Course Notes for Large Cardinals in Set Theory

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Acknowledgments

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Chapter 1

Background and Motivations

1.1 A Review of Some Basic Notions of Set Theory

Large cardinals are axioms of set theory that build upon the more commonly used axioms.

Definition 1.1. The *Zermelo-Fraenkel* axioms are the following:

1. *Extensionality:* $\forall x, y(z \in x \leftrightarrow z \in y) \to x = y$, i.e. sets are defined by their elements.

2. *Foundation:* $\forall x, \exists z \in x, x \cap z = \emptyset$.

3. *Union:* Unions of sets are sets.

4. *Pairing:* If x and y are sets, then $\{x, y\}$ is a set.

5. *Power Set:* Power sets of sets are sets.

6. *Separation:* If $\varphi(v, \bar{w})$ is a formula with parameters $\bar{w}$ and x is a set, then $y = \{z \in x : \varphi(z, \bar{w})\}$ is a set.

7. *Replacement:* If F is a class function, then for any set X, $\{F(x) : x \in X\}$ is a set.

8. *Choice:* If X is a set, there is a function $F : \mathcal{P}(X) \to X$ such that for all nonempty $Y \subset X$, $F(Y) \in Y$.

9. *Infinity:* The natural numbers are a set, i.e. there is an infinite set.

The abbreviation ZF will refer all of the axioms without the axiom of choice, and ZFC will refer to all the axioms including choice.

When we speak of a model of set theory we are talking about a structure in which the Zermelo-Fraenkel axioms are true. Various subtleties arise when we consider various subtheories of ZF, but these are outside the scope of the course.

Definition 1.2. α is an *ordinal* if it is a set such that:

1. It is *transitive*, meaning that if $\beta \in \gamma \in \alpha$, then $\beta \in \alpha$.

2. It is well-ordered by $\in$, i.e. $\in$ is a linear order and every subset of α has a minimal element. ($\in$ and $<$ are usually used interchangably in the context of ordinals.)

A *successor* is an ordinal of the type $\alpha = \beta \cup \{\beta\} := \beta + 1$ and a *limit* ordinal takes the form $\alpha = \bigcup_{\beta \in \alpha} \beta := \sup_{\beta < \alpha} \beta$.

Example 1.3. Every natural number can be represented as an ordinal: $0 = \emptyset,\ 1 = \{\emptyset\},\ 2 = \{\emptyset, \{\emptyset\}\},\ 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc. We write the set of natural numbers as the limit ordinal $\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}, \ldots\}$. $\omega + 1 := \omega \cup \{\omega\}$ is an infinite successor.

Theorem 1.4. *Any well-ordered set can be put in bijection with an ordinal.*

Ordinals are a generalization of the natural numbers, and the most important usage of ordinal numbers is in the definitions of transfinite induction:

Theorem 1.5. *Suppose P is a property such that if P holds for all $\beta < \alpha$, then P holds for α. Then P holds for all ordinal numbers.*

Definition 1.6. A *cardinal* number is an ordinal that does not inject onto a smaller ordinal, i.e. one that is "bigger" than all ordinals preceding it. The α^{th} infinite cardinal is denoted either ω_α or $\aleph_\alpha$. We write $\aleph_0 = \omega_0 = \omega$.

Definition 1.7. A cardinal κ is *regular* if for any cardinal $\lambda < \kappa$ and any function $f : \lambda \to \kappa$, the range of f is bounded in κ. Otherwise κ is *singular*.

Example 1.8.

- ω is regular because finite functions have finite images.
- For any cardinal, κ^+ (the least cardinal greater than κ) is regular: Recall that $|\kappa \times \kappa| = \kappa$ for all infinite cardinals κ. If $f : \kappa \to \kappa^+$, then $f(\alpha) < \kappa^+$ for all $\alpha < \kappa$, i.e. $|f(\alpha)| < \kappa^+$. Therefore $\bigcup_{\alpha \in \kappa} f(\alpha)$ has cardinality κ. Hence $\operatorname{im} f \subseteq \beta$ for some $\beta < \kappa^+$.
- $\aleph_\omega$ is singular because $\omega < \aleph_\omega$ and the map $f : n \mapsto \aleph_n$ is unbounded in $\aleph_\omega$.

Definition 1.9. A cardinal κ is a *strong limit* if for every cardinal $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Example 1.10.

- ω is a strong limit.
- If ω_α is the α^{th} cardinal, then $\omega_{\alpha+1}$ is not a strong limit because $|P(\omega_\alpha)| > \omega_\alpha$.
- Let $\kappa_0 = \omega$, and for every n let $\kappa_{n+1} = |2^{\kappa_n}|$. Then $\kappa^* := \bigcup_{n<\omega} \kappa_n$ is a strong limit: if $\lambda < \kappa^*$, then $\lambda \leq \kappa_n$ for some n, so $2^\lambda \leq \kappa_{n+1} < \kappa^*$.

1.2 Regarding the Consistency of Inaccessible Cardinals

Remark 1.11. The only example of an uncountable strong limit that we have given is singular.

Definition 1.12. An uncountable cardinal κ is *inaccessible* if it is both regular and a strong limit.

Oftentimes we *define* a structure using transfinite induction.

Set theorists conceive of all sets as being members of the Von Neumann hiearchy:

Definition 1.13.

- $V_0 = \emptyset$

- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
- If α is a limit, $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.
- $V = \bigcup_{\alpha \text{ an ordinal}} V_\alpha$

Remark 1.14. V is a proper class, not a set.

Theorem 1.15. *If κ is inaccessible, then V_κ satisfies the axioms of* ZFC*.*

Fact 1.16. *If α is a limit ordinal, then V_α satisfies all of the axioms of* ZFC *other than replacement.*

Example 1.17. Most of the axioms are uncontroversial here.

Proof of Theorem 1.15. The tricky part is verifying replacement, which states that the image of every function with a set as a domain is a set. So we want to show that if f is a function that takes values in V_κ and $\operatorname{dom} f \in V_\kappa$, then $\operatorname{im} f \in V_\kappa$.

We claim that if $\alpha < \kappa$, then $|V_\alpha| < \kappa$. Prove this by induction on $\alpha < \kappa$. For every α, we assume as the inductive hypothesis that if $\beta < \alpha$ then $|V_\beta| < \kappa$. If α is a successor, then $\alpha = \beta + 1$, so $|V_\alpha| = |\mathcal{P}(V_\beta)| < \kappa$ since κ is a strong limit. If α is a limit, then $|V_\alpha| = |\bigcup_{\beta<\alpha} V_\alpha| < \kappa$ because $\beta \mapsto |V_\beta| < \kappa$ and κ is regular.

It follows that if $x \in V_\kappa$ then $|x| < \kappa$: If $x \in V_\kappa$ then there is some $\alpha < \kappa$ such that $x \in V_\alpha$, so $|x| \leq |V_\alpha| < \kappa$.

Therefore, if f is a function and $\operatorname{dom} f \in V_\kappa$, then since $|\operatorname{dom} f| < \kappa$ we have $|\operatorname{im} f| < \kappa$. $\operatorname{im} f \subseteq V_\kappa$ since f takes values in V_κ, so for every $x \in \operatorname{im} f$ let α_x be such that $x \in V_{\alpha_x}$. Let $\beta = \bigcup_{x \in \operatorname{im} f} \alpha_x$, so $\operatorname{im} f \subseteq V_\beta$. $\beta < \kappa$ because κ is regular, so $\operatorname{im} f \in V_{\beta+1} \subseteq V_\kappa$. □

Theorem (Gödel's Second Incompleteness Theorem)**.** ZFC *cannot prove its own consistency.*

Theorem (Gödel's Completeness Theorem)**.** *A theory has a model if and only if it is consistent.*

Corollary 1.18. ZFC *cannot prove the existence of an inaccessible cardinal.*

Corollary 1.19. Con(ZFC+ *"there exists an inaccessible cardinal"*) *implies* Con(ZFC+ *"there* do not *exist any inaccessible cardinals"*).

Proof. Let κ be the least inaccessible and consider V_κ, which contains no inaccessible cardinals. □

Remark 1.20. $V_\omega = \bigcup_{n<\omega} V_n$ proves that the negation of the infinity axiom is consistent.

1.3 Measures

The are some connections between the measures used in analysis and those used for set theory.

Question (Lebesgue). *Find a function $m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ such that:*

1. $m(\mathbb{R}) = \infty$ *and* $m([0, 1]) < \infty$*;*
2. $m(X) = m(X + r)$ *for all* $r \in \mathbb{R}$*;*
3. *if* $\langle X_n : n \in \mathbb{N}\rangle \subset \mathcal{P}(\mathbb{R})$ *is a sequence of pairwise disjoint sets, then* $m\left(\bigcup_{n\in\mathbb{N}} X_n\right) = \sum_{n\in\mathbb{N}} m(X_n)$.

Theorem (Vitali). *Assuming* AC*, no.*

Remark 1.21. With AC, this is still an interesting question.

Definition 1.22. An uncountable cardinal κ is *measurable* if there is a non-principal κ-complete ultrafilter U on κ. In other words, there is $U \subset \mathcal{P}(\kappa)$ such that the following hold:

1. For all $\alpha < \kappa$, $\{\alpha\} \notin U$;
2. If $X \in U$ and $X \subseteq Y$, then $Y \in U$;
3. If $X \subset \kappa$ and $X \notin U$, then $\kappa \setminus X \in U$;
4. If $\lambda < \kappa$ and $\langle X_\xi : \xi < \lambda\rangle \subset U$, then $\bigcap_{\xi<\lambda} X_\xi \in U$.

We often call U a *measure* on κ.

Remark 1.23. If we did not require κ to be uncountable in this definition, ω would be measurable.

Theorem 1.24. *If κ is measurable, then κ is inaccessible.*

Proof. Let U be a measure on κ. First we show that κ is regular. We make two assertions that we leave as exercises: (1) if X is bounded in κ, i.e. if there is some $\beta < \kappa$ such that $X \subset \beta$, then $X \notin U$; (2) if $\kappa = \bigcup_{\xi<\lambda} X_\xi$ for some $\lambda < \kappa$, then there is some $\xi < \lambda$ such that $X_\xi \in U$. Now suppose that there is a function $f : \lambda \to \kappa$ for some $\lambda < \kappa$. For each ξ, let $X_\xi = f(\xi) \setminus \bigcup_{\eta<\xi} f(\eta)$. Hence by (1) we know that $X_\xi \notin U$ for all $\xi < \lambda$. If f were unbounded in κ, then we would contradict (2).

Now we show that κ is inaccessible. Suppose for contradiction that there is some $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Let $\langle f_\alpha : \alpha < \kappa \rangle$ be a sequence of distinct functions from λ to $\{0, 1\}$. For each $\xi < \lambda$, if $\{\alpha < \kappa : f_\alpha(\xi) = 0\} \in U$ then define $X_\xi = \{\alpha < \kappa : f_\alpha(\xi) = 0\}$ and define $\epsilon_\xi = 0$. Otherwise, it must be the case that $\{\alpha < \kappa : f_\alpha(\xi) = 1\} \in U$, and so we define X_ξ to be this set and define $\epsilon_\xi = 1$. Then we take $X := \bigcap_{\xi<\lambda} X_\xi$, and we will have $X \in U$. However, we then find that $\alpha \in X$ if and only if $f_\alpha(\xi) = \epsilon_\xi$ for all $\xi < \lambda$. By the distinctness of the functions $\langle f_\alpha : \alpha < \kappa \rangle$, this implies that X is a singleton, which is a contradiction. □

When it comes to the so-called "measure problem," large cardinals are in fact provably necessary.

Theorem (Solovay, Shelah)**.** *The following statements are equiconsistent:*

- ZF + *"all sets of reals are Lebesgue-measurable".*
- ZFC + *"there is an inaccessible cardinal".*

1.4 The Tree Property

Definition 1.25. A *tree* is a partially ordered set T with an order $<_T$ such that for every $x \in T$, $\{y \in T : y \leq x\}$ is well-ordered.

- The α^{th} *level* of T is the set $T_\alpha = \{x \in T : \text{ot}\{y \in T : y < x\} = \alpha\}$.
- The *height* of T is $\sup\{\alpha : \exists x \in T, \text{ot}\{y \in T : y < x\} = \alpha\}$.

Example 1.26. The set T of functions $f : \alpha \to \{0, 1\}$ for $\alpha < \omega_1$ is a tree where $g \leq_T f$ if $g = f \upharpoonright \text{dom} f$. If $\text{dom} f = \alpha$, then f is in the α^{th} level of T. T has height ω_1.

Definition 1.27. A κ-tree T is a tree of height κ and levels of size strictly less than κ.

Example 1.28. The previous example is *not* an ω_1-tree because the ω^{th} level has size 2^ω. But the set of functions from ordinals $n < \omega$ would be an ω-tree.

Definition 1.29. If T is a tree with ordering $\leq_T$ and $x \in T$, y is a *descendent* of x if $x \leq_T y$. We say y is an *immediate descendent* of x if it is a descendent of x and there are no $z \in T$ with $x <_T z <_T y$.

Lemma 1.30 (König's Lemma)**.** *If T is an ω-tree then it has an infinite branch, i.e. there is a set $b \subseteq T$ such that for every n, $|T_n \cap b| = 1$.*

Proof. Construct $b = \{b_n : n < \omega\}$ by induction on n using the inductive hypothesis that for all $m < n$, b_m has infinitely many descendents. Assume without loss of generality that T has a root b_0, i.e. a unique element on level 0. For every $n > 0$, b_n has finitely many immediate descendants $\{x_i : i < k\}$. Let $P_i = \{y \in T : x_i \leq y\}$. At least one P_i must be infinite by the pigeonhole principle, so let $b_{n+1} = x_i$. □

Definition 1.31. A cardinal κ satisfies the *tree property* if every κ-tree T has an unbounded branch b such that for all $\alpha < \kappa$, $|b \cap T_\alpha| = 1$.

Example 1.32. ω satisfies the tree property.

Theorem 1.33. *ω_1 does not satisfy the tree property.*

Proof. We will construct a tree T by induction such that the α^{th} level T_α will consist of sequences of rational numbers of order-type α. In other words, elements of T will take the form $\langle q_\beta : \beta < \alpha \rangle \subset \mathbb{Q}$ where if $\gamma < \beta < \alpha$, then $q_\gamma < q_\beta$. Since there are only countably many rational numbers, this tree will not have an unbounded branch. (Having a cofinal branch would be equivalent to having a sequence $\langle q_\beta : \beta < \omega_1 \rangle$ such that for all $\alpha < \omega_1$, $\langle q_\beta : \beta < \alpha \rangle \in T_\alpha$.)

Our inductive hypothesis is the following: For all $\beta < \alpha$, $x \in T_\beta$, and $\sup x < q \in \mathbb{Q}$, there is a sequence y of rationals of order-type α such that $x \subseteq y$ and $\sup y \leq q$.

Zero Case: First let $T_0 = \emptyset$.

Successor Case: If $\alpha = \beta + 1$, then let $T_\alpha = \{x \frown \langle q \rangle : x \in T_\beta, q \in \mathbb{Q}, \sup x \leq q\}$. It is fairly immediate to see that if T_β satisfies the inductive hypothesis, then so will T_α.

Limit Case: Suppose α is a limit ordinal.

Claim. *For every $x \in \bigcup_{\beta<\alpha} T_\alpha$ and every $q \geq \sup x$, there is a sequence of rationals $y_{x,q}$ of order-type α such that $x \subseteq y_{x,q}$, $\sup y_{x,q} \leq q$, and for* all *$\beta < \alpha$, $y_{x,q} \upharpoonright \beta \in T_\beta$.*

Assuming the claim is true, we can let $T_\alpha = \{y_{x,q} : x \in \bigcup_{\beta<\alpha} T_\beta, q \geq \sup y_{x,q}\}$ where the $y_{x,q}$'s witness the claim, so T_α is countable.

Proof of Claim. Suppose $q \in \mathbb{Q}$ and $x \in \bigcup_{\beta<\alpha} T_\alpha$ and that the order type of x is $\gamma < \alpha$. Since α is a limit, there is a sequence $\langle \alpha_n : n < \omega \rangle$ such that $\sup_{n<\omega} \alpha_n = \alpha$ and $\alpha_0 > \gamma$. Let q_n be a sequence of rational numbers so that $\lim_{n<\omega} q_n = q$. Then for every n, let y_{n+1} witness the inductive hypothesis for q_{n+1} and y_n, i.e. $y_{n+1} \supset y_n$ and $\sup y_{n+1} \leq q_{n+1}$. Then $\bigcup_{n<\omega} y_n$ witnesses the claim. □

This finishes the construction. □

Theorem (Mitchell and Silver)**.** *The following are equiconsistent:*

- *ω_2 has the tree property.*
- *There is a cardinal κ that is inaccessible and has the tree property.*

Theorem 1.34. *If κ is measurable, then it satisfies the tree property.*

Proof. Exercise. □

Remark 1.35. If κ is measurable, then there are many inaccessibles below κ!

Chapter 2

Large Cardinals and Filters

In this section we will introduce Mahlo cardinals, further develop the theory of measurable cardinals, and we will also introduce weakly compact cardinals. Most importantly, we will discuss the concept of embedding characterizations of large cardinals.

2.1 Stationary Sets

Here we will discuss an important concept that is intertwined with the study of large cardinals.

Definition 2.1. A function f whose domain is a subset of the ordinals is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in \mathrm{dom}(f) \setminus \{0\}$.

Remark 2.2. Obviously we have a regressive funtion f with domain ω: Just let $f(n) = n - 1$. But can we get a non-constant regressive function with domain $\aleph_1$?

Definition 2.3. The *cofinality* of an ordinal δ is the least ordinal γ such that there exists an unbounded function $f : \gamma \to \delta$. We denote this $\mathrm{cf}(\delta) = \gamma$. We call such a function *cofinal* in δ.

Observation 2.4. *If γ is any ordinal such that $\gamma = \mathrm{cf}(\delta)$, then γ is in fact a regular cardinal.*

Definition 2.5. Let κ be an uncountable regular cardinal. A subset $C \subseteq \kappa$ is *club* in κ (or *a* club in κ) if:

1. C is unbounded in κ, i.e. $\forall\beta < \kappa, \exists\alpha \in C$, $\alpha > \beta$;

2. C is *closed*, i.e. if $\langle\alpha_\xi : \xi < \lambda\rangle \subset C$ with $\lambda < \kappa$, then $\sup_{\xi<\lambda}\alpha_\xi \in C$.

The set $\{X \subset \kappa : X \text{ contains a club}\}$ is called *the club filter on* κ.

Example 2.6. Consider (1) the set of limit ordinals in κ or perhaps (2) $\kappa \setminus \alpha$ for any $\alpha < \kappa$.

Remark 2.7. We can define clubs in limit ordinals that are not cardinals.

