SQUARES AND UNCOUNTABLY SINGULARIZED CARDINALS

MAXWELL LEVINE AND DIMA SINAPOVA

Abstract. It is known that if κ is inaccessible in V and W is an outer model of V such that $(\kappa^+)^V = (\kappa^+)^W$ and $\mathrm{cf}^W(\kappa) = \omega$, then $\square_{\kappa,\omega}$ holds in W. (Many strengthenings of this theorem have been investigated as well.) We show that this theorem does not generalize to uncountable cofinalities: There is a model V in which κ is inaccessible, such that there is a forcing extension W of V in which $(\kappa^+)^V = (\kappa^+)^W$ and $\omega < \mathrm{cf}^W(\kappa) < \kappa$, while in W, $\square_{\kappa,\tau}$ fails for all $\tau < \kappa$. We make use of Magidor's forcing for singularizing an inaccessible κ to have uncountable cofinality. Along the way, we analyze stationary reflection in this model, and we show that it is possible for $\square_{\kappa,\mathrm{cf}(\kappa)}$ to hold in a forcing extension by Magidor's poset if the ground model is prepared with a partial square sequence.

1. Introduction

Singular cardinals are a topic of great interest in set theory, largely because they are subject to both independence results and intricate ZFC constraints. Here we focus on the subject of singularized cardinals: cardinals that are regular in some inner model and singular in some outer model. Large cardinals are in fact necessary to consider such situations, so in actuality we consider cardinals that are *inaccessible* in some inner model and singular in some outer model.

Singularized cardinals were originally sought in order to prove the consistency of the failure of the Singular Cardinals Hypothesis—the idea was to blow up the powerset of a cardinal and then singularize it—and obtaining singularized cardinals was a significant problem in its own right. Prikry resolved this problem using the hypothesis of a measurable cardinal [13]. (It follows from the Dodd-Jensen Covering Lemma that this hypothesis is necessary [4].) The result required a strikingly different forcing notion—Prikry forcing—which uses a normal measure to guide a countable sequence through the formerly measurable cardinal.

There are consequences when cardinals are singularized with regard to variations of Jensen's square principle. This principle, denoted $\square_\kappa$, expresses a non-compact relationship between κ and its successor. More precisely, $\square_\kappa$ asserts the existence of a coherent sequence of clubs singularizing points $\alpha < \kappa^+$. A hierarchy of intermediate square principles $\square_{\kappa,\lambda}$ for $1 \leq \lambda \leq \kappa$ was introduced by Schimmerling. The principle $\square_{\kappa,\lambda}$ weakens $\square_\kappa$ by allowing λ-many guesses for the clubs of each α, so a smaller λ indicates a stronger form of non-compactness between κ and κ^+ [14]. In Gödel's Constructible Universe L, $\square_\kappa$ holds at every cardinal κ, so the square hierarchy is a yardstick that allows us to compare given model to a canonical "L-like" inner model. Failure of square principles at a singular cardinal κ has many applications for obtaining lower bounds for consistency results and in infinitary combinatorics in general.

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With a strong enough large cardinal hypothesis, one can obtain failure of $\square_\kappa$ after forcing with the Prikry poset. However, this is not the case for the weaker square principles. Cummings and Schimmerling proved that Prikry forcing at κ adds a $\square_{\kappa,\omega}$-sequence [3]. This result is implicit in work of Džamonja and Shelah [5], and independently by Gitik [6], who provide the basis of the following abstract theorem: If κ is inaccessible in an inner model V, and there is an outer model $W \supset V$ such that $(\kappa^+)^V = (\kappa^+)^W$ and $(\operatorname{cf}\kappa)^W = \omega$, then $\square_{\kappa,\omega}$ holds in W. Magidor and Sinapova generalized this further to the situation where all of the cardinals in an interval $[\kappa,\nu]$, where ν is regular in V, are singularized to have cofinality ω while $(\nu^+)^V = (\kappa^+)^W$ [12].

