H^{2,0}(S)$ and j an involution on T that induces a non-trivial action on $H^{2,0}(T)$. If we define $k : S \times T \to S \times T$ by $k(s,t) = (i(s), j(t))$ then*

$$X = \frac{\widetilde{S \times T}}{k}$$

is a Calabi-Yau manifold.

Proof : similar to the Borcea-Voisin proof. Clearly X is compact, and the canonical bundle is seen to be trivial if we look at as constructed from the trivial bundles of S and T. □

We have already calculated the positive and negative eigenspace cohomologies for the involutions i and j, it remains to combine them with the Künneth formula and add in contributions from the exceptional divisors. Let N be the number of fixed curves C_r of i (again using Lemma 10.1), and g_r denote the genus of C_r. Let M be the number of fixed curves C'_s of j, and let g'_s denote the genus of C'_s. Let N' be the total genus of the fixed curves of i, and M' the total genus of the fixed curves of j. Combining S^+ with T^+ and then S^- with T^- gives us

$$H^*\left(\frac{S \times T}{k}\right) : \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & a & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & c & & b & & c & & 1 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & a & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

where $a = 20 + N - N' + M - M'$, $b = 204 + 2(N - N')(M - M')$ and $c = 20 + N' - N + M - M'$. The fixed points of k are the curves $C_r \times C'_s$. Again we are blowing up by $\mathbb{CP}^1$. Let $D_{r,s}$ denote the exceptional divisor of the fixed curve $C_r \times C'_s$. We can get the Hodge diamond for $D_{r,s}$ by using the Künneth formula to combine the Hodge diamonds

$$H^*(C_r): \begin{matrix} & 1 & \\ g_r & & g_r \\ & 1 & \end{matrix} \qquad H^*(C'_s): \begin{matrix} & 1 & \\ g'_s & & g'_s \\ & 1 & \end{matrix},$$

and by 9.8 we are blowing up using $\mathbb{CP}^1$, giving the Hodge diamonds

$$H^*(\mathbb{CP}^1) \begin{matrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{matrix} \qquad H^*(C_r \times C'_s): \begin{matrix} & & & 1 & & & \\ & g_r + g'_s & & & & g_r + g'_s & \\ g_r g'_s & & & 2 + 2g_r g'_s & & & g_r g'_s \\ & g_r + g'_s & & & & g_r + g'_s & \\ & & & 1 & & & \end{matrix}.$$

Summing each Hodge number over r and s gives total contribution from blow up

$$\begin{matrix} & & & & & & 0 & & & & & & \\ & & & & 0 & & & & 0 & & & & \\ & & 0 & & & & NM & & & & 0 & & \\ & 0 & & & MN' + NM' & & & & MN' + NM' & & & 0 & \\ 0 & & N'M' & & & & 2NM + 2N'M' & & & & N'M' & & 0 \\ & 0 & & & MN' + NM' & & & & MN' + NM' & & & 0 & \\ & & 0 & & & & NM & & & & 0 & & \\ & & & & 0 & & & & 0 & & & & \\ & & & & & & 0 & & & & & & \end{matrix}.$$

Adding this contribution to the Hodge diamond for the orbifold $(S \times T)/k$ gives us

$$H^*\left(\frac{\widetilde{S \times T}}{k}\right): \begin{matrix} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & \alpha & & 0 & & \\ & 0 & & \delta & & \delta & & 0 & \\ 1 & & \gamma & & \beta & & \gamma & & 1 \\ & 0 & & \delta & & \delta & & 0 & \\ & & 0 & & \beta & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{matrix}$$

with

$$\begin{aligned} \alpha &= 20 + N - N' + M - M' + NM \\ \beta &= 2NM + 2N'M' + 204 + 2(N - N')(M - M') \\ \gamma &= N'M' + 20 + N' - N + M - M' \\ \delta &= MN' + NM'. \end{aligned}$$

[9] tells us we can replace (S, T, i, j) with (S', T', i', j') s.t. the construction above yields a mirror manifold to the one above.