Proposition 2.8. *The club filter is κ-complete. In other words, if $\langle C_\xi : \xi < \lambda\rangle$ are clubs in κ and $\lambda < \kappa$, then $\bigcap_{\xi<\lambda} C_\xi$ is a club in κ. (In particular, the club filter is a filter.)*

Proof. Closure of $\bigcap_{\xi<\lambda} C_\xi$ is straightforward from the definitions.

For unboundedness, we will first argue that the intersection of any two clubs C and D in κ is unbounded. Fix $\delta < \kappa$. Using the unboundedness of C and D, define by induction sequences $\langle\alpha_n : n < \omega\rangle \subset C$ and $\langle\beta_n : n < \omega\rangle \subset D$ such that $\alpha_0 \geq \delta$ and $\alpha_n < \beta_n < \alpha_{n+1}$ for all $n < \omega$. Then we can see that $\sup_{n<\omega}\alpha_n = \sup_{n<\omega}\beta_n = \gamma$. (This is known as "interleaving.") By closure of C, we know that $\gamma = \sup_{n<\omega}\alpha_n \in C$, and by closure of D, we know that $\sup_{n<\omega}\beta_n = \gamma \in D$, and thus $\gamma \in C \cap D$.

Now let us do the general argument. We will argue that $\bigcap_{\xi<\eta} C_\xi$ is unbounded in κ by induction on $\eta < \kappa$.

- The statement is of course trivial if we are taking only one club, so that gives us the base case.

- Suppose that we are considering

$$\bigcap_{\xi<\eta+1} C_\xi = \left(\bigcap_{\xi<\eta} C_\xi\right) \cap C_{\xi+1}.$$

  The first part is a club by our inductive hypothesis, and the intersection of everything is a club by the same argument we used for two clubs.

- Now suppose we are considering $\bigcap_{\xi<\eta} C_\xi$ where η is a limit ordinal. By induction, $\bigcap_{\xi<\zeta} C_\xi$ is a club for all $\zeta < \eta$. Therefore we can assume without loss of generality that $C_\zeta \subseteq C_\xi$ for all $\xi < \zeta$, i.e. the clubs are "nested." Now define a sequence $\langle \alpha_\xi : \xi < \eta \rangle$ to be an increasing sequence above some fixed $\delta < \kappa$ such that $\alpha_\xi \in C_\xi$ for all $\xi < \eta$. If $\beta = \sup_{\xi<\eta} \alpha_\xi$, then $\beta < \kappa$ by regularity. Because of nestedness, $\alpha_\xi \in C_\zeta$ for all $\zeta \leq \xi$, and so $\beta = \sup_{\zeta \leq \xi < \eta} \alpha_\xi \in C_\zeta$ for all $\zeta < \eta$.

This finishes the proof. □

Definition 2.9. Let κ be an uncountable regular cardinal and let $\langle X_\alpha : \alpha < \kappa \rangle$ be a collection of subsets of κ. Then $\triangle_{\alpha<\kappa} X_\alpha := \{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$ is the *diagonal intersection* of this collection. A filter F on κ is *normal* if for all $\langle X_\alpha : \alpha < \kappa \rangle \subset F$, $\triangle_{\alpha<\kappa} X_\alpha \in F$.

Remark 2.10. We do not necessarily have $\triangle_{\alpha<\kappa} X_\alpha \subseteq X_\alpha$ for all $\alpha < \kappa$: Consider the example where $X_\alpha = \kappa \setminus \alpha$ for all $\alpha < \kappa$.

Proposition 2.11. *If κ is an uncountable regular cardinal and $\langle C_\alpha : \alpha < \kappa \rangle$ is a collection of clubs in κ, then $\triangle_{\alpha<\kappa} C_\alpha$ is a club in κ. (In other words, the club filter is normal.)*

Proof. Notice that the diagonal intersection is the same if we replace each C_α with $\bigcap_{\beta \leq \alpha} C_\beta$. Hence, as in the last proof, we can assume without loss of generality that $C_\beta \subseteq C_\gamma$ for $\gamma \leq \beta$.

Closure: Consider $\langle \gamma_\xi : \xi < \eta \rangle \subset \triangle_{\alpha<\kappa} C_\alpha$ be a strictly increasing sequence where η is a limit ordinal, and let $\sup_{\xi<\eta} \gamma_\xi = \gamma^*$. By the definition of the diagonal intersection, we need to show that $\gamma^* \in \bigcap_{\beta<\gamma^*} C_\beta$.

The definition of diagonal intersections already tells us that $\gamma_\xi \in \bigcap_{\beta<\gamma_\xi} C_\beta$ for all $\xi < \eta$. Using nestedness, this means that $\gamma_\zeta \in C_{\gamma_\xi}$ for all $\zeta \in (\xi, \eta)$, which implies that $\gamma^* = \sup_{\zeta<\eta} \gamma_\zeta = \sup_{\xi \leq \zeta < \eta} \gamma_\zeta \in C_{\gamma_\xi}$ for all $\xi < \eta$. Again using nestedness, we conclude that $\gamma^* \in C_\beta$ for all $\beta < \gamma^*$.

Unboundness: Given $\beta < \kappa$, we will inductively define a sequence $\langle \gamma_n : n < \omega \rangle$ as follows: Let γ_0 be any ordinal in the interval (β, κ). Given γ_n, choose $\gamma_{n+1} \in (\gamma_n, \kappa)$ to be an element of $\bigcap_{\alpha<\gamma_n} C_\alpha$, which we know is a club. Then let $\gamma^* = \sup_{n<\omega} \gamma_n$.

Of course, γ^* is larger than β, so we just need to show that $\gamma^* \in \triangle_{\alpha<\kappa} C_\alpha$, i.e. that $\gamma^* \in C_\alpha$ for all $\alpha < \gamma^*$. Given some particular $\alpha < \gamma^*$, there is some n such that $\alpha < \gamma_n$. Then we see that $\gamma_m \in C_\alpha$ for all $m > n$. As in our previous reasoning, $\gamma^* \in C_\alpha$. □

Definition 2.12. Let κ be regular uncountable. We say that $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all clubs $C \subset \kappa$.

Example 2.13. Given a regular uncountable κ, all clubs in κ are stationary. Also, $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is stationary.

Observation 2.14. *If $S \subset \kappa$ is stationary, then S is unbounded in κ.*

Theorem 2.15 (Fodor's Lemma)**.** *Let κ be regular uncountable and let $S \subset \kappa$ be stationary. If f is a regressive function with domain S, then there is a stationary subset $S' \subseteq S$ and some $\gamma < \kappa$ such that for all $\alpha \in S'$, $f(\alpha) = \gamma$.*

Proof. Suppose otherwise. Then for all $\gamma < \kappa$, there is some club C_γ such that for all $\alpha \in C_\gamma \cap S$, $f(\alpha) \neq \gamma$. (We are sort of jumping past a step here.) Now take $C := \triangle_{\gamma<\kappa} C_\gamma$, which we now know is a club. Let $\delta \in C \cap S \neq \emptyset$, and let $f(\delta) = \gamma < \delta$. By the definition of diagonal intersections, $\delta \in \bigcap_{\alpha<\delta} C_\alpha$, meaning that $\delta \in C_\gamma$, but this contradicts the way we defined C_γ. □

Corollary 2.16. *There is no non-constant regressive function with domain $\aleph_1$.*

Definition 2.17. A regular cardinal κ is *Mahlo* if

$$\{\delta < \kappa : \delta \text{ is an inaccessible cardinal}\}$$

is stationary in κ.

Observation 2.18. *The consistency of a Mahlo cardinal implies the consistency of an inaccessible cardinal.*

Exercise. *The following are equivalent:*

1. *κ is Mahlo (as defined above).*

2. *κ is inaccessible and $\{\delta < \kappa : \delta$ is a regular cardinal$\}$ is stationary.*

2.2 Measurable Embeddings

Now we will further develop the theory of measurable cardinals.

Definition 2.19. We say that U is a *normal measure* on an uncountable regular cardinal κ if it is a normal κ-complete nonprincipal ultrafilter. The means:

- U is an ultrafilter;
- for all $\alpha < \kappa$, $\{\alpha\} \notin U$;
- for all $\lambda < \kappa$, $\langle X_\xi : \xi < \lambda \rangle \subset U$, $\bigcap_{\xi<\lambda} X_\xi \in U$;
- for all $\langle X_\xi : \xi < \kappa \rangle \subset U$, $\triangle_{\xi<\kappa} X_\xi \in U$.

Exercise. *The following are equivalent for an uncountable regular cardinal κ:*

1. *U is a non-principal κ-complete normal filter on κ;*
2. *U is a non-principal κ-complete filter on κ such that for all $X \in U$ and all regressive functions f with domain X, there is some $\gamma < \kappa$ and some $X' \subseteq X$ such that for all $\alpha \in X'$, $f(\alpha) = \gamma$.*

Theorem 2.20. *If κ is measurable, then there is a normal measure on κ.*

Proof. Fix a κ-complete non-principal ultrafilter U on κ. We will define a κ-complete non-principal ultrafilter D on κ which will also be normal.

We will define a relation $<^*$ on the set of functions $f : \kappa \to \kappa$ as follows: $f <^* g$ if and only if $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in U$. Furthermore, write $f =^* g$ if and only if $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$. Using the fact that U is an ultrafilter, we can see that $<^*$ is transitive and is a total ordering, meaning that if $f, g : \kappa \to \kappa$, then either $f <^* g$, $g <^* f$, or $f =^* g$.

Furthermore, we can see that $<^*$ is well-founded: Suppose for contradiction that $f_0 >^* f_1 >^* \ldots >^* f_n >^* \ldots$. Let $X_n := \{\alpha < \kappa : f_n(\alpha) > f_{n+1}(\alpha)\}$. So $X_n \in U$ by definition of this ordering. Let $X := \bigcap_{n<\omega} X_n$, so $X \in U$ by κ-completeness. Then if $\alpha \in X$, we have a descending sequence

$$f_0(\alpha) > f_1(\alpha) > \ldots > f_n(\alpha) > \ldots,$$

which is a contradiction.

Now we let $f : \kappa \to \kappa$ be the $<^*$-least function such that for all $\gamma < \kappa$, $\{\alpha < \kappa : f(\alpha) > \gamma\} \in U$. The set of such functions is non-empty because it contains the diagonal function $d : \alpha \mapsto \alpha$, and it contains a $<^*$-least element because this is a well ordering. We then define D to be the collection of $X \subseteq \kappa$ such that $f_{-1}(X) := \{\alpha < \kappa : f(\alpha) \in X\} \in U$.

We must then show that D is a non-principal κ-complete normal filter. Upwards closure is fairly immediate from the definition of D. Using the fact that $f_{-1}(\bigcap_{\xi<\lambda} X_\xi) = \bigcap_{\xi<\lambda} f_{-1}(X_\xi)$, we get κ-completeness as well. We have guaranteed that $f_{-1}(\{\gamma\}) \notin U$ by the definition of f, hence we have non-principality.

We are left to prove normality. Using the above exercise, we will consider some $X \in D$ and a regressive function h with domain X. Let g be defined on $f_{-1}(X)$ so that $g(\alpha) = h(f(\alpha))$. We know that $f_{-1}(X) \in U$, and that if $\alpha \in f_{-1}(X)$, then $g(\alpha) < f(\alpha)$ by the regressive-ness of h. Hence $g <^* f$, and so there is some $Z \in U$ and $\gamma < \kappa$ such that $Z := \{\alpha < \kappa : g(\alpha) \leq \gamma\} \in U$. Applying κ-completeness, we find that there is some $Z' \in U$ and some δ such that $\{\alpha < \kappa : g(\alpha) = \delta\} = Z'$. Let $X' = X \cap \{f(\alpha) : \alpha \in Z'\}$. Then h is constant on X' with value δ. $\square$

Exercise. *Let κ be a measurable cardinal and let U be a normal measure on κ. Prove the following:*

1. *If $C \subset \kappa$ is a club, then $C \in U$.*

2. *If $X \in U$, then X is stationary in κ.*

Let us now review ultrapowers.

Definition 2.21. Let U be an ultrafilter on some set S, and for a fixed language $\mathcal{L}$ and all $i \in S$ let $\mathfrak{M}_i = (M_i, \ldots)$ be an $\mathcal{L}$-structure. For the $f, g \in \prod_{i\in S} M_i$, we write $f =_U g$ if and only if $\{i \in S : f(i) = g(i)\} \in U$. Let $[f]_U$ (or just $[f]$ when the context is clear) denote the equivalence class of $f \in \prod_{i\in S} M_i$ under the equivalence relation $=_U$.

The the *ultraproduct* of $\mathfrak{M}_i$, $i \in S$ $\mathfrak{M}_i$, $i \in S$, which is denoted $\prod_{i\in S} \mathfrak{M}_i/U$, is the structure with the following properties:

- Its underlying set is $\{[f]_U : f \in \prod_{i\in S} M_i\}$.

- For any n-ary relation R in $\mathcal{L}$, we have a relation R_U in $\prod_{i \in S} \mathfrak{M}_i/U$ where

$$R_U([f_1]_U, \ldots, [f_n]_U) \text{ holds} \iff \{i \in S : R(f_1(i), \ldots, f_n(i)) \text{ holds}\} \in U.$$

- Functions and constants in $\prod_{i \in S} \mathfrak{M}_i/U$ are defined analogously.

If $\mathfrak{M}_i = \mathfrak{M}$ for all $i \in S$, then $\prod_{i \in S} \mathfrak{M}_i/U$ is an *ultrapower* of $\mathfrak{M}$ and is denoted $\mathrm{Ult}(\mathfrak{M}, U)$.

Fact 2.22 (Łos' Theorem)**.** *For any formula $\varphi(v_1, \ldots, v_n)$, we have*

$$\prod_{i \in S} \mathfrak{M}_i/U \models \varphi([f_1]_U, \ldots, [f_n]_U) \Leftrightarrow \{i \in S : \mathfrak{M}_i \models \varphi(f_1(i), \ldots, f_n(i))\} \in U.$$

Moreover, if $\mathrm{Ult}(\mathfrak{M}, U)$ is an ultrapower, $x \in M$, and c_x is the function such that $c_x(i) = x$ for all $i \in S$, then the map $x \mapsto [c_x]_U$ is an elementary map $\mathfrak{M} \to \mathrm{Ult}(\mathfrak{M}, U)$.

Now we want to apply the ultrapower concept to measurable cardinals and V. A quick reminder about why we are not worried about using the proper class V:

Proposition 2.23 (Reflection Principle)**.** *If $\varphi(v_1, \ldots, v_n)$ is a formula and α is an ordinal, then there is some $\beta \geq \alpha$ such that for any $x_1, \ldots, x_n \in V_\beta$, we have*

$$V_\beta \models \varphi(x_1, \ldots, x_n) \iff V \models \varphi(x_1, \ldots, x_n).$$

Using conjunctions, we can extend this to any finite number of formulas.

Definition 2.24. Consider an elementary embedding $j : V \to M \subset V$. We say that δ is a *critical point* of j if $j(\alpha) = \alpha$ for all $\alpha < \delta$, but $j(\delta) > \delta$.

Remark 2.25. Suppose instead we defined a critical point to be the least ordinal δ such that $j(\delta) \neq \delta$. Then we would still conclude (using elementarity) that $j(\delta) > \delta$.

Theorem 2.26 (Scott)**.** *The following are equivalent:*

1. *κ is a measurable cardinal.*

2. *There is an elementary embedding $j : V \to M$ with critical point κ where M is a transitive subclass of V.*

Proof of 1 $\implies$ 2. Given a measurable cardinal, let U be a measure on κ. (Usually we will want U to be normal.) The elementary embedding $j = j_U : V \to M$ will be an ultrapower map using U. We need to consider some details though.

Showing that the ultrapower is well-defined: For any $f, g : \kappa \to V$, we write $f =^* g$ if and only if $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$, and we write $f \in^* g$ if and only if $\{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in U$.

We need equivalence classes to be sets. Recall that the *rank* of $x \in V$ is the least γ such that $x \in V_\gamma$. For every function $f : \kappa \to V$, we will define $[f]$ to be the *set* of g *of minimal rank* such that $f =^* g$. Hence $[f]$ itself will be a set and not a proper class. (This is known as "Scott's Trick.")

We then define $\mathrm{Ult}(V, U)$ to be the model consisting of $[f]$ for all $f : \kappa \to V$ (hence class-many) with the relation $\in^*$ (where we abuse notation to write $[f] \in^* [g]$ where $f \in^* g$ for any representative).

Showing that the ultrapower is well-founded: To show that $\mathrm{Ult}(V, U)$ is well-founded, we use an argument analogous to the one in Theorem 2.20 above. It can be seen that $\in^*$ is an extensional relation as well. It follows that the Mostowski collapse π gives an isomorphism $\mathrm{Ult}(V, U) \cong M$ where M is a transitive class model. From now on, we abuse notation to use $[f]$ to refer to $\pi([f])$.

Defining the embedding: Given any $x \in V$, let $c_x : \kappa \to V$ be defined as $c_x(\alpha) = x$ for all $\alpha < \kappa$. We let $j = j_U$ be defined by $j(x) = [c_x]_U$. (Again, this is technically $\pi([c_x]_U)$.) This gives us $j : V \to M$.

Arguing that the critical point of the embedding is κ: To argue that $j(\gamma) = \gamma$ for all $\gamma < \kappa$, we use induction. Clearly $j(0) = 0$, and by elementarity $j(\alpha + 1) = j(\alpha) + 1 = \alpha + 1$. By elementarity, $\alpha < \beta$ implies $j(\alpha) < j(\beta)$, so $\alpha \leq j(\alpha)$ for all ordinals. Hence if $\gamma < \kappa$ is a limit then we just need to show that $j(\gamma) \leq \gamma$. If $[f] = \gamma < j(\gamma)$, then $\{\alpha < \kappa : f(\alpha) < \gamma\} \in U$, so using κ-completeness there is some $\beta < \gamma$ such that $\{\alpha < \kappa : f(\alpha) = \beta\} \in U$, so $[f] = \beta$, a contradiction. This shows that $j(\gamma) = \gamma$.

Then we need to argue that $j(\kappa) > \kappa$. Let $d : \kappa \to \kappa$ be given by $d(\alpha) = \alpha$ for all $\alpha < \kappa$. We can argue that $\kappa \leq [d]$ because $\alpha \leq [d]$ for all $\alpha < \kappa$. Moreover, $\{\alpha < \kappa : d(\alpha) = \alpha < \kappa = c_\kappa(\alpha)\} \in U$, so $[d] < j(\kappa)$. □

Proof of 2 $\Longrightarrow$ 1. Let $j : V \to M$ be a non-trivial embedding. (So we are so far only assuming that there is some $x \in V$ such that $j(x) \neq x$.