For these constructions, the substantial part is describing the $\square_{\kappa,\omega}$-sequence at points $\alpha \in \lim(\kappa^+)$ such that $(\operatorname{cf}\alpha)^V < \kappa$, using so-called "pseudo Prikry sequences." Points α such that $\operatorname{cf}^V(\alpha) = \kappa$ have cofinality ω in W, and so a single cofinal sequence in α of order-type ω suffices because it is vacuously coherent: There are no limit points, so there is nothing to check.

The question, then, is what happens when κ is singularized to have uncountable cofinality. The initial conjecture was that the above theorems generalize to give $\square_{\kappa,\operatorname{cf}(\kappa)}$ in the outer model. This was in part supported by results showing that the above-mentioned pseudo Prikry sequences do indeed exist in the setting of uncountable cofinality. However, it turns out a $\square_{\kappa,\operatorname{cf}\kappa}$ sequence does not necessarily exist in the outer model. The main result of this paper is:

Theorem 1. *Assuming large cardinals, there is a model V in which κ is regular, which has an outer model $W \supset V$, such that:*

- $(\kappa^+)^V = (\kappa^+)^W$,
- $\omega < (\operatorname{cf}\kappa)^W < \kappa$, *and*
- $\square_{\kappa,\tau}$ *fails in W for all $\tau < \kappa$.*

The model V is one in which a Mahlo cardinal μ has been collapsed to be κ^+. Then W is an extension by Magidor's forcing for singularizing a cardinal to have uncountable cofinality while preserving cardinals [10]. Because κ is inaccessible in V, there is a $\square^*_\kappa$-sequence in V, and so $\square^*_\kappa$ because the successor of κ is preserved. Hence, the conclusion in our theorem is close to being sharp. We also describe a situation where, under the right preparation of the ground model, the Magidor forcing adds a $\square_{\kappa,\operatorname{cf}(\kappa)}$-sequence.

In addition to investigating square principles, we give a thorough analysis of stationary reflection in the setting of uncountably singularized cardinals where cardinals are preserved in the outer model. Stationary reflection is another instance of compactness—it follows from large cardinals and fails in L. We show both the ZFC implications in the abstract setting, without assuming the specific of the Magidor poset, and also the extent of reflection that can consistently hold after forcing with Magidor's poset.

The main result falls in the category of theorems exhibiting different behavior for singular cardinals of countable versus uncountable cofinality. These differences are not limited to square sequences. The most well-known example is Silver's result that GCH cannot fail for the first time at a singular cardinal of uncountable cofinality [15]. In contrast, Magidor showed shortly therafter that it is indeed possible for GCH to fail for the first time at $\aleph_\omega$ [11]. Another example is in the computation of the powerset of a singular cardinal with respect its ordinal definable subsets; see [2].

This paper is organized as follows. In Section 2 we give some background definitions and describe the Magidor forcing. In Section 3 we analyze the implications on stationary reflection, and also provide a model that is prepared so that the Magidor poset does add $\square_{\kappa,\mathrm{cf}(\kappa)}$. In Section 4, we prove Theorem 1.

2. Preliminaries

In this section we introduce some necessary background. Let τ be a regular cardinal. We will use $\mathrm{cof}(\tau)$ to denote points of cofinality τ and $\mathrm{cof}(<\tau)$ to denote points of cofinality less than τ. We say that a forcing poset is τ-closed if decreasing sequence of length less that τ have a lower bound.

2.1. **Basic Notions of Compactness and Non-Compactness.**

Definition 1. Given a cardinal τ, a stationary subset $S \subset \tau$, and a point $\rho \in \tau \cap \mathrm{cof}(>\omega)$ (i.e. a point ρ of uncountable cofinality), we say that S *reflects at* ρ if $S \cap \rho$ is stationary as a subset of ρ. And S *reflects* if it reflects at some $\rho \in \tau \cap \mathrm{cof}(>\omega)$.