The second modification we make is to use automorphisms of order other than 2. We start with automorphisms of order 4. Let E be an elliptic curve, and ϕ be the automorphism on E induced by the automorphism $z \mapsto -z$ on $\mathbb{C}$ as before. We have 4 fixed points which we label p_1, p_2, p_3, p_4, and the eigenspaces

$$H^*(E^{+1}): \begin{matrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{matrix} \qquad\qquad H^*(E^{-1}): \begin{matrix} & 0 & \\ 1 & & 1 \\ & 0 & \end{matrix}$$

as before. We now need to look for order 4 automorphisms on K3 surfaces that induce a non-trivial action on $H^{2,0}$ (this is necessary for the resulting manifold to be Calabi-Yau). The tactic is to look for K3s that admit an involution α inducing a non-trivial action on $H^{2,0}$, and find an order 4 automorphism β that

- induces a trivial action on $H^{2,0}$ and
- commutes with α.

Under these conditions, $\gamma := \beta \circ \alpha$ will be an order 4 automorphism inducing a non-trivial action on $H^{2,0}$. From [2] we see many tractable examples of K3s, and we perform the proposed construction on the zero-locus X of

$$f(x, y, z, w) = -x^2 + y^4 + z^8 + w^8$$

considered as a hypersurface in weighted projective space $W\mathbb{P}^3_{(4,2,1,1)}$, so we have $(x, y, z, w) \sim (\lambda^4 x, \lambda^2 y, \lambda z, \lambda w)\forall\lambda\mathbb{C}^*$. A complification arises from the fact that inserting $\lambda = -1$ shows $(x, y, z, w) \sim (x, y, -z, -w)$, such that for any $(x, y) \neq (0, 0)$, $W\mathbb{P}^3_{(4,2,1,1)}$ has a $\mathbb{Z}_2$-type singularity in $(x, y, 0, 0)$. We will come back to this in Proposition 11.3; for more information about weighted projective space, see [1]. [11] tells us that the map $\alpha(x, y, z, w) := (-x, y, z, w)$ is an involution acting non-trivially on $H^{2,0}(X)$. If we define $\beta(x, y, z, w) := (x, iy, -iz, w)$ then we see that β is an automorphism of order 4 on X which commutes with α.

Proposition 11.2. *β induces a trivial action on $H^{2,0}(X)$.*

Proof: We consider the induced action on the smooth (2,0) form in each of the four charts ($x \neq 0, y \neq 0, z \neq 0, w \neq 0$). As mentioned in [12], the smooth form Ω can be obtained by writing $\tilde{f}$ to be the restriction of f to the appropriate co-ordinate chart.

Chart $x \neq 0$: We choose $\lambda = x^{-\frac{1}{4}}$, giving homogeneous co-ordinates

$$(\xi, \zeta, \nu) = (\frac{y}{x^{\frac{1}{2}}}, \frac{z}{x^{\frac{1}{4}}}, \frac{w}{x^{\frac{1}{4}}}).$$

Because we're in weighted projective space, we have

$$\begin{aligned} \tilde{f} &= \lambda^8(-x^2 + y^4 + z^8 + w^8) \\ &= -1 + \xi^4 + \zeta^8 + \nu^8 \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial \nu} &= 8\nu^7 \\ \Omega &= \frac{d\zeta d\xi}{\partial \tilde{f}/\partial \nu}. \end{aligned}$$

We see that β acts on these homogenous co-ordinates by

$$(\xi, \zeta, \nu) \mapsto (i\xi, -i\zeta, \nu)$$

so

$$\beta^*(\Omega) = \frac{d(-i\zeta)d(i\xi)}{8\nu^7} = -i.i\frac{d\zeta d\nu}{8\nu^7} = \Omega$$

so in the chart $x \neq 0$, β fixes $H^{2,0}$.
Chart $y \neq 0$: We choose $\lambda = y^{-\frac{1}{2}}$, giving homogeneous co-ordinates

$$(\xi, \zeta, \nu) = (\frac{x}{y^2}, \frac{z}{y^{\frac{1}{2}}}, \frac{w}{y^{\frac{1}{2}}}).$$

and

$$\begin{aligned} \tilde{f} &= \lambda^8(-x^2 + y^4 + z^8 + w^8) \\ &= -\xi^2 + 1 + \zeta^8 + \nu^8 \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial \xi} &= -2\xi \\ \Omega &= \frac{d\zeta d\nu}{\partial \tilde{f}/\partial \xi}. \end{aligned}$$