Claim. *There is some ordinal δ such that $j(\delta) \neq \delta$, hence $j(\delta) > \delta$.*

Proof of Claim. Suppose for contradiction that j fixes all ordinals and that x is of minimal rank such that $j(x) \neq x$. In other words, $x \in V_\alpha$ and for all $y \in V_\beta$ for $\beta < \alpha$, we have $j(y) = y$. Clearly $x \neq \emptyset$ because $j(\emptyset) = \emptyset$ by elementarity. Because of minimality, $x \subseteq j(x)$ because $y \in x$ implies $y = j(y) \in j(x)$. Hence there must be some $z \in j(x) \setminus x$. Because $\mathrm{rank}(j(x)) = j(\mathrm{rank}(x)) = \mathrm{rank}(x)$, it follows that $\mathrm{rank}(z) < \mathrm{rank}(x)$ and $j(z) = z$. But then $z = j(z) \in j(x)$, so $z \in x$ by elementarity, which is our contradiction. (Note that we used $M \subseteq V$ for this proof.) □

Now let $\kappa = \delta$ witness the claim. Define

$$D = \{X \subseteq \kappa : \kappa \in j(X)\}. \tag{2.1}$$

For the remainder of the proof, we will demonstrate that D is a measure on κ. Upwards closure of D follows immediately from its definition. We know that there is no $\alpha < \kappa$ such that $\{\alpha\} \in D$ because $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\} \not\ni \kappa$. The fact that D is an ultrafilter is because $j(\kappa) = j(X) \cup j(\kappa \setminus X)$. Finally, for κ-completeness, observe that if $\vec{X} = \langle X_\xi : \xi < \lambda\rangle \subset D$ for some $\lambda < \kappa$, then the fact that $j(\lambda) = \lambda$ implies that $j(\bigcap_{\xi<\lambda} X_\xi) = \bigcap_{\xi<\lambda} j(X_\xi)$, and so if $\kappa \in j(X_\xi)$ for all $\xi < \lambda$, then $\kappa \in \bigcap_{\xi<\lambda} j(X_\xi)$. □

Proposition 2.27. *If U is a measure on κ and $j = j_U : V \to M$ is the associated measurable embedding, then the following are equivalent:*

1. *U is a normal measure;*

2. *$\kappa = [d]_U$ where $d : \alpha \mapsto \alpha$.*

Proof. Note that *2.* is equivalent to saying that $[d]_U \leq \kappa$ since we already have the other direction. For *1.* implies *2.*, observe that if $[f]_U < [d]_U$, then f is regressive on a set in U, so it is constant with some value $\gamma < \kappa$ by normality, and so we have shown that $[d]_U \leq \kappa$. For *2.* implies *1.*, we see that *2.* implies that $[f]_U < [d]_U$ implies that $[f]_U = [\alpha \to \gamma]_U$ for some $\gamma < \kappa$, which translates to the statement that any regressive function is equivalent to a constant function on a set in U. □

Theorem 2.28. *Suppose that $j : V \to M \subseteq V$ is a non-trivial elementary embedding, that D is the measure derived from j using 2.1 above, and that $j_D : V \to Ult(V, D)$ is the elementary embedding derived from D. Then there is a unique elementary embedding $k : \mathrm{Ult}(V, D) \to M$ such that for all $x \in V$, $k(j_D(x)) = j(x)$.*

Proof. For all $[f]_D \in \mathrm{Ult}(V, D)$ (where $f : \kappa \to V$), define $k([f]_D) = (jf)(\kappa)$.

First, we show that k is well-defined. If $[f]_D = [g]_D$, then $X := \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$, i.e. $\kappa \in j(X) = \{\alpha < j(\kappa) : (jf)(\alpha) = (jg)(\alpha)\}$, i.e. $k([f]_D) = (jf)(\kappa) = (jg)(\kappa) = k([g]_D)$.

Next, we show that k is elementary. If $\mathrm{Ult}(V, D) \models \varphi([f_1]_D, \ldots, [f_n]_D)$, then $X := \{\alpha < \kappa : \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in D$, so $\kappa \in j(X) = \{\alpha < j(\kappa) : \varphi((jf_1)(\alpha), \ldots, (jf_n)(\alpha))\}$ in M, so $M \models \varphi((jf_1)(\kappa), \ldots, (jf_n)(\kappa))\}$, meaning that $M \models \varphi(k([f_1]_D, \ldots, k([f_n]_D))$.

Finally, observe that

$$k(j_D(x)) = k([\alpha \mapsto x]_D) = [\alpha \mapsto j(x)](\kappa) = j(x).$$

□

Theorem 2.29. *Let $j : V \to M$ be a measurable embedding defined from a normal measure U. (We can prove the following with a non-normal measure, but this makes the notation more complicated.) Then the following are true:*

1. *$M^\kappa \subset M$, i.e. if $\vec{a} = \langle a_\xi : \xi < \kappa \rangle$ is a sequence in V such that $a_\xi \in M$ for all $\xi < \kappa$, then $\vec{a} \in M$.*

2. *$U \notin M$.*

3. *$2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.*

Proof. Fix $j : V \to M$, etc.

1. For each $\xi < \kappa$, let $f_\xi : \kappa \to V$ be such that $a_\xi = [f_\xi]_U$. Then define $g : \kappa \to V$ such that $g(\alpha) = \langle f_\xi : \xi < \alpha \rangle$. Now we need to show that $[g]_U = \vec{a}$, since by definition $[g]_U \in M$. For each α, $g(\alpha)$ is an α-sequence, and since $[d]_U = [\alpha \to \alpha]_U = \kappa$, it follows that by Łos that $[g]_U$ is a κ-sequence. For all $\alpha > \xi$, the ξ^{th} term of $g(\alpha)$ is $f_\xi(\alpha)$, and of course $\kappa \setminus \xi \in U$, so the ξ^{th} term of $[g]_U$ is $[f_\xi]_U$.

2. Suppose for contradiction that $U \in M$. Let $e : \kappa^\kappa \twoheadrightarrow j(\kappa)$ be given by $e(f) = [f]_U$. (Note that every ordinal below $j(\kappa)$ is represented by some $f : \kappa \to \kappa$.) Since $(\kappa^\kappa)^V \in M$ (by *1.*) and since $U \in M$ by our assumption, we have $e \in M$. Therefore $M \models j(\kappa) \leq 2^\kappa$, but this is a contradiction because we are supposed to have $M \models$ "$j(\kappa)$ is inaccessible" by elementarity.

3. We have $(2^\kappa)^V \leq (2^\kappa)^M$ by the fact that $P(\kappa)^V = P(\kappa)^M$ (from *1.*). We have $(2^\kappa)^M < j(\kappa)$ again by the inaccessibility of $j(\kappa)$ in M. Finally, $j(\kappa) < ((2^\kappa)^+)^V$ because all ordinals below $j(\kappa)$ are represented by some function $f : \kappa \to \kappa$. □

Corollary 2.30. *If $j : V \to M$ is a measurable embedding, then $V \neq M$.*

Fact 2.31 (Kunen)**.** *There is no non-trivial embedding $j : V \to V$.*

Fact 2.32 (Scott)**.** *There is no non-trivial embedding $j : L \to L$ defined in L where L is Gödel's "constructible universe."*

Chapter 3

Kurepa Trees

3.1 A Very Quick Review of Forcing

We will review some definitions and facts without proofs.

Definition 3.1.

1. $\mathbb{P}$ is a *poset* if it is a partially ordered set with underlying order $\leq_{\mathbb{P}}$ (we often omit the subscript) and with a maximal element $1_{\mathbb{P}}$. We will let $\mathbb{P}$ denote a poset always. Elements $p \in \mathbb{P}$ are called *conditions* and if p, q are conditions such that $q \leq p$, then we say that q is *stronger* than p, meaning that it expresses more information.

2. If $p, q \in \mathbb{P}$, we say that p and q are *compatible* and write $p \| q$ if there is some $r \in \mathbb{P}$ such that $r \leq p, q$. Otherwise we say that p and q are *incompactible* and write $q \perp q$.

3. $\mathbb{P}$ is *non-atomic* if for all $p \in \mathbb{P}$, there exist $q, r \leq p$ such that $q \perp r$. (We will always assume that $\mathbb{P}$ is non-atomic.)

4. $F \subset \mathbb{P}$ is a *filter* if: *(1)* for all $p, q \in F$, there is some $r \in F$ with $r \leq p, q$; and *(2)* for all $p \in F$, if $p \leq q$ then $q \in F$.

5. A subset $D \subseteq P$ is *dense* if for all $p \in \mathbb{P}, \exists q \leq p, q \in D$.

6. A filter $G \subset \mathbb{P}$ is $\mathbb{P}$-*generic over* V if for all dense subsets $D \subset \mathbb{P}$, $G \cap D \neq \emptyset$. If we say that G is "a $\mathbb{P}$-generic" then we mean that it is a $\mathbb{P}$-generic *filter*.

7. A subset $A \subset \mathbb{P}$ is an *antichain* if for all $p, q \in A$, $p \neq q$ implies $p \perp q$.

8. An antichain $A \subset \mathbb{P}$ is *maximal* if for all $p \in \mathbb{P}$, there is some $q \in A$ such that $q \| p$.

Proposition 3.2. *The following are equivalent for a poset $\mathbb{P}$ and a filter $G \subset \mathbb{P}$:*

1. *G is $\mathbb{P}$-generic over V.*

2. *$G \cap D \neq \emptyset$ for every* open dense *subset of $\mathbb{P}$ in V, meaning every dense subset of $\mathbb{P}$ in V such that $p \in D$ and $q \leq p$ implies $q \in D$.*

3. *If $p \in G$ and D is dense below p (i.e. $\forall q \leq p, \exists r \in D, r \leq q$), then $G \cap D \neq \emptyset$.*

4. *$G \cap A \neq \emptyset$ for every maximal antichain $A \subset \mathbb{P}$ with $A \in V$.*

Using these definitions, we can define forcing extensions.

Definition 3.3. Fix a poset $\mathbb{P}$

1. We induct on rank to define $\mathbb{P}$-*names* $\dot{x}$ as sets consisting of ordered pairs $(\dot{y}, p)$ where $\dot{y}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

2. If $\dot{x}$ is a $\mathbb{P}$-name and G is $\mathbb{P}$-generic over V, then $\dot{x}[G] = \{(\dot{y}[G], p) : p \in G\}$ (where this is again defined by induction on rank).

3. If G is $\mathbb{P}$-generic over V, then the model $V[G]$ consists of $\dot{x}[G]$ for all $\mathbb{P}$-names $\dot{x} \in V$.

Fact 3.4. *Given a filter $\mathbb{P}$ in V, we can always produce an extension $V[G]$ where G is $\mathbb{P}$-generic over V. Moreover, $V[G] \models \mathsf{ZFC}$.*

The phrasing here is somewhat vague. The justification for this fact is that given a finite set of statements $\varphi_1, \ldots, \varphi_n$, we can obtain a model M of these statements that is both countable and transitive. Then we can use what is occasionally known as Sikorsky's Lemma to find a filter G that is generic over this model. Since any proof will involve only finitely many statements, it is more convenient to consider forcing extensions over the Von Neumann universe V.

Theorem 3.5. *The forcing relation $\Vdash$ has the following key properties:*

1. *If G is $\mathbb{P}$-generic over V, then $V[G] \models \varphi(\dot{x}_1[G], \ldots, \dot{x}_n[G])$ if and only if there is some $p \in G$ such that $p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$.*

2. *"$p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$" is definable in V.*

3. *If $p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$ and $q \leq p$, then $q \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$.*

3.2 Product Forcing and the Lévy Collapse

Definition 3.6. Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets. Then $\mathbb{P} \times \mathbb{Q}$ is the following poset:

1. $\mathbb{P} \times \mathbb{Q}$ consists of ordered pairs (p, q) such that $p \in \mathbb{P}, q \in \mathbb{Q}$;

2. $1_{\mathbb{P}\times\mathbb{Q}}$ is $(1_{\mathbb{P}}, 1_{\mathbb{Q}})$;

3. if $(p_0, q_0), (p_1, q_1) \in \mathbb{P}\times\mathbb{Q}$, then $(p_0, q_0) \leq (p_1, q_1)$ if and only if $p_0 \leq_{\mathbb{P}} p_1$ and $q_0 \leq_{\mathbb{Q}} q_1$.

Theorem 3.7. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets. Then the following are true:*

1. *If K is $\mathbb{P} \times \mathbb{Q}$-generic over V, $G = \{p \in \mathbb{P} : \exists q \in \mathbb{Q}, (p, q) \in K\}$, and $H = \{q \in \mathbb{Q} : \exists p \in G, (p, q) \in K\}$, then G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$.*

2. *If G is $\mathbb{P}$-generic over V and that H is $\mathbb{Q}$-generic over $V[G]$, then $K = G \times H = \{(p, q) : p \in G, q \in H\}$ is $\mathbb{P} \times \mathbb{Q}$-generic over V.*

Moreover, the same is true if we reverse the roles of $\mathbb{P}, G$ and $\mathbb{Q}, H$.

Proof of 1. First we show that G is $\mathbb{P}$-generic over V. Suppose that $D \subseteq \mathbb{P}$ is dense and in V. We want to show that $G \cap D \neq \emptyset$. Let $D' = \{(p, q) : p \in D, q \in \mathbb{Q}\}$. One can argue that D' is dense in $\mathbb{P} \times \mathbb{Q}$, and therefore there is $(p_0, q_0) \in D' \cap (G \times H)$. In particular, $p_0 \in G \cap D$.

We still need to show that H is $\mathbb{Q}$-generic over $V[G]$, so suppose that $D \subseteq \mathbb{Q}$ is dense and $D \in V[G]$. This means that there is a $\mathbb{P}$-name $\dot{D}$ such that $\dot{D}_G = D$ and some $p_0 \in G$ such that $p_0 \Vdash$ "D is dense in $\mathbb{Q}$". Let $q_0 \in H$ be arbitrary and let

$$D' = \{(p, q) \in \mathbb{P} \times \mathbb{Q} : (p, q) \leq (p_0, q_0), p \Vdash_{\mathbb{P}} q \in \dot{D}\}.$$

We can argue that D' is dense in $\mathbb{P} \times \mathbb{Q}$ below (p_0, q_0): If $(p, q) \leq (p_0, q_0)$, then $p \Vdash_{\mathbb{P}}$ "$\exists r \in \dot{D}, r \leq q$". Hence there are $p' \leq p$ and $q' \leq q$ such that $p' \Vdash_{\mathbb{P}}$ "$q' \in \dot{D}$". Hence $(p', q') \in D'$.

Since D' is dense below $(p_0, q_0) \in G \times H$, we find $(p, q) \leq (p_0, q_0)$ such that $(p, q) \in (G \times H) \cap D'$. This means that $q \in H \cap D$. □

Proof of 2. It is fairly apparent that $G \times H$ as defined is a filter. We want to show that it is generic. Let $D \subset \mathbb{P} \times \mathbb{Q}$ be dense. Define $D_{\mathbb{Q}} = \{q \in \mathbb{Q} : \exists p \in G, (p, q) \in D\}$. Note that $D_{\mathbb{Q}} \in V[G]$.

We will argue that $D_{\mathbb{Q}}$ is dense. To do this, let $q_0 \in \mathbb{Q}$. let $D_{\mathbb{P}} = \{p \in \mathbb{P} : \exists q \in \mathbb{Q}, q \leq q_0, (p, q) \in D\}$. This is dense by the density of D. Hence there is some $p' \in D_{\mathbb{P}} \cap G$ which is witnessed by $q' \in \mathbb{Q}$, so $(p', q') \in D$ and in particular $q' \leq q_0$.

Now let $q \in H \cap D_{\mathbb{Q}}$ using the genericity of H over V. By the definition of $D_{\mathbb{Q}}$, there is some $p \in G$ such that $(p, q) \in D$. In other words, $(p, q) \in (G \times H) \cap (\mathbb{P} \times \mathbb{Q})$. □

Example 3.8. $\mathrm{Add}(\kappa, 1) \cong \mathrm{Add}(\kappa, 1) \times \mathrm{Add}(\kappa, 1)$.

Definition 3.9. We say that a poset $\mathbb{P}$ has the *countable chain condition* if every antichain $A \subset \mathbb{P}$ is at most countable. We say that a poset $\mathbb{P}$ has the κ-chain condition (sometimes abbreviated as the κ-cc) if every antichain $A \subset \mathbb{P}$ has size strictly less than κ. (Hence $\mathbb{P}$ has the countable chain condition precisely when $\mathbb{P}$ has the $\aleph_1$-chain condition.)

Definition 3.10.

- We say that a poset $\mathbb{P}$ is countably closed if every $\leq_{\mathbb{P}}$-decreasing sequence $\langle p_n : n < \omega \rangle$ has a lower bound $q \in \mathbb{P}$, meaning that $q \leq p_n$ for all $n < \omega$.
- We say that a poset $\mathbb{P}$ is κ-closed if every regular cardinal $\lambda < \kappa$, every $\leq_{\mathbb{P}}$-decreasing sequence $\langle p_i : i < \lambda \rangle$ has a lower bound $q \in \mathbb{P}$.

Definition 3.11. Let λ and κ be regular cardinals with $\lambda < \kappa$. Then $\mathrm{Col}(\lambda, \kappa)$ is the poset of partial functions p from λ to κ ordered by reverse inclusion. In other words:

- $p \in \mathrm{Col}(\lambda, \kappa)$ if and only if $\operatorname{dom} p \subset \lambda$, $\forall \alpha \in \operatorname{dom} p$, $p(\alpha) \in \kappa$, and $|p| < \lambda$.
- $p \leq_{\mathrm{Col}(\lambda,\kappa)} q$ if and only if $q \subseteq p$.

Proposition 3.12. *The poset* $\mathrm{Col}(\lambda, \kappa)$ *forces* κ *to have cardinality* λ. *Also,* $\mathrm{Col}(\lambda, \kappa)$ *preserves cardinals up to and including* λ, *and if* $\kappa^{<\lambda} = \kappa$, *then* $\mathrm{Col}(\lambda, \kappa)$ *preserves* κ^+ *and all cardinals above* κ^+.

Definition 3.13. Suppose that λ is regular and κ is inaccessible. Then $\mathrm{Col}(\lambda, <\kappa)$ consists of conditions p such that:

1. $\operatorname{dom} p \subset \kappa \times \lambda$;
2. if $(\delta, \alpha) \in \operatorname{dom} p$, then $p(\delta, \alpha) \in \delta$;
3. $|p| < \lambda$.

And $p \leq_{\mathrm{Col}(\lambda,<\kappa)} q$ if and only if $q \subseteq p$.