Note that the notion of stationarity in ordinals of countable cofinality does not quite make sense, which is why we only speak of reflection at points of uncountable cofinality. We can also talk about reflection in ordinals that are not cardinals.

Definition 2. We say that $\langle \mathcal{C}_\alpha \mid \alpha \in \lim(\kappa^+)\rangle$ is a $\square_{\kappa,\lambda}$ sequence if for all limit $\alpha < \kappa^+$:

(1) each $C \in \mathcal{C}_\alpha$ is a club subset of α with $\mathrm{ot}(C) \leq \kappa$;
(2) for every $C \in \mathcal{C}_\alpha$, if $\beta \in \lim(C)$, then $C \cap \beta \in \mathcal{C}_\beta$;
(3) $1 \leq |\mathcal{C}_\alpha| \leq \lambda$.

A $\square_{\kappa,<\lambda}$-sequence has essentially the same definition, but where the third point is replaced by $1 \leq |\mathcal{C}_\alpha| < \lambda$. Note that $\square_{\kappa,1}$ is just the original Jensen's $\square_\kappa$, and $\square_{\kappa,\kappa}$ is the weak square $\square^*_\kappa$.

We will also consider the notion of *partial square*:

Definition 3. Let $S \subset \kappa^+$ be a stationary set. A sequence $\langle C_\alpha \mid \alpha \in S\rangle$ is a partial square sequence if each C_α is a club subset of α with $\mathrm{ot}(C_\alpha) \leq \kappa$, and for all $\alpha, \beta \in S$, for all $\gamma \in \lim(C_\alpha) \cap \lim(C_\beta)$, we have that $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

In general, square principles imply failure of stationary reflection:

Fact 2. *If $\square_\kappa$ holds, for every stationary $S \subset \kappa^+$, there is a stationary $T \subset S$ that does not reflect.*

On the other hand, stationary reflection follows from large cardinals. For example, if κ is measurable then every stationary subset of κ reflect. For stronger instances of reflection recall also the definition of a supercompact cardinal:

Definition 4. A cardinal κ is *μ-supercompact* if there is an elementary embedding $j : V \to M \subset V$ such that,

(1) j has critical point κ;
(2) $j(\kappa) > \mu$;
(3) $M^\mu \subset M$.

And κ is *supercompact* if it is μ-supercompact for all $\mu \geq \kappa$.

Given a μ-supercompact cardinal κ and a corresponding embedding $j : V \to M$, it can be shown that the point $\rho := \sup j[\mu]$ is a reflection point of $j(S)$ for any stationary $S \subset \mu \cap \mathrm{cof}(< \kappa)$, and so we have the following:

Fact 3. *If κ is supercompact and $\mu > \kappa$ is regular, then for every $\mu, \tau < \kappa$ and every sequence $\langle S_i : i < \tau \rangle$ of stationary subsets of $\mu \cap \mathrm{cof}(< \kappa)$, there is some $\rho < \mu$ with $\mu \leq \mathrm{cf}(\rho)$ and $\mathrm{cf}(\rho)$ a successor cardinal, such that for all $i < \tau$, $S_i \cap \rho$ is stationary.*

Moreover, any such embedding can be lifted over a small enough forcing, hence:

Fact 4. *If κ is supercompact in V and $|\mathbb{P}| < \kappa$, then κ is supercompact in $V^{\mathbb{P}}$.*

2.2. **Magidor Forcing.** Starting from a cardinal κ with Mitchell-order λ, the Magidor forcing adds a normal sequence of ground model-inaccessible cardinals $\langle \kappa_i \mid i < \lambda \rangle$ cofinal in κ. The sequence is added with conditions that determine only finitely many points at a time, so it is added in a non-monotonic way and is guided by a sequence of measures rather than just one.