We see that β acts on these homogenous co-ordinates by

$$(\xi, \zeta, \nu) \mapsto (-\xi, \frac{-i}{\sqrt{i}}\zeta, \frac{1}{\sqrt{i}}\nu)$$

so

$$\beta^*(\Omega) = \frac{-i}{\sqrt{i}}.\frac{1}{\sqrt{i}}\frac{d\zeta d\nu}{2\xi} = \Omega$$

so in the chart $y \neq 0$, β fixes $H^{2,0}$.
Chart $z \neq 0$: We choose $\lambda = z^{-1}$, giving homogeneous co-ordinates

$$(\xi, \zeta, \nu) = (\frac{x}{z^4}, \frac{y}{z^2}, \frac{w}{z}).$$

and

$$\begin{aligned} \tilde{f} &= \lambda^8(-x^2 + y^4 + z^8 + w^8) \\ &= -\xi^2 + \zeta^4 + 1 + \nu^8 \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial \xi} &= -2\xi \\ \Omega &= \frac{d\zeta d\nu}{\partial \tilde{f}/\partial \xi}. \end{aligned}$$

We see that β acts on these homogenous co-ordinates by

$$(\xi, \zeta, \nu) \mapsto (\xi, -\zeta, -\nu)$$

so

$$\beta^*(\Omega) = \frac{d(-\zeta)d(-\nu)}{-2\xi} = (-1).(-1)\frac{d\zeta d\nu}{-2\xi} = \Omega$$

so in the chart $z \neq 0$, β fixes $H^{2,0}$.
Chart $w \neq 0$: We choose $\lambda = w^{-1}$, giving homogeneous co-ordinates

$$(\xi, \zeta, \nu) = (\frac{x}{w^4}, \frac{y}{w^2}, \frac{z}{w}).$$

and

$$\begin{aligned}\tilde{f} &= \lambda^8(-x^2+y^4+z^8+w^8)\\ &= -\xi^2+\zeta^4+\nu^8+1\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial\tilde{f}}{\partial\xi} &= -2\xi\\ \Omega &= \frac{d\zeta d\nu}{\partial\tilde{f}/\partial\xi}.\end{aligned}$$

We see that β acts on these homogenous co-ordinates by

$$(\xi,\zeta,\nu) \mapsto (\xi, i\zeta, -i\nu)$$

so

$$\beta^*(\Omega) = \frac{d(i\zeta)d(-i\nu)}{-2\xi} = (i).(-i)\frac{d\zeta d\nu}{-2\xi} = \Omega$$

so in the chart $w \neq 0$, β fixes $H^{2,0}$. □
This shows us that $\gamma = \beta \circ \alpha$ is an order 4 automorphism inducing a non-trivial action on $H^{2,0}$. The fixed points of γ are those where

$$(\lambda^4 x, \lambda^2 y, \lambda z, \lambda w) = (-x, iy, -iz, w).$$

for some $\lambda \in \mathbb{C}^*$ and (x,y,z,w) lies in the zero locus of f.

Proposition 11.3. *γ has exactly two fixed points on X. On a smooth model $\widetilde{X}$ of X, γ has exactly four fixed points.*

Proof: we analyse the different possible values of λ.

- $w \neq 0$ $\Rightarrow \lambda = 1 \Rightarrow x = y = z = 0$. However, the point $(0,0,0,w)$ cannot lie on the zero locus of f, so every fixed point must have $w = 0$.
- $z \neq 0$ $\Rightarrow \lambda = -i \Rightarrow x = y = 0$. The point $(0,0,z,0)$ doesn't lie on the zero locus of f, so every fixed point must have $z = 0$.
- $y \neq 0$ $\Rightarrow \lambda = \sqrt{i}$, so we look for fixed points $(x,y,0,0)$ in the zero locus of f, i.e. points s.t. $x^2 = y^4$.