Proposition 3.14. *Consider the Lévy Collapse* $\mathrm{Col}(\lambda, <\kappa)$ *where* λ *is regular and* κ *is inaccessible. The following are true:*

1. $\mathrm{Col}(\lambda, <\kappa)$ *is* λ*-closed.*
2. $\mathrm{Col}(\lambda, <\kappa)$ *satisfies the* κ*-chain condition.*
3. $\mathrm{Col}(\lambda, <\kappa)$ *forces* κ *to be the successor of* λ.
4. *Let* $\delta < \kappa$ *be an ordinal. Let* $\mathbb{P}_\delta = \{p \in \mathrm{Col}(\lambda, <\kappa) : \operatorname{dom} p \subset \delta \times \lambda\}$ *and let* $\mathbb{P}^\delta = \{p \in \mathrm{Col}(\lambda, <\kappa) : \operatorname{dom} p \subset (\kappa \setminus \delta) \times \lambda\}$. *Then* $\mathrm{Col}(\lambda, <\kappa) \cong \mathbb{P}_\delta \times \mathbb{P}^\delta$.
5. *If* G *is* $\mathrm{Col}(\lambda, <\kappa)$*-generic over* V *and* $F : \gamma \to \mathrm{ON} \in V[G]$ *is a function where* $\gamma < \kappa$, *then there is some* $\delta < \kappa$ *such that* $F \in V[G_\delta]$ *(where* G_δ *is the induced* $\mathbb{P}_\delta$*-generic where* $\mathbb{P}_\delta$ *is defined above).*

Proof. 1. If $\langle p_\xi : \xi < \eta \rangle$ is a descending sequence of conditions in $\mathrm{Col}(\lambda, <\kappa)$, then $\bigcup_{\xi<\eta} p_\xi$ is a condition in $\mathrm{Col}(\lambda, <\kappa)$ and is a lower bound.

2. Suppose $X \subseteq \mathrm{Col}(\lambda, <\kappa)$ has size κ. By the Δ-System Lemma, there is some d of cardinality λ and a subset $X' \subset X$ of cardinality κ such that for all $p, q \in X'$ with $p \neq q$, $\operatorname{dom}(p) \cap \operatorname{dom}(q) = d$. If $\delta = \sup\{\gamma : (\alpha, \gamma) \in d\}$, then there no more than δ^λ-many conditions in $\mathrm{Col}(\lambda, <\kappa)$ with domain d, and since $\delta^\lambda < \kappa$ this means that by the Pigeonhole Principle there is some $X'' \subset X'$ of cardinality κ and some $\bar{p} : d \to \kappa$ such that for all $p \in X''$, $p \restriction d = \bar{p}$. It follows that for any $p, q \in X''$, we have some $r \leq p, q$, specifically $r = p \cup q$. Therefore X cannot be an antichain.

3. In a forcing extension by $\mathrm{Col}(\lambda, <\kappa)$, we can define a surjection from λ to δ for every $\delta < \kappa$. By the κ-chain condition, $\mathrm{Col}(\lambda, <\kappa)$ does not collapse κ.

4. The map is just $p \mapsto (p \restriction (\delta \times \lambda), p \restriction ((\kappa \setminus \delta) \times \lambda))$.

5. Let $\dot{F}$ be a $\mathrm{Col}(\lambda, <\kappa)$-name for F. For $\xi < \gamma$, let $D_\xi = \{p \in \mathrm{Col}(\lambda, <\kappa) : \exists \zeta \in \mathrm{ON}, p \Vdash \dot{F}(\xi) = \zeta\}$. These D_ξ's can be thinned out to maximal antichains A_ξ that must have cardinality less than κ. So $\delta = \sup_{\xi<\gamma} |A_\xi|$ will have cardinality less than κ by regularity. All conditions in $\bigcup_{\xi<\gamma} A_\xi$ will be contained in $\mathbb{P}_{\delta'}$ for some $\delta' < \kappa$ (as defined in the statement of the proposition). □

3.3 Defining Kurepa Trees

Definition 3.15. We say that a $\aleph_1$-tree T is a *Kurepa tree* if there are more than κ-many cofinal branches of T. In other words,

$$|\{b : \forall \alpha < \kappa, |b \cap T_\alpha| = 1\}| \geq \aleph_2.$$

The *Kurepa hypothesis*, denoted KH, it the assertion that a Kurepa tree exists.

This definition can be generalized in the natural way to refer to a κ-Kurepa tree—so a plain Kurepa tree is an $\aleph_1$-Kurepa tree. We can also define the generalized Kurepa hypothesis at κ: KH_κ.

3.4 Obtaining a Model of $\neg\mathsf{KH}$

Lemma 3.16 (Silver)**.** *Suppose that $\mathbb{P}$ is a countably closed poset and that T is a tree of uncountable regular height λ and with countable levels. Then $\mathbb{P}$ does not add any cofinal branches to T. In other words, if G is $\mathbb{P}$-generic over V and $b \in V[G]$ is a totally $<_T$-ordered set such that $|b \cap T_\alpha| = 1$ for all $\alpha < \lambda$, then $b \in V$.*

Proof. Suppose that $\bar{p} \in \mathbb{P}$, $\dot{b}$ is a $\mathbb{P}$-name, and $\bar{p} \Vdash$ "$\dot{b}$ is a branch in T". We want to show that there is some $a \in V$ and some $q \leq \bar{p}$ such that $q \Vdash$ "$\dot{b} = \check{a}$". Assume for contradiction that this is not the case.

Claim. *If $q \leq \bar{p}$, $x \in T$ with $q \Vdash x \in \dot{b}$, and $\beta < \lambda$, then there are $q_0, q_1 \leq q$, $\alpha \in (\beta, \lambda)$, and $x_0, x_1 \geq_T x$ such that $x_0 \neq x_1$ and $q_i \Vdash$ "$\dot{b} \cap T_\alpha = \{x_i\}$" for $i \in \{0, 1\}$.*

Proof of Claim. Otherwise, we have the negation of this statement, which implies that there is some $\bar{q} \leq \bar{p}$, $\bar{\beta} < \lambda$, and $\bar{x} \in T$ with $q \Vdash$ "$\bar{x} \in \dot{b}$" such that for all $\alpha < \lambda$, all $q_0, q_1 \leq \bar{q}$, and all $x_0, x_1 \geq_T \bar{x}$, if $q_i \Vdash$ "$\dot{b} \cap T_\alpha = \{x_i\}$ for $i \in \{0, 1\}$, then $x_0 = x_1$. Let $a = \{x \in T : \exists q \leq \bar{q}, q \Vdash x \in \dot{b}\}$. We can argue that $\bar{q} \Vdash$ "$\dot{b} = \check{a}$" because there is always some $q \leq \bar{q}$ deciding $\dot{b} \cap T_\alpha$, and this value must be unique. But we were specifically assuming (in the main proof, outside this claim) that there was no $a \in V$ and $q \leq \bar{p}$ with $q \Vdash$ "$\dot{b} = \check{a}$". We therefore have the claim. □

Now we will define a set $\{q_s : s \in 2^{<\omega}\}$ (meaning s is a finite binary string) of conditions in $\mathbb{P}$, a set $\{\alpha_n : n < \omega\}$ of ordinals below λ, and a set $\{x_s : s \in 2^{<\omega}$ of points in T with the following properties:

1. If $s \sqsubseteq t$ (meaning that there is some $n < \omega$ such that $s = t \restriction n$), then $q_t \leq_{\mathbb{P}} q_s$ and $x_s \leq_T x_t$.
2. For all $s \in 2^{<\omega}$ and $|s| = n$, then $q_s \Vdash T_{\alpha_n} \cap \dot{b} = \{x_s\}$.
3. If $s, t \in 2^{<\omega}$, $|s| = |t| = n$, and $s \neq t$, then $x_s \neq x_t$.

We define the q_s's by induction on $|s|$. We let $q_\emptyset$ be any condition below $\bar{p}$ such that for some $\alpha_\emptyset$ and $x_\emptyset$, $q_\emptyset \Vdash T_{\alpha_\emptyset} \cap \dot{b} = \{x_\emptyset\}$. If we have defined q_s, x_s, and α_n, then apply the claim to find $q'_{s^\frown 0}, q'_{s^\frown 1} \leq q_s$, distinct $x'_{s^\frown 0}$ and $x'_{s^\frown 1}$, and some $\alpha_* > \alpha_n$ such that $q_{s^\frown i} \Vdash \dot{b} \cap T_{\alpha_*} = x'_{s^\frown i}$. Then we can choose $q_{s^\frown i} \leq q'_{s^\frown i}$, $x_{s^\frown i} \geq x'_{s^\frown i}$, and α_{n+1} greater than these α_*'s so that $x_{s^\frown i} \in T_{\alpha_{n+1}}$. Hence everything is "evened out" and we have defined the q_s's, x_s's, and α_n's.

Now let $\gamma = \sup_{n<\omega} \alpha_n$ and observe that $\gamma < \lambda$. For each $f \in 2^\omega$, use the countable closure of $\mathbb{P}$ to find some q_f such that $q_f \leq q_{f \restriction n}$ for all $n < \omega$ and there is some x_f such that $q_f \Vdash \dot{b} \cap T_\gamma = \{x_f\}$. Then if $f \neq g$, it follows that $x_f \neq x_g$. But there are more than countably many $f \in 2^\omega$ and only countably many elements in T_γ, so we have found a contradiction. □

Proposition 3.17. *Suppose that κ is inaccessible and $\mathbb{P}$ is a poset such that $|\mathbb{P}| < \kappa$. If G is $\mathbb{P}$-generic over V, then $V[G] \models$ "κ is inaccessible".*

Proof. It is immediate that $\mathbb{P}$ has the κ-chain condition and therefore preserves regularity of κ. It remains to show that κ is a strong limit in the extension.

If $\dot{X}$ is a $\mathbb{P}$-name and $\Vdash_{\mathbb{P}}$ "$\dot{X} \subset \delta$", then we say that $\dot{X}$ is a *nice name* if there are maximal antichains $A_\alpha \subset \mathbb{P}$ below p for $\alpha < \delta$ such that all elements of $\dot{X}$ take the form $\langle \check{\alpha}, p \rangle$ for some $p \in A_\alpha$. One can argue that if $\Vdash_{\mathbb{P}}$ "$\dot{X} \subset \delta$", then there is some $\dot{Y}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{Y} = \dot{X}$" and $\dot{Y}$ is a nice name.

Observe that if $\delta < \kappa$, then there are at most $\delta^{|\mathbb{P}|}$-many nice $\mathbb{P}$-names for subsets of δ and $|\mathbb{P}|^\delta < \kappa$. Therefore $V[G] \models$ "$2^\delta < \kappa$". □

Theorem 3.18. *Suppose κ is inaccessible and $\mathbb{P} := \mathrm{Col}(\aleph_1, < \kappa)$ is the Lévy Collapse. If G is $\mathbb{P}$-generic over V, then $V[G] \models \neg\mathsf{KH}$.*

Proof. Suppose that T is a $\aleph_1$-tree in $V[G]$. Then T can be coded as a subset of $\aleph_1$. Therefore, by point *(5)* of Proposition 3.14 we find some $\delta < \kappa$ for the factoring $\mathbb{P} \cong \mathbb{P}_\delta \times \mathbb{P}^\delta$ such that $T \in V[G_\delta]$ (where G_δ is induced from G using this isomorphism).

A subtle but important point is that $\mathbb{P}^\delta$ remains countably closed in $V[G_\delta]$ since $\mathbb{P}_\delta$ is countably closed. Therefore, by Silver's Lemma, all cofinal branches of T are already in $V[G_\delta]$. By Proposition 3.17, $V[G_\delta] \models$ "$2^{\aleph_1} < \kappa$". Hence $\mathbb{P}^\delta$ adds a surjection from $\aleph_1$ to the set of branches of T, so there are at most $\aleph_1$-many branches of T in $V[G]$. □

Chapter 4

Gödel's Model L

Here we will discuss Gödel's constructible universe.

4.1 The Definition of L and its Basic Properties

We need to deal with definability in the model-theoretic sense. We will do so somewhat informally. We know that X is definable in a structure $(M, \ldots)$ if there is a formula $\varphi(v, \bar{w})$ and a set of parameters $\bar{b}$ such that X is the set of $a \in M$ satisfying $\varphi(a, \bar{b})$. It takes a good amount of work to justify notions of definability can be expressed in the language of set theory because one must consider what it means to refer to a formula φ which is not obviously expressible as a set. For our purposes here, we are assuming the existence of some Gödel coding $\ulcorner \varphi \urcorner$ of formulae φ.

Definition 4.1. Let A be a set. Then $\mathrm{Def}(A)$ is the set of sets definable over $(A, \in)$ using elements from A as parameters.

Proof. This follows from the fact that all finite subsets of some A are definable from parameters in A. $\square$

Definition 4.2.

- $L_0 = \emptyset$;
- $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$;

- if α is a limit, then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

Gödel's Constructible Universe L is the class model consisting of x such that for some ordinal, $x \in L_\alpha$.

Proposition 4.3. *If $n < \omega$, then $L_n = V_n$.*

Proposition 4.4. *The following facts are true for all $\alpha \in \mathrm{ON}$:*

1. $\forall \beta < \alpha$, $L_\beta \subseteq L_\alpha$.
2. $L_\alpha \subseteq V_\alpha$;
3. L_α *is transitive;*
4. $L_\alpha \cap \mathrm{ON} = \alpha$;
5. *if* $\alpha \geq \omega$ *then* $|L_\alpha| = |\alpha|$.

Proof. The first three boil down to transfinite induction, and the fourth basically starts the induction at ω.

1. This is immediate from the definition.

2. Induct on $\alpha \in \mathrm{ON}$. This is clear if $\alpha = 0$ or if α is a limit. If $\alpha = \beta + 1$, then $L_\alpha = L_{\beta+1} = \mathrm{Def}(L_\beta) \subset \mathcal{P}(L_\beta) \subset \mathcal{P}(V_\beta) = V_{\alpha+1}$.

3. Again, induct on $\alpha \in \mathrm{ON}$. Suppose that $x \in L_\alpha$. If $\alpha = 0$, then the statement holds vacuously, so we can assume that $\alpha > 0$. Now suppose moreover that α is the *least* ordinal such that $x \in L_\alpha$, which implies that α is not a limit, and so $\alpha = \beta + 1$. By the definition of L_α, we see that $x \subset L_\beta \subset L_\alpha$.

4. By induction again. Note that *2.* implies that $L_\alpha \cap \mathrm{ON} \subseteq \alpha$, so we just need to show that $\alpha \subseteq L_\alpha$.

We have $L_0 \cap \mathrm{ON} = \emptyset \cap \mathrm{ON} = \emptyset = 0$.

If α is a limit, then $L_\alpha \cap \mathrm{ON} = (\bigcup_{\beta<\alpha} L_\beta) \cap \mathrm{ON} = \bigcup_{\beta<\alpha}(L_\beta \cap \mathrm{ON}) = \bigcup_{\beta<\alpha} \beta = \alpha$ by induction. (There is some abuse of notation here because ON is a proper class, but the reasoning works.)

Suppose that $\alpha = \beta+1$. Then by induction $\beta = \{\gamma \in L_\beta : \gamma \text{ is an ordinal}\}$ and so $\beta \in L_\alpha$, so $\alpha \subset L_\alpha$.

5. By the above proposition, L_n is finite for all $n < \omega$. Therefore, L_ω is countable since the L_n's are also getting larger.

Now we induct on $\alpha \geq \omega$, where we have shown the base case of $|L_\omega| = |\omega|$. For the limit case α, we use that $|L_\alpha| = |\bigcup_{\beta<\alpha} L_\alpha| = \bigcup_{\beta<\alpha} |\beta| = |\alpha|$.

The most important part of the induction is the successor case. Suppose $\alpha = \beta+1$. There are $|\beta|^{<\omega} = |\beta|$-many possible formulas with parameters in L, and there are only countably many formulas in the language of set theory. Therefore, $|\operatorname{Def}(L_\beta)| = |\beta|$, meaning that $|L_\alpha| = |\beta| = |\beta+1| = |\alpha|$. $\square$

Definition 4.5. The Δ_0 formulae are the smallest collection of formulae generated from the collection of atomic formulae using the following operations:

- If φ is Δ_0, then $\exists v \in u, \varphi(v, \bar{w})$ and $\forall v \in u, \varphi(v, \bar{w})$ are Δ_0.
- If φ is Δ_0, then so is $\neg\varphi$. If φ and ψ are Δ_0, then so are $\varphi \wedge \psi$ and $\varphi \vee \psi$.

Example 4.6. Some Δ_0 notions include: $x \subset y$; x is transitive; f is a function; f is an injective function; x is an ordinal; etc.

Proposition 4.7. *Suppose $W \subset V$ are transitive models of set theory, $\varphi(\bar{v})$ is a Δ_0 formula, and $\bar{a} \in W$. Then $V \models \varphi(\bar{a})$ if and only if $W \models \varphi(\bar{a})$. (Sometimes we write this as $\varphi(\bar{a}) \iff \varphi^W(\bar{a})$.)*

Theorem 4.8. $L \models \mathsf{ZF}$.

Proof. Remember that L is transitive, so Δ_0 formulas will be absolute between V and L.

- *Extensionality:* This follows from L being transitive.
- *Pairing:* If $a, b \in L$, there is some α such that $a, b \in L$. Then $\{a, b\}$ is definable, so $c \in L_{\alpha+1}$.
- *Regularity:* It is enough to show that if $y \in L$ is nonempty, then there is some $x \in L \cap y$ such that $y \cap x = \emptyset$. Since $x \in L$ by transitivity and "$x \cap y = \emptyset$" is Δ_0 we are done.
- *Union:* If $X \in L_\alpha$ and $Y = \bigcup X$, then $Y \subset L_\alpha$ by transitivity. The rest follows from definability and the fact that we are only concerned with Δ_0 notions.
- *Infinity:* We assert that ω is definable in a Δ_0 way because it does not have any limit ordinals. (This is in contrast to $\aleph_1$, for example.) This shows that $\omega \in L$, specifically $\omega \in L_{\omega+1}$.

- *Powerset:* Consider $x \in L_\alpha$ and $y := \mathcal{P}(x) \cap L$. Let β be large enough that $y \subset L_\beta$. Then $y \in L_{\beta+1}$ because "$z \subset x$" defines y. The main observation is that $L \models$ "$y = \mathcal{P}(x)$".
- *Separation:* Given a formula φ, and $x, \bar{p} \in L$, we want to show that $y := \{z \in x : L \models f(z, \bar{p})\} \in L$. We apply the Reflection Principle to the L_α-hierarchy to find some α with $x, \bar{p} \in L_\alpha$ and $y = \{z \in x : L_\alpha \models \varphi(z, \bar{p})\}$. Hence $y \in L_{\alpha+1}$.
- *Replacement:* This is similar to proving separation for L. We apply the replacement axiom in V and then use the Reflection Principle.

□

4.2 Important Conceptual Tools for L

Theorem 4.9. $L \models \mathsf{AC}$. *More specifically, there is a well ordering $<_L$ on L such that for every limit ordinal δ and $x \in L_\delta$, we have $y <_L x$ if and only if $y \in L_\delta$ and $L_\delta \models y <_L x$.*

In particular, $L \models \mathsf{ZFC}$.

Sketch. We make sure that the ordering on $L_{\alpha+1} \setminus L_\alpha$ end-extends the ordering on L_α. If α is a limit, then the ordering on L_α is a union of the orderings on L_β for $\beta < \alpha$. This means that most of the work is in finding an order for the "new" elements in $L_{\alpha+1} \setminus L_\alpha$ for every α. Every element is defined from a formula φ and a parameter set $\bar{p}$. Hence one needs use the ordering of L_α to define an ordering on $L_\alpha^{<\omega}$, and then combine this with an ordering on formula φ that uses the inductive definition of formula construction. □

We need to recall some notions that will allows us to justify the existence of a formula defining L.