Fix some regular $\lambda < \kappa$ for the following discussion. We write $\trianglelefteq$ to denote the Mitchell order on normal measure. Let $\vec{U} = \langle U_\xi : \xi < \lambda \rangle$ be a sequence of normal measures on κ that are increasing with respect to the Mitchell order. For $\eta < \xi$, let r^ξ_η represent U_η in the ultrapower V^κ / U_ξ. Let,

$$X_\xi = \{\delta < \kappa : \forall \eta < \xi, \forall \zeta < \eta, r^\xi_\eta(\delta), r^\xi_\zeta(\delta) \text{ are normal measures on } \delta \text{ and } r^\xi_\zeta(\delta) \trianglelefteq r^\xi_\eta(\delta)\} \in U_\xi.$$

Let Y_0 be the set of inaccessibles below κ and for $0 < \xi < \lambda$ let,

$$Y_\xi = \{\delta \in X_\xi : \forall \eta < \xi \forall \zeta < \eta, [r^\eta_\zeta \restriction \delta]_{r^\xi_\eta(\delta)} = r^\xi_\zeta(\delta)\}.$$

It is a fact that $Y_\xi \in U_\xi$ for all $\xi < \lambda$, although this is a nontrivial lemma in Magidor's original study.

Definition 5. Conditions in the Magidor forcing $\mathbb{M}(\vec{U})$ are pairs of functions (f, A) such that

(1) $\mathrm{dom}\, f \in [\lambda]^{<\omega}$ and $\mathrm{dom}\, A = \lambda \setminus \mathrm{dom}\, f$;
(2) $\forall \xi \in \mathrm{dom}\, f$, $f(\xi) \in Y_\xi$, $f(\xi) > \lambda$, and f is also strictly increasing;
(3) $\forall \xi \in \mathrm{dom}\, A$, if $\xi < \max \mathrm{dom}\, f$ and $\eta = \min\{(\mathrm{dom}\, f) \setminus (\xi + 1)\}$, then $A(\xi) \in r^\eta_\xi(f(\eta))$,
(4) $\forall \xi \in \mathrm{dom}\, A(\xi)$, if $\xi > \max \mathrm{dom}\, f$, then $A(\xi) \in U_\xi$ and $A(\xi) \subset Y_\xi \setminus (f(\eta)+1)$ where is the maximum element of $\mathrm{dom}\, f$ below ξ.

If $(f, A), (g, B) \in \mathbb{M}(\vec{U})$, then $(g, B) \leq (f, A)$ if:

(1) $f \subset g$;
(2) $\forall \xi \in \mathrm{dom}\, g \setminus \mathrm{dom}\, f$, $g(\xi) \in A(\xi)$;
(3) $\forall \xi \in \lambda \setminus \mathrm{dom}\, g$, $B(\xi) \subset A(\xi)$.

We can denote the poset by $\mathbb{M}$ if it is clear which sequence $\vec{U}$ is being used to define it. If $(f, A) \in \mathbb{M}$, then $\mathrm{stem}(f, A) = f$.

We denote $(f \restriction (\xi+1), A \restriction (\xi+1))$ by $(f, A)_\xi$ and $(f \setminus (f \restriction (\xi+1)), A \setminus (A \restriction (\xi+1)))$ by $(f, A)^\xi$.

Fact 5 (Prikry Lemma)**.** *Let $(f, A) \in \mathbb{M}$, let $\varphi(v)$ be a formula with one free variable, let $\xi \in \operatorname{dom} f$, and let δ be an ordinal such that $\eta \in \operatorname{dom} g \setminus (\xi + 1)$ implies $g(\eta) > \delta$. Then there is a condition $(f, B) \leq (f, A)$ such that $(f, B)_\xi = (f, A)_\xi$ and such that for all $\gamma < \delta$, if $(g, C) \leq (f, A)$ and (g, C) decides $\varphi(\gamma)$, then $(g, C)_\xi^\frown (f, B)^\xi$ decides $\varphi(\gamma)$ the same way.*

Let $\mathbb{M}_{\xi,\beta} = \{(f, A)_\xi : (f, A) \in \mathbb{M}, \xi \in \operatorname{dom} f \wedge f(\xi) = \beta\}$ and let $\mathbb{M}^{\xi,\beta} = \{(f, A)^\xi : (f, A) \in \mathbb{M}, \xi \in \operatorname{dom} f \wedge f(\xi) = \beta\}$. Observe that for β suited to the definition of $\mathbb{M}$, we have $\{(f, A) \in \mathbb{M} : \xi \in \operatorname{dom} f \wedge f(\xi) = \beta\} \cong \mathbb{M}_{\xi,\beta} \times \mathbb{M}^{\xi,\beta}$.