There are two cases to consider. If $x = y^2$ then set $\lambda = y^{-\frac{1}{2}}$, so $(x,y,0,0) = (\lambda^4 x, \lambda^2 y, 0, 0) = (1,1,0,0)$ in $W\mathbb{P}_{(4,2,1,1)}$. If $x = -y^2$ then the same calculation yields $(x,y,0,0) = (-1,1,0,0)$ which is clearly not the same as $(1,1,0,0)$. Thus we have exactly two fixed points of γ. Label these q_1 and q_2. Note that $q_1, q_2 \in X$ are $\mathbb{Z}_2$-type singularities on X by the remark made before Proposition 11.2, since $q_1, q_2 \in W\mathbb{P}_{(4,2,1,1)}$ are singular. Explicit blow up shows that γ induces a $\mathbb{Z}_4$-type automorphism on a smooth model $\widetilde{X}$ of X with exactly two fixed points ${q_i}^{(1)}, {q_i}^{(2)}$ over each q_i. □.

Corollary 11.4. *Let κ be the order 4 automorphism on $E \times X$ defined by $\kappa(e,x) := (\phi(e), \gamma(x))$. Then κ has 16 fixed points on $E \times X$.*

Proof : easy to see that the fixed points are the points $p_i \times {q_j}^{(k)}$.□
We now calculate the fixed points of κ^2. Clearly κ^2 has no action on E, so we examine the action of γ^2 on X. $\gamma^2(x,y,z,w) = (x,-y,-z,w)$, so we can as before find fixed points by solving $(\lambda^4 x, \lambda^2 y, \lambda z, \lambda w) = (x, -y, -z, w)$.

Proposition 11.5. *γ^2 has exactly 8 fixed points on $\widetilde{X}$, and 4 of them, labelled t_i, are identified in pairs under γ (say $t_1 \sim t_3$ and $t_2 \sim t_4$).*

Proof: If $w \neq 0$, we get $\lambda = 1$ so $y = z = 0$. So we look for points $(x, 0, 0, w)$ in the zero locus of f. Thus $x^2 = w^8$. If $x = w^4$, we get the point $(1, 0, 0, 1)$ and if $x = -w^4$ we get $(-1, 0, 0, 1)$. We note that $\gamma(1, 0, 0, 1) = (-1, 0, 0, 1)$ so γ identifies these two points.
If $z \neq 0$ then $\lambda = -1$ so $y = w = 0$, so the points $(x, 0, z, 0)$ are fixed. We see that $(1, 0, 1, 0)$ and $(-1, 0, 1, 0)$ are the points in the zero locus of f. We note that $\gamma(1, 0, 1, 0) = (-1, 0, -i, 0) = (-1, 0, 1, 0)$ so γ identifies these two points.
If $y \neq 0$ then $\lambda = \pm i$. In either case $z = w = 0$ so we are left with the 2 points that were fixed under γ. No additional fixed points of γ^2 are introduced by the blow up $\widetilde{X} \longrightarrow X$. □.
By abuse of notation, in the following we write X for $\widetilde{X}$.

Corollary 11.6. *The fixed points of κ^2 on $E \times X$ are the curves $E \times t_i$ and $E \times \{q_j\}^{\{}(k)\}$.*

Proof: Clear from the fact that $\kappa^2 = (\phi^2, \gamma^2)$, and ϕ^2 is the identity so all of E is fixed.□
Now we calculate the Hodge diamonds of the $+1, -1, +i, -i$ eigenspaces of the action of γ on X. We note that because E has only $+1$ and -1 eigenspaces, we need not calculate the $\pm i$ eigenspaces for X, as they will have no contribution to the final Hodge diamond.
Denote the eigenspaces of the action of γ^2 on X by X^+ and X^-, and the eigenspaces of the action of γ on X by $X^{+1}, X^{-1}, X^{+i}, X^{-i}$. Since $H^*(X^+) = H^*(X^{+1}) \oplus H^*(X^{-1})$, we will calculate the cohomology of X^+ to help determine the cohomologies of X^{+1} and X^{-1}. It is easy to check that γ^2 acts trivially on $H^{2,0}(X)$, so the (2,0)- and (0,2)- forms will lie in X^+. We calculate

$$\begin{aligned}\chi(X/\gamma^2) &= \frac{1}{2}\chi(X - \bigcup_{i=1}^{4}\{t_i\} - \bigcup_{j,k}\{q_j^{(k)}\}) + \sum_{i=1}^{4}\chi(t_i) + \sum_{j,k}\chi(q_j^{(k)}) \\ &= \frac{1}{2}(24 - 4 - 4) + 4 + 4 = 16\end{aligned}$$

and so we get $16 = 1 + 1 + h^{1,1} + 1 + 1$ and thus $h^{1,1} = 12$ and we get

$$H^*(X^+):\ \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 12 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \qquad H^*(X^-):\ \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 8 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{matrix}\ .$$