Theorem 4.10. *The function $\alpha \mapsto L_\alpha$ is defined by a Δ_1 formula and is therefore absolute.*[1]

[1] This theorem is mostly a black box for this course. A truly rigorous treatment is presented in Keith Devlin's textbook *Constructibility.* Even though the book is extremely well-written, it contains mistakes in the part of the book that works towards

Sketch. Verifying $X = L_\alpha$ is mostly a matter of showing that generating $L_{\alpha+1}$ from L_α can be rigorously defined in a Σ_1 way. This means that there is a function that outputs all of the definable sets, which requires defining the notion of definability and formalizing it properly. □

Corollary 4.11. *There is a sentence σ such that if M is a transitive set, then $(M, \in) \models \sigma$ if and only if there is some $\alpha \in \mathrm{ON}$ such that $M = L_\alpha$.*

Corollary 4.12. *If W is any inner model of set theory, then $L \subseteq W$ as a class.*

Corollary 4.13. *There is no cardinal κ such that $L \models$ "κ is measurable."*

Proof. Otherwise we could take a measurable embedding $j : L \to M$. But we showed that $M \subsetneq L$, contradicting minimality of L. □

4.3 GCH and Relative Constructibility

Definition 4.14. Recall that if M is a set, then the *Mostowski collapse* of M is the image of the function $\pi : M \to \bar{M}$ inductively defined by the formula $\pi(x) = \{\pi(y) : y \in x \cap M\}$. Sometimes we write the Mostowksi collapse as $\pi(M)$.

Theorem 4.15 (The Condensation Lemma)**.** *For all sets M and limit ordinals α, if $M \prec L_\alpha$, then there is some $\beta \leq \alpha$ such that $\pi(M) \cong L_\beta$.*

Sketch. This uses Corollary 4.11 above together with the Mostowski collapse. □

Proposition 4.16. *If $X \subset M$ is transitive and $\pi : M \to \bar{M}$ is the Mostowski collapse of M, then $\pi(z) = z$ for all $z \in X$.*

Definition 4.17. The *generalized continuum hypothesis*, abbreviated GCH, is the statement that for all cardinals κ, $2^\kappa = \kappa^+$.

Theorem 4.18. *$L \models$ GCH.*

this theorem. The mistakes are pointed out in a review by Lee Stanley. Adrian Mathias corrected these mistakes in a paper called "Weak Systems of Gandy, Jensen and Devlin." A helpful discussion can be found here: `https://mathoverflow.net/questions/77734/devlins-constructibility-as-a-resource`. Other textbooks that lack these mistakes seem to gloss over the difficult issues inherent in "defining definability."

Proof. First we need to make use of the fact that $|L_\alpha| = |\alpha|$ for all infinite ordinals α.

Claim. *If κ is a cardinal, then $|L_{\kappa^+}| = \kappa^+$.*

If κ is a cardinal in L, then it is a cardinal in V, so $|L_{\kappa^+}| = \kappa^+$. Therefore it is enough to show that $\mathcal{P}(\kappa) \cap L \subseteq L_{\kappa^+}$.

Fix some $X \subset \kappa$. Let M be the Skolem hull of $\{X\} \cup \kappa$ inside L_α where α is limit ordinal large enough for L_α to contain X. Then let N be the Mostowski collapse of M. By the Condensation Lemma, there is some $\beta \leq \alpha$ such that $N = L_\beta$. Since $|M| = \kappa$, it follows that $|N| = \kappa$, and so we in fact have $\beta < \kappa^+$. Moreover, because $\kappa \cap M = \kappa$ is transitive, it follows that $\pi(X) = X$, and therefore that $X \in M = L_\beta$, and finally that $X \in L_{\kappa^+}$. □

Definition 4.19. Let A be a set. Then let $\mathrm{Def}_A(M)$ consists of all subsets of M definable over the structure $(M, \in, M \cap A)$. This allows us to define *relative constructibility*:

- $L_0[A] = \emptyset$;
- $L_{\alpha+1}[A] = \mathrm{Def}_A(L_\alpha[A])$;
- $L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A]$ if α is a limit ordinal.

Then the model $L[A]$ consists of all sets contained in $L_\alpha[A]$ for some ordinal α.

Theorem 4.20. *Certain facts about L generalize to $L[A]$ for all sets A, as in the following:*

- *$L[A]$ is transitive, plus everything from Proposition 4.4.*
- *$L[A] \models \mathsf{ZF}$.*
- *$L[A] \models$ "$\exists X, V = L[X]$".*
- *There is a well-ordering $<_{L[A]}$ of $L[A]$ definable within $L[A]$.*
- *$L[A]$ is minimal in the sense that if $M \supset \mathrm{ON}$ is a transitive class model and $A \cap M \in M$, then $L[A] \subset M$.*

Sketch. Point *(1)* is relatively straightforward. Note that when considering the operation $\mathrm{Def}_A(M)$, the extra predicate $A \cap M$ does not increase the size of the language. Point *(2)* is also relatively straightfoward.

Points *(3)* through *(5)* use the following:

Claim. *If $\bar{A} = L[A] \cap A$, then for all ordinals α we have $L_\alpha[A] = L_\alpha[\bar{A}]$ and $\bar{A} \in L[\bar{A}]$.*

Once we understand this claim, we can see how the necessary arguments can be done "internally" in $L[A]$.

We prove the claim by induction on α. The only substantial case is the successor case. Observe that if $U = L_\alpha[A]$, then $A \cap U = A \cap U \cap L[A] = \bar{A} \cap U$, so $\mathrm{Def}_A(U) = \mathrm{Def}_{A\cap U}(U)$. It follows that $L_{\alpha+1}[A] = \mathrm{Def}_A(U) = \mathrm{Def}_{A\cap U}(U) = \mathrm{Def}_{\bar{A}}(U) = L_{\alpha+1}[\bar{A}]$. □

Theorem 4.21 (Relative Condensation Lemma)**.** *If $M \prec L_\alpha[A]$ for a limit ordinal α, then there is some $\beta \leq \alpha$ such that $\pi(M) \cong L_\beta[B]$ where $B = \pi''[A \cap M]$.*

Theorem 4.22. *If κ is a cardinal in $L[A]$ such that $A \subseteq \kappa$, then $L[A] \models 2^\lambda = \lambda^+$ for all cardinals $\lambda \geq \kappa$ in $L[A]$.*

If $V \models \mathsf{AC}$, then this can generalize to "all sets" since we can code any set as a subset of an ordinal using Gödel's pairing function.

Chapter 5

The Diamond Principle and its Variants

Definition 5.1. We say that the *diamond principle* $\diamondsuit$ holds if there exists a sequence $\langle S_\alpha : \alpha < \aleph_1 \rangle$ such that $S_\alpha \subset \alpha$ for all $\alpha < \aleph_1$ and such that for all $X \subset \aleph_1$, the set

$$\{\alpha < \aleph_1 : X \cap \alpha = S_\alpha\}$$

is stationary. Such a sequence is referred to as a $\diamondsuit$-sequence.

Exercise. $\diamondsuit$ *implies* CH.

Theorem 5.2. $L \models \diamondsuit$.

Proof. Work in L and define a $\diamondsuit$-sequence by induction on $\alpha < \aleph_1$. Actually, we will define a sequence of pairs (S_α, C_α) here $S_\alpha \subseteq \alpha$ and C_α is a club in α for all $\alpha < \aleph_1$. Let $S_0 = C_0 = \emptyset$ and let $S_{\alpha+1} = C_{\alpha+1} = \alpha + 1$ for all $\alpha < \aleph_1$. If α is a limit, we define the pair as follows:

> (S_α, C_α) is the $<_L$-least pair such that $S_\alpha \subseteq \alpha$, C_α is a club in α, and $S_\alpha \cap \xi \neq S_\xi$ for all $\xi \in C_\alpha$, and if no such pair exists then $S_\alpha = C_\alpha = \alpha$.

We claim that $\langle S_\alpha : \alpha < \aleph_1 \rangle$ is a $\diamondsuit$-sequence. Otherwise, let (X, C) be the $<_L$-least pair such that $X \subseteq \aleph_1$ and $C \subseteq \aleph_1$ is a club such that for all $\alpha \in C$, $X \cap \alpha \neq S_\alpha$.

Since $\vec{A} = \langle (S_\alpha, C_\alpha) : \alpha < \aleph_1 \rangle$ is an $\aleph_1$-sequence of pairs of subsets of $\aleph_1$, one can argue that $\vec{A} \in L_{\omega_2}$, and it also satisfies the same definition in $L_{\aleph_2}$. Also, $(X, C) \in L_{\omega_2}$ as well. Now take $M \prec L_{\aleph_2}$ to be countable. Since $\vec{A}$ and (X, C) are definable without parameters, $\vec{A}$, $(X, C) \in M$ (as elements).

We claim that $M \cap \aleph_1$ is an ordinal in M. If $\alpha \in \aleph_1 \cap M$, then there is a surjection $f : \omega \twoheadrightarrow \alpha$ and without loss of generality, $f \in M$. Also we have $\omega \subset M$ by elementarity. Therefore $\alpha = f''\omega \subset M$.

Let $\delta = \aleph_1 \cap M < \aleph_1$. Let $N = \pi(M)$ be the Mostowski collapse of N, so by the Condensation Lemma we have $N = L_\gamma$ for some $\gamma < \aleph_1$. In particular, we have $\gamma > \delta$. Note that since $M \prec L_{\omega_2}$, $\omega_1 \in M$. We can "compute" some values of π using the fact that $\delta \subset M$ is transitive:

- $\pi(\aleph_1) = \{\pi(\alpha) : \alpha \in M \cap \aleph_1\} = \{\alpha : \alpha \in M \cap \aleph_1\} = \delta$.
- $\pi(X) = \{\pi(\alpha) : \alpha \in M \cap X\} = \{\pi(\alpha) : \alpha \in X \cap \delta\} = \{\alpha : \alpha \in X \cap \delta\} = X \cap \delta$.
- Similarly, $\pi(C) = C \cap \delta$.
- Also similarly, $\pi(\vec{A}) = \{(S_\alpha, C_\alpha) : \alpha < \delta\}$.

So L_γ satisfies that $(X \cap \delta, C \cap \delta)$ is the $<_L$-least pair (Z, D) such that $Z \subset \delta$, $D \subseteq \delta$ is a club, and $Z \cap \xi \neq S_\xi$ for all $\xi \in D$. By absoluteness of $<_L$ in this context, this statements holds in L. Therefore we have $X \cap \delta = S_\delta$, and since $C \cap \delta$ is a club in δ, we have $\delta \in C$. But this contradicts the assumption that (X, C) was a counterexample to the definition of $\Diamond$, so we are done. $\square$

Definition 5.3. We say that *strong diamond*, denoted $\Diamond^+$ holds, if there exists a $\Diamond^+$-sequence, which is a sequence $\langle \mathcal{S}_\alpha : \alpha < \aleph_1 \rangle$ such that the following hold:

1. $\forall \alpha$, $\mathcal{S}_\alpha \subseteq \mathcal{P}(\alpha)$ and $\mathcal{S}_\alpha$ is countable;
2. for every $X \subseteq \aleph_1$, there is a club $C \subseteq \aleph_1$ such that for all $\alpha \in C$, $X \cap \alpha \in \mathcal{S}_\alpha$ *and* $C \cap \alpha \in \mathcal{S}_\alpha$.

We could be more explicit and write $\Diamond^+$ as $\Diamond^+_{\aleph_1}$. There are natural generalizations $\Diamond^+_\kappa$ for $\kappa > \aleph_1$ as well.

Proposition 5.4. *The principle $\diamondsuit$ is equivalent to the existence of a sequence $\langle \mathcal{S}_\alpha : \alpha < \aleph_1 \rangle$ such that the following hold:*

1. *$\forall \alpha$, $\mathcal{S}_\alpha \subseteq \mathcal{P}(\alpha)$ and $\mathcal{S}_\alpha$ is countable;*
2. *for every $X \subseteq \aleph_1$, there is a stationary set $S \subseteq \aleph_1$ such that for all $\alpha \in S$, $X \cap \alpha \in \mathcal{S}_\alpha$.*

Corollary 5.5. $\diamondsuit^+ \implies \diamondsuit$

Remark 5.6. Observe that it is necessary to use $\mathcal{S}_\alpha \subset \mathcal{P}(\alpha)$ in the definition of $\diamondsuit^+$ rather than $S_\alpha \subset \alpha$. Consider what would happen if enumerate point *(2)* of the definition for $X, Y \subset \aleph_1$ such that $X \cap Y = \emptyset$.

Theorem 5.7. $L \models \diamondsuit^+$.

Proof. We work in L. First we need to define the $\diamondsuit^+$-sequence.

Claim. *The set $\{\delta < \omega_1 : L_\delta \prec L_{\omega_1}\}$ is a club.*

Proof of Claim. The fact that this set is closed is basic model theory. To see that it is unbounded, consider some $\beta < \omega_1$ and let $M = \mathrm{Sk}^{L_{\omega_1}}(\beta + 1)$ be the Skolem hull of $\beta + 1$ generated inside inside L_{ω_1}, which we can do using $<_L$. Then $\beta + 1 \subseteq M$ and M is transitive by a homework exercise (Exercise 2, Sheet 5), so therefore $\pi(M) = M = L_\delta$ for some $\delta < \omega_1$ by the Condensation Lemma, and in particular $\delta > \beta$. □

So given $\alpha < \omega_1$, let $q(\alpha) > \alpha$ be the minimal ordinal below ω_1 such that $L_{q(\alpha)} \prec L_{\omega_1}$. Then let $\mathcal{S}_\alpha = \mathcal{P}(\alpha) \cap L_{q(\alpha)}$. We claim that $\langle \mathcal{S}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit^+$-sequence. The sets $\mathcal{S}_\alpha$ are evidently countable, so our main work is to show that we have the guessing properties we need. Therefore let us fix some $A \subseteq \omega_1$ for the rest of the proof. Observe that $A \in L_{\omega_2}$ since we proved that $\mathcal{P}(\kappa)^L \subset L_{\kappa^+}$ when proving $L \models \mathsf{GCH}$.

Now we will define the club C_A that will witness guessing. For $\gamma < \omega_1$ we let $M_{A,\gamma} = \mathrm{Sk}^{L_{\omega_2}}(\{A\} \cup \gamma)$. Observe that $M_{A,\gamma}$ is countable and that $M_{A,\gamma} \cap \omega_1 \in \omega_1$ by an argument similar to Exercise 2 of Sheet 5 (M will contain a surjection to any ordinal in $M_{A,\gamma} \cap \omega_1$). Hence we define $C_A = C = \{\alpha < \omega_1 : M_{A,\alpha} \cap \omega_1 = \alpha\}$. This is a club: closure is fairly clear, and unboundedness follows from an interleaving argument where we take $M_{A,\alpha_n} \cap \omega_1 < \alpha_{n+1}$, then $\alpha_n < M_{A,\alpha_{n+1}} \cap \omega_1 < \alpha_{n+2}$, etc.

Claim. *For all $\alpha \in C_A$, $A \cap \alpha \in \mathcal{S}_\alpha$.*

Proof of Claim. Let $\alpha \in C_A$ and let N be the Mostowski collapse of $M = M_{A,\alpha}$. By the Condensation Lemma, $N = \pi(M_{A,\alpha}) = L_\delta$ for some $\delta < \omega_1$. Since $M_{A,\alpha} \cap \omega_1 = \alpha$, we have $\pi_M(\omega_1) = \alpha$, so $L_\delta = N \models$ "α is uncountable". But since $\alpha \in L_{q(\alpha)} \prec L_{\omega_1}$, we also have $L_{q(\alpha)} \models$ "α is countable", so it follows that $q(\alpha) > \delta$. Since we knew that $A \cap \alpha = \pi_M(A \cap \alpha) \in L_\delta$, it follows that $A \cap \alpha \in P(\alpha) \cap L_{q(\alpha)} = \mathcal{S}_\alpha$. □

Claim. *For all $\alpha \in C_A = C$, $C \cap \alpha \in \mathcal{S}_\alpha$.*

Proof of Claim. Let δ be as in the above claim. Because L_{ω_2} is a model of $\mathsf{ZFC} - \mathsf{Powerset}$, and therefore the same is true of $L_{q(\alpha)}$, we can do model theory within the structure $L_{q(\alpha)}$ in terms of definability and so on. Moreover, $L_{q(\alpha)}$ contains the sets L_δ, α, and $A \cap \alpha$, so we can define

$$\hat{C} = \{\beta < \alpha : \mathrm{Sk}^{L_\delta}(\{A \cap \alpha\} \cup \beta) \cap \alpha = \beta\} \in L_{q(\alpha)}.$$

We would like to show that

$$\{\beta < \alpha : \mathrm{Sk}^{L_{\omega_2}}(\{A\} \cup \beta) \cap \omega_1 = \beta\} = \{\beta < \alpha : \mathrm{Sk}^{L_\delta}(\{A \cap \alpha\} \cup \beta) \cap \alpha = \beta\} \tag{5.1}$$

because this is the same as showing that $\hat{C} = C \cap \alpha$.

To prove one direction, suppose $\beta < \alpha$ and that $\mathrm{Sk}^{L_{\omega_2}}(\{A\} \cup \beta) \cap \omega_1 \neq \beta$. Then there is some $\gamma < \omega_1$ such that $\gamma \geq \beta$ and $\gamma \in \mathrm{Sk}^{L_{\omega_2}}(\{A\} \cup \beta)$. Then γ is definable in the model L_{ω_2} using parameters in $\{A\} \cup \beta$. Because $\beta < \alpha$, it follows that $\gamma < \alpha$ as well using the fact that $\alpha \in C$ is a closure point in this sense. Since $M = M_{A,\alpha} \prec L_{\omega_2}$, we see that γ is definable in M using the parameters $\{A\} \cup \beta$. Also, π_M fixes γ and the parameters in A while giving $\pi_M(A) = A \cap \alpha$, so γ is definable in L_δ using parameters in $\{A \cap \alpha\} \cup \beta$. Therefore $\mathrm{Sk}^{L_\delta}(\{A \cap \alpha\} \cup \beta) \cap \alpha \neq \beta$ because $\gamma \in \mathrm{Sk}^{L_\delta}(\{A \cap \alpha\} \cup \beta) \cap \alpha$.

The other direction for proving Equation 5.1 is analogous. □

This completes the proof that $\Diamond^+$ holds in L. □

Definition 5.8. We say that $\mathcal{F} \subset \mathcal{P}(\aleph_1)$ is a *Kurepa family* if and only if $|\mathcal{F}| \geq \aleph_2$ and for all $\alpha < \aleph_1$, $\{X \cap \alpha : X \in \mathcal{F}\}$ is countable.