By noting that $f(\xi) = \beta$ means that β is inaccessible, and by counting the cardinalities of the stems, we obtain the following:

Fact 6. *$\mathbb{M}$ has the κ^+-chain condition and $\mathbb{M}_{\xi,\beta}$ has the β^+-chain condition.*

With a bit more work, we also have the following essential fact:

Fact 7. *$\mathbb{M}$ preserves cardinals.*

3. Stationary reflection and adding squares after singularizing to uncountable cofinality

3.1. **Stationary Reflection.** In this subsection we give the ZFC constraints on obtaining stationary reflection after singularizing to uncountable cofinality. Then, using the Magidor model, we show how how much reflection is consistently possible.

The following proposition generalizes results on the Prikry model from Cummings, Foreman, and Magidor [1], except now we also must consider points below κ that may have been singularized.

Proposition 8. *Suppose that $V \subset W$ are models of set theory, κ is a cardinal in both, such that $(\kappa^+)^V = (\kappa^+)^W$, κ is regular in V, and singular in W with $\omega < \lambda = \operatorname{cf}^W(\kappa)$. Then:*

(1) The set $S_0 := \kappa^+ \cap \operatorname{cof}^V(\kappa)$ cannot reflect to any point in W such that $(\operatorname{cf} \rho)^V < \kappa$.
(2) There is a sequence of λ-many stationary subsets of $\kappa^+ \cap \operatorname{cof}^V(< \kappa)$ in W that do not reflect simultaneously to any point $\rho \notin S_0$.
(3) If $W = V^{\mathbb{M}}$, where $\mathbb{M}$ is the Magidor forcing singularizing κ to have cofinality λ, then every stationary set $S \subset \kappa^+$ from W has a stationary subset $T \subset S$ that does not reflect to any ρ such that $(\operatorname{cf} \rho)^V = \kappa$.

If W is an extension of V by the Magidor forcing, then $S_0 := \kappa^+ \cap \operatorname{cof}^V(\kappa)$ is stationary in W by the chain condition.

Proof of (1). In this case we have $(\operatorname{cf} \rho)^V > \omega$ as well, and so there is $C \in V$ that is club in ρ and such that $\operatorname{ot} C = (\operatorname{cf} \rho)^V$. But then $(\operatorname{cf} \beta)^V < \kappa$ for all $\beta \in \lim C$, and so $\lim C \cap S_0 = \emptyset$. □

Proof of (2). Working in V, for each regular $\delta < \kappa$, let $S_\delta := \kappa^+ \cap \operatorname{cof}(\delta)$. In W, let s be cofinal sequence through κ consisting of W-regular cardinals, and consider $\langle S_\delta : \delta \in s\rangle$. Note that for $\delta \in s$, S_δ remains stationary in W because δ is regular in W, and so this is a sequence of λ-many stationary subsets of κ^+. Consider a supposed reflection point $\rho < \kappa^+$. Since the W-cofinalities of the points go above λ, we have $\operatorname{cf}^W(\rho) > \lambda$. So, $\operatorname{cf}^V(\rho) < \kappa$. Now consider $A := \{\delta < \kappa : S_\delta \cap \rho \text{ is stationary}\}$, which is defined in V. Since the S_δ's are disjoint, $|A| \le (\operatorname{cf}\rho)^V < \kappa$. But if $\langle S_\delta : \delta \in s\rangle$ reflects simultaneously in ρ, then A is unbounded in κ, a contradiction. □