The Euler characteristic of X/γ is given by

$$\chi(X/\gamma) = \chi\left(\frac{X - \bigcup_{i=1}^{4}\{t_i\} - \bigcup_{j,k}\{q_j^{(k)}\}}{\gamma}\right) + \sum_{i=1}^{2}\chi(t_i) + \sum_{j,k}\chi(q_j^{(k)})$$

and since the t_i are identified in pairs under γ, instead of 4 non-fixed points we have 2 fixed points

$$\begin{aligned}\chi(X/\gamma) &= \frac{1}{4}\left(\chi(X) - \chi(\bigcup_{i=1}^{4}\{t_i\}) - \chi(\bigcup_{j,k}\{q_j^{(k)}\})\right) + \sum_{i=1}^{2}\chi(t_i) + \sum_{j,k}\chi(q_j^{(k)}) \\ &= \frac{1}{4}(24-4-4)+2+4 = 10.\end{aligned}$$

We have checked that γ acts non-trivially on $H^{2,0}(X)$, so we get the ± 1 eigenspaces to be

$$H^*(X^{+1}): \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 8 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \qquad H^*(X^{-1}): \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 4 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{matrix}\ .$$

We can now calculate the Hodge diamond of the orbifold $E \times X/\kappa$ by using the fact that

$$H^*((E\times X)/\kappa) = (H^*(E^{+1}) \otimes H^*(X^{+1}) \oplus (H^*(E^{-1}) \otimes H^*(X^{-1}))$$

which gives

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 9 & & 0 & \\ 1 & & 5 & & 5 & & 1 \\ & 0 & & 9 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}\ .$$

There is one step left, and that is to calculate the contribution from blow-up of singularities. We do this observing that

$$\frac{\widetilde{E\times X}}{\kappa} \cong \frac{\widetilde{V}}{\widetilde{\kappa}}$$

where

$$V = \frac{\widetilde{E\times X}}{\kappa^2}$$

and $\widetilde{\kappa}$ is a naturally induced automorphism of order 20. We examine the action of $\widetilde{\kappa}$ on $(\widetilde{E\times X})/\kappa^2$ where we have blown up the 8 fixed curves $E\times\{t_i\}$ and $E\times\{q_j^{(k)}\}$ with $\mathbb{CP}^1$, giving a contribution

$$\begin{matrix} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 1 & & 1 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{matrix}$$

for each. The action of $\widetilde{\kappa}$ on $(\widetilde{E\times X})/\kappa^2$ is an involution, and the fixed points are just the fixed points of κ, i.e $\{p_i\}\times\{q_j^{(k)}\}$. Thus we blow up each of these with

$\mathbb{CP}^2$, giving 16 contributions of

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}.$$

We are not finished yet, because we need to see if the action of $\widetilde{\kappa}$ affects any of the blow-ups we made of the fixed curves under κ^2. First, the blow ups of the $E \times \{t_i\}$ are paired, as before. Second, for the blow ups of the $E \times \{q_i^{(k)}\}$ the only part of the blow-up that survives is

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

for each $E \times \{q_i^{(k)}\}$. So the total contribution from blow up is this same Hodge diamond for each of the 4 fixed curves and for each of the 16 fixed points together with two contributions from two pairs of $E \times \{t_i\}$. So the total contribution from blow up is

$$\left(2\times \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 1 & & 1 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \right) + \left(4\times \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \right)$$

$$+ \left(16\times \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \right) = \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 22 & & 0 & \\ 0 & & 2 & & 2 & & 0 \\ & 0 & & 22 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

giving the final Hodge diamond of

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 31 & & 0 & \\ 1 & & 7 & & 7 & & 1 \\ & 0 & & 31 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

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