Proposition 5.9. *A Kurepa tree exists if and only if a Kurepa family exists.*

Proof. Let $(T, <_T)$ be a Kurepa tree—so it is a tree of height $\aleph_1$ with countable levels and at least $\aleph_2$-many cofinal branches. We can assume without loss of generality that $T \subset \aleph_1$ and that $\alpha <_T \beta$ implies that $\alpha < \beta$. (Such a construction can be done by inducting on the levels of T.) We then let $\mathcal{F}$ be the set of cofinal branches of T.

Now suppose that $\mathcal{F}$ is a Kurepa family. For each $X \in \mathcal{F}$, let $f_X : \alpha \mapsto X \cap \alpha$. Then let $T_\alpha = \{f_X \upharpoonright \alpha : X \in \mathcal{F}\}$. Then let $T = \bigcup_{\alpha<\aleph_1} T_\alpha$. □

Theorem 5.10. *If $\diamondsuit^+$ holds, then there is a Kurepa tree.*

Proof. Let $\langle \mathcal{S}_\alpha : \alpha < \aleph_1 \rangle$ be a $\diamondsuit^+$-sequence. We will construct a Kurepa family that satisfies the following condition:

$$\forall A \in [\aleph_1]^{\aleph_1}, \exists X \in [A]^{\aleph_1} \text{ s.t. } X \in \mathcal{F} \tag{5.2}$$

Claim. *This condition implies that $|\mathcal{F}| \geq \aleph_2$.*

Proof of Claim. Suppose for contradiction that $|\mathcal{F}| \leq \aleph_1$ and is enumerated as $\{B_\xi : \xi < \aleph_1\}$. Then define a sequence $A = \{\alpha_\xi : \xi < \omega_1\}$ such that for all ξ there is some $\beta \in B_\xi$ such that $\alpha_\xi < \beta < \alpha_{\xi+1}$. Then A witnesses the failure of 5.2.(We could actually show that $|\mathcal{F}| = 2^{\aleph_1}$.) □

Our plan is define $\mathcal{F}$ by defining $\mathcal{F}_\beta \subset \mathcal{P}(\beta)$ for $\beta < \aleph_1$ and letting $\mathcal{F}$ consist of all $X \subset \aleph_1$ such that $X \cap \beta \in \mathcal{F}_\beta$ for all $\beta < \aleph_1$. We succeed as long as the $\mathcal{F}_\beta$'s are clearly countable and 5.2 holds.

We introduce some terms to define the Kurepa family. If $C \subset \aleph_1$ and $\xi, \eta \in C$ are such that $\xi < \eta$ but $C \cap (\xi, \eta) = \emptyset$, then we say that ξ and η are *adjacent.* If $A \subset \aleph_1$ and $C \subset \aleph_1$, we define $t(A, C)$ to consist of all ordinals of the form $\min\{A \cap [\xi, \eta)\}$ if ξ and η are adjacent in C and $A \cap [\xi, \eta)$ is nonempty. Observe that $t(A, C) \subset A$ and that if C is a club and $A \subset \aleph_1$ is unbounded, then $t(A, C)$ is unbounded in $\aleph_1$.

Finally, for $\beta < \aleph_1$ we define:

$$\mathcal{F}_\beta = \left\{ a \cup t(A, C) : a \in [\beta]^{<\omega} \text{ and } A, C \in \{\emptyset\} \cup \bigcup_{\alpha \leq \beta} \mathcal{S}_\alpha \right\}.$$

It is clear that $\mathcal{F}_\beta$ is countable for all $\beta < \aleph_1$, so it remains to show that 5.2 holds. Suppose that $A \subset \aleph_1$ if unbounded. Let C witness $\diamondsuit^+$ with respect to A, and let $X = t(A, C)$. We claim that X suffices, so we fix $\beta < \aleph_1$ and hope to show that $X \cap \beta \in \mathcal{F}_\beta$. If $C \cap \beta$ is finite, then

$X \cap \beta \in [\beta]^{<\omega}$ and we are basically done. If $C \cap \beta$ is infinite, let α be the largest limit ordinal in $C \cap \beta + 1$. Then $X \cap \beta = a \cup t(A \cap \alpha, C \cap \alpha)$ for some $a \in [\beta]^{<\omega}$.

$\square$

Chapter 6

Mitchell Forcing

Here we will develop the theory needed to show that the consistency of a weakly compact cardinal implies the consistency of the tree property at $\aleph_2$.

Fact 6.1 (Specker)**.** *If κ is a cardinal and $\kappa^{<\kappa} = \kappa$, then there is a κ^+-Aronszajn tree and therefore the tree property fails at κ^+.*

Corollary 6.2. *If CH holds, i.e. $2^{\aleph_0} = \aleph_1$, then the tree property fails at $\aleph_2$.*

Corollary 6.3. *For any cardinal κ, the tree property fails at $\aleph_2$ in the model $V[\mathrm{Col}(\aleph_1, < \kappa)]$.*

6.1 Two-Step Iterations

Definition 6.4. Suppose that $(\mathbb{P}, \leq_{\mathbb{P}})$ is a poset and $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}})$ is a $\mathbb{P}$-name for a poset. Then the *two-step iteration* $\mathbb{P} * \dot{\mathbb{Q}}$ is the poset defined as follows:

1. Conditions take the form $(p, \dot{q})$ where $p \in \mathbb{P}$ and $1_{\mathbb{P}} \Vdash \dot{q} \in \dot{\mathbb{Q}}$.
2. $(p_s, \dot{q}_s) \leq_{\mathbb{P} * \dot{\mathbb{Q}}} (p_w, \dot{q}_w)$ if and only if $p_s \leq_{\mathbb{P}} p_w$ and $p_s \Vdash_{\mathbb{P}}$ "$\dot{q}_s \leq_{\dot{\mathbb{Q}}} \dot{q}_w$".

Remark 6.5. There are different ways of defining two-step iterated forcing, and we are using Jech's definition.

Theorem 6.6 (Fundamental Theorem of Two-Step Iterations)**.** *Suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration. Then:*

1. *Suppose that G is $\mathbb{P}$-generic over V and H is $\dot{\mathbb{Q}}[G]$-generic over $V[G]$. Then $K := \{(p, \dot{q}) : p \in G, \dot{q}[G] \in H\}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V.*

2. *Suppose that K is $\mathbb{P} * \dot{G}$-generic. Then $G := \{p \in \mathbb{P} : \exists \dot{q}, (p, \dot{q}) \in K\}$ is $\mathbb{P}$-generic over V and $H := \{\dot{q}[G] : \exists p \in G, (p, \dot{q}) \in K\}$ is $\dot{\mathbb{Q}}[G]$-generic over $V[G]$.*

Proof. This is analogous to the similar theorem for product forcings.

Proof of 1. To see that K is a filter, suppose that $(p_0, \dot{q}_0) \in K$ and $(p_0, \dot{q}_0) \leq_{\mathbb{P}\dot{\mathbb{Q}}} (p_1, \dot{q}_1)$. Then $p_0 \in G$ and $p_0 \leq_{\mathbb{P}} p_1$, so $p_1 \in G$. Also, $p_0 \Vdash$ "$\dot{q}_0 \leq_{\dot{\mathbb{Q}}} \dot{q}_1$", so $V[G] \models$ "$\dot{q}_0[G] \leq_{\dot{\mathbb{Q}}[G]} \dot{q}_1[G]$ where $\dot{q}_0[G] \in H$, so $\dot{q}_1[G] \in H$ by upwards closure. Hence $(p_1, \dot{q}_1) \in K$. This gives upwards closure and compatibility is similar.

Now we argue that K is generic. Suppose that $D \subset \mathbb{P} * \dot{\mathbb{Q}}$ is dense.

Claim. *The set $D' = \{\dot{q}[G] : \exists p \in G, (p, \dot{q}) \in D\} \in V[G]$ is dense in $\dot{\mathbb{Q}}[G]$.*

Proof of Claim. We first show that for each $\dot{q}_0$ such that $\Vdash_{\mathbb{P}}$ "$\dot{q}_0 \in \dot{\mathbb{Q}}$", the set $D_{\dot{q}_0} := \{p \in \mathbb{P} : \exists \dot{q}_1, p \Vdash$ "$\dot{q}_1 \leq \dot{q}_0$" and $(p, \dot{q}_1) \in D\} \in V$ is dense in $\mathbb{P}$. To see this, observe that for any $\bar{p} \in \mathbb{P}$, there is some $(p, \dot{q}_1) \in D$ such that $(p, \dot{q}_1) \leq_{\mathbb{P}*\dot{\mathbb{Q}}} (\bar{p}, \dot{q}_0)$ by the density of D. Then the fact that $p \in D_{\dot{q}_0}$ follows from the definition of $\leq_{\mathbb{P}*\dot{\mathbb{Q}}}$.

Now suppose that $\dot{q}_0[G] \in \dot{\mathbb{Q}}[G]$. Find $\bar{p} \in G$ such that $\bar{p} \Vdash$ "$\dot{q}_0 \in \dot{\mathbb{Q}}$". Then apply the above paragraph to find $p \leq \bar{p}$ such that $p \in D_{\dot{q}_0} \cap G$ and suppose $\dot{q}_1$ witnesses this. Then $(p, \dot{q}_1) \in D$ and $\dot{q}_1[G] \in D'$. □

Now use the density of D' to find some $\dot{q}[G] \in H \cap D'$. This means that $(p, \dot{q}) \in K \cap D$.

Proof of 2. This is very similar to the proof of the "Fundamental Theorem of Product Forcing."

□

Proposition 6.7. *If $\mathbb{P}$ is κ-closed and $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is κ-closed", then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-closed.*

Proof. Suppose that $\langle (p_\alpha, \dot{q}_\alpha) : \alpha < \delta \rangle$ is $\leq_{\mathbb{P}*\dot{\mathbb{Q}}}$-decreasing for some $\delta < \kappa$. Using that κ-closure of $\mathbb{P}$, there is some $p_* \in \mathbb{P}$ such that $p_* \leq_{\mathbb{P}} p_\alpha$ for all $\alpha < \delta$. By the definition of the ordering of $\mathbb{P} * \dot{\mathbb{Q}}$, $p_* \Vdash$ "$\langle \dot{q}_\alpha : \alpha < \delta \rangle$ is decreasing". We also have $p_* \Vdash$ "$\dot{\mathbb{Q}}$ is κ-closed", so it follows that there is some $\dot{q}_*$ such that $p_* \Vdash$ "$\dot{q}_*$ is a lower bound of $\langle \dot{q}_\alpha : \alpha < \delta \rangle$". By definition $(p_*, \dot{q}_*)$ is a lower bound of $\langle (p_\alpha, \dot{q}_\alpha) : \alpha < \delta \rangle$. □

6.2 Defining the Mitchell Forcing

Definition 6.8. We recall some notation for Cohen posets.

- If λ is a cardinal, then $\mathrm{Add}(\lambda, 1)$ consists of partial functions $p : \lambda \to 2$ such that $|p| < \lambda$ where $p \leq q$ precisely when $p \supseteq q$.
- Let λ be an infinite cardinal and let $\delta \geq \lambda$ be any ordinal. Then $\mathrm{Add}(\lambda, \delta)$ consists of partial functions $p : \delta \times \lambda \to 2$ such that $|p| < \lambda$.
- If $\kappa^\lambda = \kappa$, then $\mathrm{Add}(\lambda, \kappa)$ is λ^+-Knaster.
- If $p \in \mathrm{Add}(\lambda, \delta)$ and $\gamma < \delta$, then $p \restriction \gamma$ will be used to refer to $p \restriction \gamma \times \lambda$.

Definition 6.9. Fix some regular κ. Mitchell forcing $\mathbb{M}(\omega, \kappa)$ is the poset consisting of pairs (p, q) such that the following hold:

1. $p \in \mathrm{Add}(\omega, \kappa)$.
2. q is a function such that:
   a) $\mathrm{dom}(q) \subseteq \kappa$;
   b) $\mathrm{dom}(q)$ is countable;
   c) $\forall \alpha \in \mathrm{dom}(q)$, $\Vdash_{\mathrm{Add}(\omega,\alpha)} q(\alpha) \in \dot{\mathrm{Add}}(\omega_1, 1)$.

The order relation $(p_s, q_s) \leq (p_w, q_w)$ holds if and only if the following hold:

(i) $p_s \leq_{\mathrm{Add}(\omega,\kappa)} p_w$;

(ii) $\mathrm{dom}(q_s) \supseteq \mathrm{dom}(q_w)$;

(iii) for all $\alpha \in \mathrm{dom}\, q_w$, $p_s \restriction \alpha \Vdash$ "$q_s(\alpha) \leq_{\dot{Add}(\omega_1,1)} q_w(\alpha)$".

We will often refer to $\mathbb{M}(\omega, \kappa)$ simply as $\mathbb{M}$.

Proposition 6.10. *If κ is inaccessible, then the Mitchell forcing $\mathbb{M}(\omega, \kappa)$ has the κ-chain condition. In fact, it is κ-Knaster.*

Proof. Suppose (p_ξ, q_ξ) are conditions in $\mathbb{M}(\omega,\kappa)$ for $\xi < \kappa$. Then it follows from two applications of the Δ-System Lemma (also using the fact that inaccessibility of κ gives us $\lambda^\omega < \kappa$ for all $\lambda < \kappa$) that there is some unbounded $I \subset \kappa$, a finite set d_1, and a countable set d_2 such that for all $\xi, \eta \in I$, $\mathrm{dom}(p_\xi) = \mathrm{dom}(p_\eta) = d_1$ and $\mathrm{dom}(q_\xi) \cap \mathrm{dom}(q_\eta) = d_2$.

Since the p_ξ's are functions into $\{0,1\}$, there are only finitely many possibilities for $p_\xi \restriction d_1$ for an arbitrary $\xi \in I$. Similarly, since for an arbitrary $\xi \in I$ we have $\Vdash_{\mathrm{Add}(\omega,\alpha)} q_\xi(\alpha) \in \dot{\mathrm{Add}}(\omega_1, 1)$ for each $\alpha \in d_2$ and $|\mathrm{Add}(\omega,\alpha)| < \kappa$, there are strictly *fewer* than κ-many possibilities for $q_\xi(\alpha)$. Therefore we can apply the Pigeonhole Principle to I to find $J \subset I$ with $|J| = \kappa$, $\bar{p}$, and $\bar{q}$ such that for any $\xi \in J$, $p_\xi \restriction d_1 = \bar{p}$ and $q_\xi \restriction d_2 = \bar{q}$. This means that for any $\xi, \eta \in J$, (p_ξ, q_ξ) and (p_η, q_η) are compatible. □

Therefore, if G is $\mathbb{M}(\omega,\kappa)$-generic over V, then $V[G] \models$ "κ is a cardinal".

6.3 Projections

In this section, we will show that $\mathbb{M}(\omega,\kappa)$ preserves $\aleph_1$ and κ and collapses all cardinals in the interval $(\aleph_1, \kappa)$ It then follows that $V[\mathbb{M}(\omega,\kappa)] \models$ "$\kappa = \aleph_2$". For the rest of this section, let $\mathbb{M} = \mathbb{M}(\omega,\kappa)$.

Definition 6.11. We say that $\pi : \mathbb{P} \to \mathbb{Q}$ is a *projection* if the following hold:

1. For all $p_0 \leq_{\mathbb{P}} p_1$, $\pi(p_0) \leq_{\mathbb{Q}} \pi(p_1)$.
2. For all $p \in \mathbb{P}$, if $q \leq \pi(p)$, then there is some $p' \leq p$ such that $\pi(p') \leq q$.

Proposition 6.12. *Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection and G is $\mathbb{P}$-generic over V. Then $H := \{q \in \mathbb{Q} : \exists p \in G, \pi(p) \leq q\}$ is $\mathbb{Q}$-generic over V.*

Lemma 6.13. *$V[\mathbb{M}(\omega,\kappa)] \models$ "$2^\omega = \kappa$".*

Proof. The map $\pi : (p,q) \mapsto p$ is a projection $\mathbb{M} \to \mathrm{Add}(\omega,\kappa)$. It is clear that this is order-preserving. Moreover, if $p' \leq \pi(p,q) = \pi(p)$, then $(p',q) \leq_{\mathbb{M}} (p,q)$ and $\pi(p',q) = p'$, which gives us the second requirement. Therefore it follows that $V[\mathrm{Add}(\omega,\kappa)] \subset V[\mathbb{M}(\omega,\kappa)]$. This shows that

$V[\mathbb{M}(\omega,\kappa)] \models$ "$2^\omega \geq \kappa$". But we can also find that since $|\mathbb{M}(\omega,\kappa)| = \kappa$, there are at most $\kappa^\omega = \kappa$-many nice names for subsets of ω, so therefore $V[\mathbb{M}(\omega,\kappa)] \models$ "$2^\omega \leq \kappa$" as well. □

Lemma 6.14. *$V[Add(\omega,\alpha)][Add(\omega_1,1)] \models$ "$|\alpha| \leq \omega_1^V$".*

Proof. Since $V[\mathrm{Add}(\omega,\alpha)] \models$ "$|2^\omega| = |\alpha|$" for $\alpha \geq \omega_1$ and $\mathrm{Add}(\omega_1,1)$ does not add new subsets of ω, it is enough to argue that in general $V[\mathrm{Add}(\omega_1,1)] \models$ "$2^\omega = \omega_1$".

Let G be $\mathrm{Add}(\omega_1,1)$-generic over V and let $A = \bigcup G$ be the generic function $\omega_1 \to \{0,1\}$. Working in $V[G]$, define $F : 2^\omega \to \omega_1$ as follows: if $s \in 2^\omega$, then $F(s)$ is the least limit ordinal $\beta < \omega_1$ such that for all $n < \omega$, $A(\beta+n) = s(n)$. The function F is injective from 2^ω to ω_1: if $s \neq t$, then there is some $n < \omega$ such that $s(n) \neq t(n)$, so if we had $F(s) = F(t) = \beta$, then it would follow that $A(\beta+n) = s(n) \neq t(n) = A(\beta+n)$.

Therefore it is enough to show that F is actually defined for all $s \in 2^\omega$. For all $s \in 2^\omega$, work in V to define $D_s \subseteq \mathrm{Add}(\omega_1,1)$ to consist of all $p \in \mathrm{Add}(\omega_1,1)$ such that $\exists \beta < \omega_1, \forall n < \omega, (\beta+n) \in \operatorname{dom} p$ and $\forall n < \omega, p(\beta+n) = s(n)$. It enough to show that D_s is dense because if G is $\mathrm{Add}(\omega_1,1)$-generic over V and $p \in G \cap D_s$, then $F(s)$ is defined.