Proof of (3). Working in W, let $\langle \kappa_\xi : \xi < \lambda\rangle$ converge to κ, let $T_\xi := \{\alpha \in S : (\operatorname{cf}\alpha)^W < \kappa_\xi\}$, and let $\eta < \lambda$ be such that $T := T_\eta$ is stationary. Given any $\rho < \kappa^+$ such that $(\operatorname{cf}\rho)^V = \kappa$, there is a club $C = \{\delta_\xi : \xi < \lambda\}$ such that for each limit $\xi < \lambda$, $(\operatorname{cf}\delta_\xi)^V = \kappa_\xi$. (Specifically, C is the image of the generic sequence under a strictly increasing cofinal function $f : \kappa \to \rho$ in V.) Then $\lim C \setminus \delta_\eta$ is disjoint from T. □

Now we establish the positive reflection results. Along the way we will use the facts mentioned in Section 2, fact 3 and fact 4.

Lemma 9. *Suppose G is $\mathbb{M}$-generic and there is some $(f, A) \in G$ and $\bar\xi \in \operatorname{dom} f$ such that $f(\bar\xi) = \beta$. Let G_0 be the projection of G onto $\mathbb{M}_{\bar\xi,\beta}$. Suppose also that $V[G_0] \models \beta < \operatorname{cf}\rho < \kappa$, $\operatorname{cf}\rho$ is not in the generic sequence added by G, and $V[G_0] \models$ "$S \subset \rho$ is stationary". Then $V[G] \models$ "S stationary in ρ".*

Proof. It is enough to show that if $C \in V[G]$ is a club in ρ then there is a club $D \subset C$ such that $D \in V[G_0]$. In fact, we will find such a club in V since $\mathbb{M}_{\bar\xi,\beta}$ has the β^+-chain condition and $\beta < \operatorname{cf}\rho$, and so it follows that $(\operatorname{cf}\rho)^{V[G_0]} = (\operatorname{cf}\rho)^V$.

Let (f, A) force that $\dot C$ is a club in ρ. Strengthening if necessary, we can assume that there is a $\xi < \lambda$ such that $\xi, \xi + 1 \in \operatorname{dom} f$ and $f(\xi) < \operatorname{cf}\rho < f(\xi + 1)$.

Pick a club $E = \{\rho_i : i < \operatorname{cf}\rho\} \subset \rho$ from V. Apply the Prikry Lemma 5 with respect to the formulas "$\rho_i \in \dot C$" for $i < \operatorname{cf}(\rho)$ to obtain $(f, B) \le (f, A)$, such that:

- $(f, B)_\xi = (f, A)_\xi$,
- for every $i < \operatorname{cf}(\rho)$, if $(h, J) \le (f, A)$ forces "$\rho_i \in \dot C$", then so does $(h, J)_\xi{}^\frown(f, B)^\xi$.

Now for each $i < \operatorname{cf}\rho$ let $\alpha_i \ge i$ and $(h^i, J^i) \le (f, A)$ be such that $(h^i, J^i) \Vdash$ "$\rho_{\alpha_i} \in \dot C$". Here we use that the two clubs are forced to intersect in a club. We have $h^i(\xi) = f(\xi)$ for each i, so that set of possible $(h^i, J^i)_\xi$'s has size less than $f(\xi)$. Since $f(\xi) < \operatorname{cf}\rho$, there is a fixed $(h, J)_\xi$ and an unbounded $X \subset \operatorname{cf}\rho$ such that $(h^i, J^i)_\xi = (h, J)_\xi$ for all $i \in X$.

Then $(h, J)_\xi{}^\frown(f, B)^\xi \Vdash$ "$\rho_{\alpha_i} \in \dot C$", for all $i \in X$. Define,

$$D = \{\rho_i : (h, J)_\xi{}^\frown(f, B)^\xi \Vdash \rho_i \in \dot C\}.$$

It follows that D is closed because $\dot C$ is forced to be closed, and D is unbounded because $\{\rho_{\alpha_i} : i \in X\} \subset D$. □

Theorem 10. *Suppose that κ is κ^+-supercompact and $\mathbb{M}$ is the Magidor forcing to change the cofinality of κ to λ. Then in $V^{\mathbb{M}}$, for every $\mu < \kappa$ and every $\tau < \lambda$, every sequence $\langle S_i : i < \tau\rangle$ of stationary subsets of $\kappa^+ \cap \operatorname{cof}^V(<\kappa)$ reflects simultaneously to an ordinal of cofinality greater μ.*

Proof. Let G be $\mathbb{M}$-generic and let $\langle S_i : i < \tau\rangle \in V[G]$ be a sequence of stationary subsets of $\kappa^+ \cap \operatorname{cof}^V(<\kappa)$. Let $\dot S_i$ for $i < \tau$ be the respective $\mathbb{M}$-name.

For $\xi < \lambda$, let $S_i^\xi = \{\alpha \in S_i : \exists (f, A) \in G, \operatorname{dom} f \subset \xi, (f, A) \Vdash \alpha \in \dot S_i\}$. Since $S_i = \bigcup_{\xi<\lambda} S_i^\xi$, there is some ξ_i such that $S_i^{\xi_i}$ is stationary. Let $\eta = \sup_{i<\tau} \xi_i < \lambda$. Let $(\bar f, \bar A) \in G$ be a condition such that $\eta \in \operatorname{dom}\bar f$ and let $\beta := \bar f(\eta)$. Let G_0 be the $\mathbb{M}_{\eta,\beta}$-generic induced by G and the factorization $\mathbb{M}_{\eta,\beta} \times \mathbb{M}^{\eta,\beta}$.

Now we work in $V[G_0]$. For every $i < \tau$, define

$$T_i := \{\alpha < \kappa^+ : \exists (f, A) \in \mathbb{M}/G_0, \operatorname{dom} f \subset \eta, (f, A) \Vdash \alpha \in \dot{S}_i\}.$$

In $V[G]$, for each i, $S_i^{\xi_i} \subset T_i$, so T_i is stationary. Since $|\mathbb{M}_{\eta,\beta}| < \kappa$, κ is still κ^+-supercompact in $V[G_0]$ by Fact 4. By Fact 3 we can find an ordinal $\rho < \kappa^+$ such that $T_i \cap \rho$ is stationary for all $i < \tau$, $\operatorname{cf} \rho$ is a successor cardinal, and $\beta < \operatorname{cf} \rho < \kappa$.

We claim that each S_i reflects at ρ in $V[G]$. Continue working in $V[G_0]$. Let $C \subset \rho$ be a club in V such that $\operatorname{ot} C = \operatorname{cf} \rho$; then $T_i' := T_i \cap C$ is still stationary in ρ. For each $i < \tau$ and $\alpha \in T_i'$, let $(f_{i,\alpha}, A_{i,\alpha}) \in \mathbb{M}/G_0$ witness membership in T_i, i.e.

- $(f_{i,\alpha}, A_{i,\alpha})$ forces $\alpha \in \dot{S}_i$,
- $\operatorname{dom} f_{i,\alpha} \subset \eta$.

Then each, $(f_{i,\alpha}, A_{i,\alpha})^\eta$ has no stem, so we can take a lower bound in $\mathbb{M}^{\eta,\beta}$ for the sequence $\langle (f_{i,\alpha}, A_{i,\alpha})^\eta \mid i < \tau, \alpha \in T_i' \rangle$ by intersecting measure-one sets. Then for all $i < \tau$, $(h, B) \Vdash_{\mathbb{M}/G_0} \Vdash T_i' \subset \dot{S}_i$.

By density we can get such a condition $(h, B) \in G$. I.e. $V[G] \models T_i' \subset S_i$.