We argue that D_s is dense: If $p \in \mathrm{Add}(\omega_1,1)$, then $|p| < \omega_1$, so choose a limit ordinal $\beta < \omega_1$ large enough that $\operatorname{dom} p \subset \beta$. Let p' be a condition so that $p' \upharpoonright \operatorname{dom} p = p$, $\beta + n \in \operatorname{dom}(p')$ for all $n < \omega$, and $p'(\beta+n) = s(n)$ for all $n < \omega$. Then $p' \leq_{\mathrm{Add}(\omega_1,1)} p$ and $p' \in D_s$. □

Lemma 6.15. *For all cardinals $\alpha < \kappa$, $\Vdash_{\mathbb{M}(\omega,\kappa)}$ "$|\alpha| \leq \aleph_1^V$".*

Proof. First we use:

Claim. *For all $\alpha < \kappa$, there is a projection $\pi : \mathbb{M}(\kappa,\omega) \to Add(\omega,\alpha) * \dot{A}dd(\omega_1,1)$.*

Proof of Claim. The projection is defined as $\pi : (p,q) \mapsto (p \upharpoonright \alpha, q(\alpha))$ if $\alpha \in \operatorname{dom}(q)$ and $\pi : (p,q) \mapsto (p \upharpoonright \alpha, 1_{\dot{\mathrm{Add}}(\omega_1,1)})$ otherwise. To see that this fulfills the first requirement, suppose that $(p_0,q_0) \leq_{\mathbb{M}} (p_1,q_1)$. Then we can see that $(p_0 \upharpoonright \alpha) \leq_{\mathrm{Add}(\omega,\alpha)} p_1 \upharpoonright \alpha$. If $\alpha \in \operatorname{dom}(q_0)$, then $p_0 \upharpoonright \alpha \Vdash$ "$q_0(\alpha) \leq q_1(\alpha)$ from the definition of the Mitchell forcing.

To see that the second requirement is fulfilled, suppose that $(r,s) \leq \pi(p,q)$. This means that $r \leq_{\mathrm{Add}(\omega,\alpha)} p \upharpoonright \alpha$, and we know that $r \in \mathrm{Add}(\omega,\alpha)$, so $p' := r \cup p$ is well-defined as a function in $\mathrm{Add}(\omega,\kappa)$. Let q' be such that

$\mathrm{dom}(q') = \mathrm{dom}(q) \cup \{\alpha\}$, $q'(\alpha) = s$, and $q'(\beta) = q(\beta)$ for all $\beta \in \mathrm{dom}(q)$. Then we can see that $(p', q') \leq_{\mathbb{M}} (p, q)$ and $\pi(p', q') \leq (r, s)$. □

Then this follows from Lemma 6.14. □

Definition 6.16. The *termspace forcing* $\mathbb{Q}$ is the poset consisting of functions q such that $\mathrm{dom}(q) \subseteq \kappa$, $\mathrm{dom}(q)$ is countable, and $\forall \alpha \in \mathrm{dom}(q)$, $\Vdash_{\mathrm{Add}(\omega,\alpha)} q(\alpha) \in \dot{\mathrm{Add}}(\omega_1, 1)$.

The ordering is as follows: $q_s \leq q_w$ if $\mathrm{dom}(q_w) \subseteq \mathrm{dom}(q_s)$ and for all $\alpha \in \mathrm{dom}\, q_w$, $1_{\mathrm{Add}(\omega,\alpha)} \Vdash$ "$q_s(\alpha) \leq_{\dot{\mathrm{Add}}(\omega_1,1)} q_w(\alpha)$".

Remark 6.17. The difference between point *(iii)* in Definition 6.8 and the ordering of the termspace forcing is extremely important.

Lemma 6.18. *The termspace forcing* $\mathbb{Q}$ *from Definition 6.16 is countably closed.*

The following is an important elementary fact of forcing:

Proposition 6.19 (Mixing Principle)**.** *Let* $\mathbb{P}$ *be any poset. If we have* $p \Vdash \exists v, \varphi(v, \dot{a}_1, \ldots, \dot{a}_k)$*, then there is some* $\mathbb{P}$*-name* σ *such that we have* $p \Vdash \varphi(\sigma, \dot{a}_1, \ldots, \dot{a}_k)$*.*

Proof of Lemma 6.18. Let $\langle q_n : n < \omega \rangle$ be a descending sequence of conditions in $\mathbb{Q}$. Let $d_n = \mathrm{dom}(q_n)$ and let $d = \bigcup_{n<\omega} \mathrm{dom}(q_n)$. Then d will be an at most countable subset of κ. For all $\alpha \in d$, there will be some n_α such that $\alpha \in \mathrm{dom}(q_n)$ for all $n \geq n_\alpha$. For each $\alpha \in \mathrm{dom}(q_n)$, we have

$$\Vdash_{\mathrm{Add}(\omega,\kappa)} \text{"}\langle q_n(\alpha) : n_\alpha \leq n < \omega \rangle \text{ is descending in } \mathrm{Add}(\omega_1, 1)\text{"}.$$

Hence

$$\Vdash_{\mathrm{Add}(\omega,\kappa)} \text{"there is a lower bound of } \langle q_n(\alpha) : n_\alpha \leq n < \omega \rangle\text{"}.$$

By the Mixing Principle, we therefore have $\dot{c}_\alpha$ such that

$$\Vdash_{\mathrm{Add}(\omega,\kappa)} \text{"}\dot{c}_\alpha \text{ is the lower bound of } \langle q_n(\alpha) : n_\alpha \leq n < \omega \rangle\text{"}.$$

More specifically, we inductively construct a maximal antichain $A_\alpha \subset \mathrm{Add}(\omega, \kappa)$ such that for all $u \in \mathrm{Add}(\omega, \kappa)$, there is some $\dot{c}_u^\alpha$ such that

$u \Vdash$ "$\dot{c}^\alpha_u$ is a lower bound of $\langle q_n(\alpha) : n_\alpha \leq n < \omega\rangle$". For each $\alpha \in d$, let $\dot{c}_\alpha = \{\langle\sigma, u'\rangle : \exists u \in A_\alpha, u' \leq u, u' \Vdash \sigma \in \dot{c}^\alpha_u\}$.

Now we let q be the condition with domain d such that for all $\alpha \in d$, $q(\alpha) = \dot{c}_\alpha$. We see that q is a lower bound of $\langle q_n : n < \omega\rangle$. □

Lemma 6.20. *There is a projection $\pi : Add(\omega, \kappa) \times \mathbb{Q} \twoheadrightarrow \mathbb{M}(\omega, \kappa)$.*

Proof. The projection π is just the identity: $\pi(p, q) = (p, q)$. However, the ordering in $\mathbb{M}$ is different from that of $\mathrm{Add}(\omega, \kappa) \times \mathbb{Q}$, so there is still work to do. Observe that $(p_0, q_0) \leq_{\mathrm{Add}(\omega,\kappa)\times\mathbb{Q}} (p_1, q_1)$ implies that $(p_0, q_0) \leq_{\mathbb{M}} (p_1, q_1)$, so the projection is automatically order-preserving.

It is left to show that if $(r, s) \leq_{\mathbb{M}} \pi(p_0, q_0)$, then there is some condition $(p_1, q_1) \leq_{\mathrm{Add}(\omega,\kappa)\times\mathbb{Q}} (p_0, q_0)$ such that $\pi(p_1, q_1) \leq_{\mathbb{M}} (r, s)$. We will choose $p_1 := r$, so we just need to find q_1. We know that for all $\alpha \in \mathrm{dom}(q_0)$, it is the case that $\alpha \in \mathrm{dom}(s)$ as well, and moreover that $r \restriction \alpha \Vdash s(\alpha) \leq_{\dot{\mathrm{Add}}(\omega_1,1)} q_0(\alpha)$.

We define $\mathrm{dom}(q_1) = \mathrm{dom}(s)$. For each $\alpha \in \mathrm{dom}(s) \setminus \mathrm{dom}(q_0)$, let $q_1(\alpha) = s(\alpha)$. For each $\alpha \in \mathrm{dom}(q_0)$, we define

$$q_1(\alpha) = \{\langle\sigma, u\rangle : u \leq r \restriction \alpha, u \Vdash \sigma \in s(\alpha) \text{ or } u \perp r \restriction \alpha, u \Vdash \sigma \in q_0(\alpha)\}.$$

This gives us both of our requirements:

- $1_{\mathrm{Add}(\omega,\alpha)} \Vdash q_1(\alpha) \leq_{\dot{\mathrm{Add}}(\omega_1,1)} q_0(\alpha)$;
- $r \restriction \alpha = p_1 \restriction \alpha \Vdash q_1(\alpha) \leq_{\dot{\mathrm{Add}}(\omega_1,1)} s(\alpha)$.

□

Lemma 6.21 (Easton)**.** *Suppose that $\mathbb{P}$ has the κ-chain condition and $\mathbb{Q}$ is κ-closed. Then $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ has the κ-chain condition".*

Proof. Suppose that $\bar{q} \in \mathbb{Q}$ and $\bar{q} \Vdash_{\mathbb{Q}}$ "$\dot{A}$ is an antichain in $\mathbb{P}$ of size $\geq \kappa$". We will find an antichain $A^* \subset \mathbb{P}$ such that $A \in V$ and $|A| = \kappa$. By induction on $\xi < \kappa$ we define a $\leq_{\mathbb{Q}}$-decreasing sequence $\langle q_\xi : \xi < \kappa\rangle$ below $\bar{q}$ and the antichain $A^* = \langle p_\xi : \xi < \kappa\rangle$.

The induction works as follows: In the successor case where q_ξ and p_ξ have been defined, use the fact that $q_\xi \Vdash$ "$|\dot{A}| \geq \kappa$" to choose $q_{\xi+1} \leq q_\xi$ and $p_{\xi+1} \in \dot{A}$ and such that $q_{\xi+1} \Vdash$ "$\forall \eta \leq \xi,\, p_{\xi+1} \neq p_\eta$". It ξ is a limit and

q_η, p_η have been defined for all $\eta < \nu$, first use the κ-closure of $\mathbb{Q}$ to find a lower bound q^*_ξ of $\langle q_\eta : \eta < \xi \rangle$, then choose q_ξ and p_ξ as an the successor case.

Finally, we can argue in V that $\langle p_\xi : \xi < \kappa \rangle$ is an antichain in $\mathbb{P}$ is an antichain. The construction was done so that $p_\eta \neq p_\xi$ for all $\eta < \xi < \kappa$. Furthermore, since $q_\xi \Vdash$ "$\dot{A}$ is an antichain" for all $\xi < \kappa$, it follows that for all $\eta < \xi < \kappa$, we have $p_\eta \perp p_\xi$. □

Lemma 6.22. *$\mathbb{M}(\omega, \kappa)$ preserves $\aleph_1$ (i.e. $\aleph_1^V$).*

Proof. Because the projection $\pi : \mathrm{Add}(\omega, \kappa) \times \mathbb{Q} \twoheadrightarrow \mathbb{M}(\kappa, \omega)$ that implies that $V[\mathbb{M}(\omega, \kappa)] \subset V[\mathrm{Add}(\omega, \kappa)][\mathbb{Q}]$, it is enough to show that the product $\mathrm{Add}(\omega, \kappa) \times \mathbb{Q}$ preserves $\aleph_1$.

Let $\mu = \aleph_1^V$ and let $G \times H$ be $\mathrm{Add}(\omega, \kappa) \times \mathbb{Q}$-generic over V. Recall that we can write $V[G \times H] = V[G][H] = V[H][G]$ where G is $\mathrm{Add}(\omega, \kappa)$-generic over $V[H]$ and H is $\mathbb{Q}$-generic over V. We know that $V[H] \models$ "$\mu = \aleph_1''$ because closed posets always preserve $\aleph_1$. We know that $V[H] \models$ "$\mathrm{Add}(\omega, \kappa)^V$ has the countable chain condition" by Easton's Lemma above, and so therefore $V[H][G] \models$ "$\mu = \aleph_1$" because posets with the countable chain condition always preserve $\aleph_1$. □

6.4 Quotients and Lifted Embeddings

Theorem 6.23 (Silver)**.** *Let $j : V \to M$ be an elementary embedding with critical point κ. Suppose that G is $\mathbb{P}$-generic and H is a $j(\mathbb{P})$-generic such that $j``G := \{j(p) : p \in G\} \subseteq H$. Then $j(G) = H$ and moreover in $V[H]$ there is an elementary embedding $j^* : V[G] \to M[j(G)] = M[H]$ such that $j^* \restriction V = j$.*

Remark 6.24. We typically write j^* as j.

Proof. Of course, any element of $V[G]$ takes the form $\dot{x}[G]$ where $\dot{x}$ is a $\mathbb{P}$-name in V. This allows us to define j^* as $j^* : \dot{x}_G \mapsto j(\dot{x})_H$.

First we argue that this is well-defined. Suppose that $V[G] \models$ "$\dot{x}[G] = \dot{y}[G]$". Then there is some $p \in G$ such that $p \Vdash$ "$\dot{x} = \dot{y}$". Since $j(p) \in H$ and $j(p) \Vdash$ "$j(\dot{x}) = j(\dot{y})$", it follows that $j^*(\dot{x}[G]) = j(\dot{x})[H] = j(\dot{y})[H] = j^*(\dot{y}[G])$.

Next, we need to argue that this is an elementary embedding. (From the definition, it is clear that $j^* \restriction V = j$.) Suppose that $V[G] \models \varphi(\dot{x}_1[G], \ldots, \dot{x}_k[G])$.

Then there is some $p \in G$ such that $p \Vdash^V_{\mathbb{P}} \varphi(\dot{x}_1, \ldots, \dot{x}_k)$, and so we continue with an argument analogous to the one in the previous paragraph to find that $j(p) \Vdash^M_{j(\mathbb{P})} \varphi(j(\dot{x}_1), \ldots, j(\dot{x}_k))$ and so $M[H] \models \varphi(j^*(\dot{x}_1[G]), \ldots, j^*(\dot{x}_k[G]))$.

Finally, to see that $j(G) = H$, consider the fact that there is a canonical name $\Gamma = \{(\check{p}, p) : p \in \mathbb{P}\}$ for the generic filter. Therefore $j(G) = j(\Gamma)[H] = \{(\check{p}, p) : p \in j(\mathbb{P})\}[H] = H$. □

The next proposition serves as an example of this theorem:

Proposition 6.25. *Let κ be measurable as witnessed by $j : V \to M$, let $\mathbb{P} = Add(\omega, \kappa)$, and let G be $Add(\omega, \kappa)$-generic over V. Then there is an extension $V[G][K]$ of $V[G]$ in which a lift $j : V[G] \to M[j(G)] = M[G][K]$ is defined.*

Remark 6.26. Observe that in this proposition, κ is no longer measurable in $V[G]$.

Proof. Observe that $M \models$ "$j(\mathbb{P}) = \mathrm{Add}(j(\omega), j(\kappa)) = \mathrm{Add}(\omega, j(\kappa))$", meaning that in M, $\mathrm{Add}(\omega, j(\kappa))$ consists of finite partial functions $j(\kappa) \times \omega \to \{0, 1\}$ that are ordered by reverse inclusion. This definition is absolute enough that $V \models$ "$j(\mathbb{P}) = \mathrm{Add}(\omega, j(\kappa))$".

Let $\mathbb{P}'$ be the poset of finite partial functions from $(j(\kappa) \setminus \kappa) \times \omega \to \{0, 1\}$, ordered by reverse inclusion. Then we can see that there is an isomorphism $\mathrm{Add}(\omega, j(\kappa)) \cong \mathrm{Add}(\omega, \kappa) \times \mathbb{P}'$ given by $p \mapsto (p \restriction (\kappa \times \omega), p \restriction ((j(\kappa) \setminus \kappa) \times \omega))$.

Now let G be $\mathrm{Add}(\omega, \kappa)$-generic over V and let K be $\mathbb{P}'$-generic over $V[G]$. Observe that $j(p) = p$ for all $p \in \mathrm{Add}(\omega, \kappa)$, so that if we identify $\mathrm{Add}(\omega, \kappa)$ with its isomorphic copy in $\mathrm{Add}(\omega, \kappa) \times \mathbb{P}'$, then $j''G \subset G \times K$. Furthermore, $G \times K$ is equivalent to an $\mathrm{Add}(\omega, j(\kappa))$-generic over V, so we can apply Silver's Lifting Lemma to obtain $j : V[G] \to M[G][K]$. □

Lemma 6.27. *Let $j : V \to M$ be a non-trivial elementary embedding with critical point κ and let $\mathbb{M} = \mathbb{M}(\omega, \kappa)$. Then there is a projection $\pi : j(\mathbb{M}) \to \mathbb{M}$.*

Proof. First, observe that we can understand the definition of $j(\mathbb{M})$ by elementarity. Considitions in $j(\mathbb{M})$ take the form (p, q) where:

1. $p \in \mathrm{Add}(\omega, j(\kappa))$;

2. $\mathrm{dom}\, q$ is a countable subset of $j(\kappa)$ and for all $\alpha \in \mathrm{dom}(q)$, $\Vdash_{\mathrm{Add}(\omega,\alpha)}$ "$q(\alpha) \in \dot{\mathrm{Add}}(\omega_1, 1)$".

Moreover, we have $(p_0, q_0) \leq_{j(\mathbb{M})} (p_1, q_1)$ precisely when:

(i) $p_0 \leq_{\mathrm{Add}(\omega, j(\kappa))} p_1$;

(ii) $\operatorname{dom} q_0 \supseteq \operatorname{dom} q_1$ and for all $\alpha \in \operatorname{dom} q_1$, $p_0 \restriction \alpha \Vdash q_0(\alpha) \leq_{\mathrm{Add}(\omega,\alpha)} q_1(\alpha)$.

Now we can define the projection π as $\pi : (p, q) \mapsto (p \restriction (\kappa \times \omega), q \restriction \kappa)$. We can observe that π is order-preserving.

Now we just need to show that if $(r, s) \leq_{\mathbb{M}} \pi(p_0, q_0)$, then there is some $(p_1, q_1) \leq_{j(\mathbb{M})} (p_0, q_0)$ such that $\pi(p_1, q_1) \leq_{\mathbb{M}} (r, s)$. To find p_1, just let $p_1 := r \cup p_0$, which is a well-defined function in $\mathrm{Add}(\omega, j(\kappa))$ because $r \restriction (\omega \times \kappa) \leq p_0$. Define q_1 such that $\operatorname{dom}(q_1) = \operatorname{dom}(s) \cup \operatorname{dom}(q_0)$. If $\alpha \in \operatorname{dom}(s)$, then let $q_1(\alpha) = s(\alpha)$. Otherwise it must be the case that $\alpha \geq \kappa$, and so we let $q_1(\alpha) = q_0(\alpha)$. Then (p_1, q_1) has the needed properties. □

Proposition 6.28. *Let κ be measurable as witnessed by $j : V \to M$, let $\mathbb{M} = \mathbb{M}(\omega, \kappa)$, and let G be $\mathbb{M}$-generic over V. Then there is an extension $V[G][H]$ of $V[G]$ in which a lift $j : V[G] \to M[j(G)]$ is defined.*

6.5 Branch Preservation Lemmas

Lemma 6.29 (Silver's Lemma)**.** *Let κ and λ be regular cardinals. Suppose that T is a κ tree, that $2^\lambda \geq \kappa$, and that $\mathbb{P}$ is λ^+-closed. Then $\mathbb{P}$ does not add cofinal branches to T.*

Proof. Generalize from Lemma 3.16, which we used for Kurepa trees. □

Lemma 6.30. *Assume that κ is regular and that T is a κ-tree with no cofinal branches. If $\mathbb{P}$ is κ-Knaster, then $\mathbb{P}$ does not add any cofinal branches to T.*

Proof. Fix $\mathbb{P}$ and T. Let $p \in \mathbb{P}$ be such that $p \Vdash_{\mathbb{P}}$ "$\dot{b}$ is a cofinal branch of T". We will find a cofinal branch of T in V.

For each $\alpha < \kappa$, find $p_\alpha \leq p$ and $x_\alpha \in T$ such that $p_\alpha \Vdash$ "$\dot{b} \cap T_\alpha = \{x_\alpha\}$. By κ-Knasterness, there is an unbounded set $I \subseteq \kappa$ (in particular, $|I| = \kappa$) such that for all $\alpha, \beta \in I$, $p_\alpha \| p_\beta$. Now let

$$a = \{t \in T : \exists\alpha \in I, t \le x_\alpha\} \in V.$$

We claim that a is a cofinal branch of T. From its definition we can see that $a \cap T_\alpha \neq \emptyset$ for all $\alpha < \kappa$. To see that it is will-ordered, observe that if $\alpha, \beta \in I$ and $\alpha < \beta$, then $x_\alpha <_T x_\beta$ because otherwise p_α and p_β would be incompatible, i.e. we would have $p_\alpha \perp p_\beta$. Specifically, if we had $x_\alpha \not<_T x_\beta$ and $r \le p_\alpha, p_\beta$ witnessed Knasterness, then we would have $r \Vdash$ "$\dot{b}$ is not linearly ordered". □

Remark 6.31. There is an improvement of this lemma, due to Carl Thomas Dean, that does not assume that T has no cofinal branches in the ground model.

Lemma 6.32 (Unger)**.** *Suppose that κ is regular and $\mathbb{P}$ is a poset and $\mathbb{P} \times \mathbb{P}$ has the κ-chain condition. Then if $T \in V$ is a tree of height κ, $\mathbb{P}$ does not add cofinal branches to T.*

Remark 6.33. This lemma is a weakening of the statement that Unger proved, which had to do with something called the κ-approximation property. Note also that we assume nothing about the width of T since we did not say that it needs to be a κ-tree.

Proof. Let T be a tree of height κ and suppose that there is some $p \in \mathbb{P}$ such $p \Vdash$ "$\dot{b}$ is a cofinal branch of T" and there is no $a \in V$ and $q \le p$ such that $q \Vdash$ "$\dot{b} = \check{a}$.

Claim. *For all $q \le p$ and $\alpha < \kappa$, there is some $\beta \in (\alpha, \kappa)$, some $q_0, q_1 \le q$, and $x_0 \neq x_1$ such that $q_i \Vdash$ "$\dot{b}(\beta) = x_i$ for $i \in \{0, 1\}$.*

Proof of Claim. This is as in the lemma of Silver that we used for Kurepa trees. If this were not the case, and there were some $\bar{q} \le p$ and $\bar{\alpha} < \kappa$ witnessing this, then we would have

$$\bar{q} \Vdash \text{“}\dot{b} = \{x \in T : \exists y \ge_T x, q \le \bar{q}, q \Vdash \text{“}y \in \dot{b}\}.$$

This is because for all $\beta \in (\bar{\alpha}, \kappa)$, there is some $q \le \bar{q}$ deciding $\dot{b}(\beta)$, and it does so uniquely. □

Now we will finish proving the lemma by building an antichain of size κ in $\mathbb{P} \times \mathbb{P}$. We define $\langle (p^0_\xi, p^1_\xi) : \xi < \kappa \rangle$ below p at the same time as an increasing sequence of ordinals $\langle \alpha_\xi : \xi < \kappa \rangle$ in κ such that for all $\xi < \kappa$, p^0_ξ

and p^1_ξ decide $\dot{b} \restriction \alpha_\xi$ the same way but decide $\dot{b}(\alpha_\xi)$ differently. We define these sequences simultaneously by a induction on $\xi < \kappa$ in which the limit and successor cases are basically identical. Suppose that for some $\eta < \xi$ we have already defined $\langle (p^0_\xi, p^1_\xi) : \xi < \eta \rangle$ and $\langle \alpha_\xi : \xi < \eta \rangle$. Let $\beta < \kappa$ be such that $\beta > \sup_{\xi < \eta} \alpha_\xi$. We find some $\bar{p}_\xi$ and x_ξ such that $\bar{p}_\xi \Vdash \dot{b}(\beta) = x$ for some $x \in T$. Observe that the implies that $\bar{p}_\xi$ forces $\dot{b}(\alpha_\eta)$ to be the predecessor of x on the α_η^{th} level of T for all $\eta < \xi$. Then we apply the claim to find $p^0_\xi, p^1_\xi \leq \bar{p}_\xi$ and some $\alpha_\xi \in (\beta, \kappa)$ such that

Now we argue that $\langle (p^0_\xi, p^1_\xi) : \xi < \kappa \rangle$ is an antichain. Suppose for contradiction that $\xi < \eta < \kappa$ and that (p^0_ξ, p^1_ξ) and (p^0_η, p^1_η) are compatible in the product ordering. Since p^0_η, p^1_η both decide $\dot{b}(\alpha_\xi)$ the same way, this implies that p^0_ξ, p^1_ξ also decide $\dot{b}(\alpha_\xi)$ the same way—otherwise (p^0_ξ, p^1_ξ) and (p^0_η, p^1_η) would be incompatible. However, this contradicts the construction, in which p^0_ξ and p^1_ξ decide $\dot{b}(\alpha_\xi)$ differently. □

Proposition 6.34. *If μ is regular and $\mathbb{P}$ is μ-Knaster, then $\mathbb{P} \times \mathbb{P}$ has the μ-chain condition.*

Proof. This is a weakening of the homework problem to prove that if $\mathbb{P}$ is μ-cc and $\mathbb{Q}$ is μ-Knaster, then $\mathbb{P} \times \mathbb{Q}$ is μ-cc. □

6.6 Putting Everything Together

Theorem 6.35. *If $V \models$ "κ is measurable", then $V[\mathbb{M}] \models$ "$\aleph_2$ has the tree property".*

Proof. Suppose G is $\mathbb{M}$-generic over V and fix an $\aleph_2$-tree $T \in V[G]$. Let $j : V \to M$ be a measurable embedding with critical point κ. Let $T \in V[G]$ be a κ-tree, or in other words, an $\aleph_2$-tree. Without loss of generality, $T \subset \kappa$.

First we argue that there is an extension $V[G][H]$ in which T has a cofinal branch b. There is a projection $\pi : j(\mathbb{M}) \to \mathbb{M}$, so let H be $j(\mathbb{M})/G := \{r \in j(\mathbb{M}) : \pi(r) \in G\}$-generic over V. Observe that for all $r \in \mathbb{M}$, $j(r) = r$ because the critical point of j is κ, and so $j''\mathbb{M} = \mathbb{M} \subset j(\mathbb{M})$. Working in $V[G][H]$, we can apply Silver's lifting lemma to extend j to $j : V[G] \to M[j(G)] = M[G][H]$. Then we define the cofinal branch b in the usual way, where we find some $z \in j(T)_\kappa$ (that is, a point in $j(T)$ on the κ^{th} level) and let $b = \{x \in T : x = j(x) <_{j(T)} z\}$.

Now we argue that in fact $b \in V[G]$. This will follow from the projection of products, which we will recall now. It was argued in the homework that, working in $V[G]$, there is a countable chain condition poset $\mathbb{P}'$ and a countably closed poset $\mathbb{Q}'$ such that there is a projection $\pi' : \mathbb{P}' \times \mathbb{Q}' \to j(\mathbb{M})/G$. Therefore can find a filter H_1 that is $\mathbb{Q}'$-generic over $V[G]$ and a filter H_2 that is $\mathbb{P}'$-generic over $V[G][H_1]$ such that $V[G][H] \subset V[G][H_1][H_2]$.

We can argue that the extension of $V[G][H_1][H_2]$ over $V[G]$ does not contain additional cofinal branches of T, which will finish the proof. In the homework it was proved that $\mathbb{P}'$ is κ-Knaster in $V[G]$, which implies that $\mathbb{P}' \times \mathbb{P}'$ has the κ-chain condition in $V[G]$, which implies by Easton's Lemma that $\mathbb{P}' \times \mathbb{P}'$ still has the κ-chain condition in $V[G][H_1]$, and so Unger's Lemma implies that that cofinal branch b could not have been added by $\mathbb{P}'$ and therefore we have $b \in V[G][H_1]$. Recall that $V[G] \models$ "$\kappa = \aleph_2 = 2^\omega$", so because of the countable closure of $\mathbb{Q}'$, Silver's Lemma implies that $b \in V[G]$. □

Definition 6.36. A collection $A \subset \mathcal{P}(S)$ of sets is an *algebra* if the following hold:

1. $X \in A \implies S \setminus X \in A$;
2. $X, Y \in A \implies X \cap Y \in A$.

An algebra A is κ-complete if $\langle X_\xi :< \lambda \rangle \subset A \implies \bigcap_{\xi<\lambda} X_\xi \in A$ for all ordinals $\lambda < \kappa$.

Definition 6.37. Let A be an algebra of subsets of S. Then an A-ultrafilter F is a filter such that for all $X \in A$, either $X \in F$ or $S \setminus X \in F$.

We use the following proposition, which is useful for the following theorem.

Proposition 6.38. *Suppose that κ is inaccessible and $X \subset V_\kappa$. Then there is a model $M \supset X$ such that M is transitive, $M \models \mathsf{ZFC} - \mathsf{Powerset}$, $\kappa \in M$, and $M^{<\kappa} \subset M$.*[1]

Proof. Choose some regular cardinal $\Theta > \kappa$; then it is the case that $H_\Theta := \{x : |\operatorname{tc}(x)| < \kappa\}$ (where $\operatorname{tc}(x)$ is the transitive closure of x) is a model of

[1] This is stated slightly differently than it was in the lecture.

$\mathsf{ZFC} - \mathsf{Powerset}$. (This is Exercise 12.13 in Jech.) Then let $\langle M_\xi : \xi < \kappa \rangle$ be an $\subseteq$-increasing sequence of elementary submodels of H_Θ of size κ such that $V_\kappa \cup \{\kappa\} \cup X \subset M_0$, and $M_\xi^{<\kappa} \subset M_{\xi+1}$. Then let $M' = \bigcup_{\xi<\kappa} M_\xi$ and let $M = \pi(M')$, the Mostowksi collapse of M'. By construction, M is a transitive set model of $\mathsf{ZFC} - \mathsf{Powerset}$ such that $|M| = \kappa$, $\kappa \in M$, and $M^{<\kappa} \subset M$. Most importantly, since $X \subset V_\kappa \subset M$, we have $\pi_M(z) = z$ for all $z \in X$, and therefore $X \subset M$. □

Theorem 6.39 (Folklore?). *Let κ be inaccessible.*[2] *Then the following are equivalent:*

1. *κ has the tree property.*
2. *Every κ-complete algebra $A \subset \mathcal{P}(\kappa)$ such that $|A| = \kappa$ has a κ-complete non-principal A-ultrafilter F.*
3. *Suppose M is a transitive set model of $\mathsf{ZFC} - \mathsf{Powerset}$ such that $|M| = \kappa$, $\kappa \in M$, $M^{<\kappa} \subset M$. Then there is an elementary embedding $j : M \to N$ where N is transitive, $|N| = \kappa$, $N^{<\kappa} \subset N$, the critical point of j is κ.*

Proof of 1. $\Longrightarrow$ 2. assuming κ is inaccessible. We will define a κ-tree T such that a cofinal branch of T will give us the filter.

First we need to establish some notation and facts. Let $\bar{F}$ be the "generalized Frechet filter" of co-bounded sets in κ, or more precisely $\bar{F} := \{X \subset \kappa : |\kappa \setminus X| < \kappa\}$ It is straightforward that this filter is closed under intersections of size less than κ. Let $\langle X_\xi : \xi < \kappa \rangle$ enumerate $\{X \subset \kappa : X \in M, |X| = |\kappa \setminus X| = \kappa\}$. Let $X_\xi^1 = X_\xi$ and let $X_\xi^0 = \kappa \setminus X_\xi$. Let ${}^{<\kappa}2$ denote $\bigcup_{\eta<\kappa} {}^\eta 2$, meaning the set of functions $s : \eta \to \{0, 1\}$ for some $\eta < \kappa$. For $s \in {}^{<\kappa}2$, we denote

$$Y_s := \bigcap_{\xi<\eta} X_\xi^{s(\xi)}.$$

Now define T to be the set of $s \in {}^{<\kappa}2$ such that Y_s has cardinality κ, and where $s <_T t$ if $s \sqsubseteq t$, i.e. $\operatorname{dom} s \subseteq \operatorname{dom} t$ and $t \restriction \operatorname{dom} s = s$. For $\eta < \kappa$, the level $T_\eta = \{s \in T : \operatorname{dom} s = \eta\}$.

[2] Some of the implication partially work if κ is not inaccessible, but it seems best to give the statement for inaccessible κ.

We need to make sure that T has height κ and levels of width strictly less than κ. By inaccessibility of κ, we have $|{}^{\eta}2| < \kappa$, so the levels will certainly have width less than κ. Now we argue that all levels of T are non-empty. Observe that if $s, t \in {}^{<\kappa}2$ are such that $\operatorname{dom} s = \operatorname{dom} t$ and $s \neq t$, then $Y_s \cap Y_t = \emptyset$. Therefore, for any $\eta < \kappa$, we can express κ as the disjoint union of Y_s for $s \in {}^{\eta}2$. By regularity of κ and the fact that $|{}^{\eta}2| < \kappa$ it follows that the set of $s \in {}^{\eta}2$ such that $|Y_s| = \kappa$ is non-empty.

Since κ has the tree property, it follows that T has a cofinal branch b. We can interpret b as a function $\kappa \to \{0, 1\}$ such that for all $\eta < \kappa$, $\bigcap_{\xi<\eta} X_{\xi}^{b(\xi)} \neq \emptyset$. Now define $F := \bar{F} \cup \{X : X = \exists\xi, X_{\xi} \text{ and } b(\xi) = 1\}$. It is fairly straightforward to verify from the construction that F is non-principal, that F measures all sets in $\mathcal{P}(\kappa) \cap M$, and that F is κ-complete. □

Proof of 2. $\Longrightarrow$ 3. First we show that we can find such an N and a j. Fix M with all of the required properties. Let F be a filter as in *2.* with respect to M. We define an embedding like the one for measurable cardinals.

We will define j as an ultrapower in terms of functions $f : \kappa \to M$ such that $f \in M$. For such functions $f, g : \kappa \to M$, let $f \in_F g$ if and only if $\{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in F$. Since M satisfies the separation schema, we have $\{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in M$, and the same is the case when we replace $\in$ with $\notin$. Hence F is "as much of an ultrafilter" as we need it to be. Moreover, we can define equivalence classes $[f]_F = \{g : \kappa \to M | g \in M, g =_F f\}$. Then we define N as the Mostowski collapse of the ultrapower $\operatorname{Ult}(M, F) = \{[f]_F | f : \kappa \to M, f \in M\}$.

The fact that N is transitive comes from the fact that it is defined as a Mostowski collapse, and the fact that $|N| = \kappa$ comes from the fact that $|M| = \kappa$. It remains to show that the critical point of j is κ and that $N^{<\kappa} \subset N$. These points follow from the same arguments that are used for measurable embeddings with no meaningful changes necessary. □

Proof of 3. $\Longrightarrow$ 1. It is enough to show that κ has the tree property since we are assuming that κ is inaccessible. Let T be a κ-tree, meaning that it has height κ and levels of width strictly less than κ. Without loss of generality, $T \subseteq \kappa$ and $<_T \subseteq \kappa \times \kappa$.

Use Proposition 6.38 to find $M \ni T$ such that M is a transitive model of $\mathsf{ZFC} - \mathsf{Powerset}$, $|M| = \kappa$, $\kappa \in M$, and $M^{<\kappa} \subset M$. Now take $j : M \to N$ witnessing *2.* and consider $j(T)$, which by elementarity is a $j(\kappa)$-tree in N. Let $z \in j(T)_{\kappa}$ and let $b = \{x \in T : x <_{j(T)} z\}$, keeping in mind that $j(x) = x$ since $T \subseteq \kappa$ and the critical point of j is κ. As in similar arguments, b is linearly ordered by $<_T$ and meets every level of T.

Since T was arbitrary, we have proved that κ has the tree property. $\square$

Theorem 6.40 (Silver)**.** *Let $j : M \to N$ be an elementary embedding with critical point κ where M is a model of* $\mathsf{ZFC} - \mathsf{Powerset}$*. Suppose that G is $\mathbb{P}$-generic and H is a $j(\mathbb{P})$-generic such that $j``G := \{j(p) : p \in G\} \subseteq H$. Then $j(G) = H$ and moreover in $M[H]$ there is an elementary embedding $j^* : M[G] \to N[j(G)] = N[H]$ such that $j^* \restriction V = j$.*

Proof. This is exactly the same as the proof with V in place of M. We state this only as an observation that we only need enough of ZFC for M to be able to interpret the forcing relation. $\square$

Theorem 6.41. *If $V \models$ "κ is weakly compact", then $V[\mathbb{M}] \models$ "$\aleph_2$ has the tree property".*

Proof. First, observe that if $\dot{T}$ is an $\mathbb{M}$-name for a κ-tree, which without loss of generality is an $\mathbb{M}$-name for a subset of κ. Therefore we can assume that $\dot{T}$ is a nice name, meaning that elements of $\dot{T}$ take that form $\langle \check{\alpha}, r \rangle$ for $\alpha < \kappa$ and $r \in \mathbb{M}$. Moreover, we can assume that $\leq_{\dot{T}}$ is a nice name in a similar sense. Then the fact that $|\mathbb{M}| = \kappa$ implies that without loss of generality, $|\dot{T}|, | \leq_{\dot{T}} | = \kappa$. Therefore, we can use Proposition 6.38 to construct a transitive set model of $\mathsf{ZFC} - \mathsf{Powerset}$ with the usual needed properties such that $\dot{T} \in M$.

If G is $\mathbb{M}$-generic over V, then G is $\mathbb{M}$-generic over M. Moreover, the dense sets deciding points of T are in M since it satisfies the separation schema. Therefore $\dot{T}[G] \in M[G]$. We can then define a lift $j : M[G] \to N[j(G)]$ in $V[G][H] = V[j(G)]$ where H is $j(\mathbb{M})/G$-generic over $V[G]$. We use that usual argument to find $b \in N[G][H] \subset V[G][H]$. Then we argue as before that $b \in V[G]$. $\square$