Since $\operatorname{cf} \rho$ is a successor, it cannot be on the generic sequence added by G. So, by lemma Lemma 9 we have that $T_i' \subset \rho$ remains stationary in $V[G]$. Therefore $S_i \cap \rho$ is stationary in $V[G]$. $\square$

Recall the following fact due to Cummings, Foreman, and Magidor [1]:

Fact 11. *If $\kappa^{<\lambda} = \kappa$ and $\square_{\kappa,<\lambda}$ holds, then every stationary subset of κ^+ contains a stationary subset that does not reflect at ordinals of cofinality greater or equal to λ.*

This together with Theorem 10 implies the following:

Corollary 12. *If κ is κ^+-supercompact and $\mathbb{M}$ is the Magidor poset for making κ have cofinality λ, then $\square_{\kappa,<\lambda}$ fails in $V^{\mathbb{M}}$.*

3.2. **Adding squares.** In this subsection, we show that under some assumptions of the inner model, singularizing to uncountable cofinality adds a $\square_{\kappa,\operatorname{cf}(\kappa)}$ sequence.

Theorem 13. *Suppose $\kappa^+ \cap \operatorname{cof}(\kappa)$ carries a partial square and κ is inaccessible in V. Then if $V \subset W$, $(\kappa^+)^W = (\kappa^+)^V$, and $\operatorname{cf}(\kappa)^W = \lambda < \kappa$, we have that $\square_{\kappa,\lambda}$ holds in W.*

We will use a theorem of Lambie-Hanson (a thorough simplification of Corollary 4.2 in [7]):

Fact 14. *Suppose that V is an inner model of W, κ is regular in V, and $(\operatorname{cf} \kappa)^W = \lambda < \kappa$, while $(\kappa^+)^V = (\kappa^+)^W$. Suppose furthermore than in V there is a sequence $\langle D_\alpha : \alpha < \kappa^+ \rangle$ of clubs in κ. Then in W, there is a club $\langle \kappa_i : i < \lambda \rangle$ of uncountable W-cofinality, such that for all $\alpha < \kappa^+$ and for all sufficiently large $i < \lambda$, $\kappa_i \in D_\alpha$.*

This fact presents the analog of the Magidor-generic sequence in a ZFC setting. The sequence $\langle \kappa_i : i < \lambda \rangle$ may be referred to as a *pseudo-Prikry sequence*.

Proof of Theorem 13. Let $\langle D_\alpha \mid \alpha \in \kappa^+ \cap \operatorname{cof}(\kappa) \rangle \in V$ be a partial square sequence. We start with a standard construction along the lines of [8]. In V we construct continuous elementary models $\langle M_\delta^\alpha \mid \delta < \kappa \rangle$, for every $\alpha \in \kappa^+ \cap \operatorname{cof}^V(<\kappa)$, such that each M_δ^α has size less than κ, $M_\delta^\alpha \cap \kappa \in \kappa$, and $\kappa, \alpha \in M_0^\alpha$.

In W fix a Pseudo-Prikry sequence $\langle \kappa_i \mid i < \lambda \rangle$ through κ, such that each κ_i has uncountable cofinality in W, and for all $\alpha < \kappa$, for all large $i < \lambda$, $\kappa_i \in \{M^\alpha_\delta \cap \kappa \mid \delta < \kappa\}$.

Then, for $\alpha \in \kappa^+ \cap \operatorname{cof}^V(< \kappa)$, define

$$\begin{aligned}\mathcal{C}_\alpha =\{ & \overline{M^\beta_\delta} \cap \alpha \mid \beta \geq \alpha, \delta < \kappa, M^\beta_\delta \text{ unbounded in both } \alpha, \beta, \\ & \text{and for some } i < \lambda, M^\beta_\delta \cap \kappa = \kappa_i\}.\end{aligned}$$

As in [8], $\langle \mathcal{C}_\alpha \mid \alpha \in \kappa^+ \cap \operatorname{cof}^V(< \kappa)\rangle$ is coherent, each $\mathcal{C}_\alpha$ consists of clubs of order type less than κ, and $1 \leq |\mathcal{C}_\alpha| \leq \lambda$. The key points in proving the last facts are:

- If M^β_δ is cofinal in β and $\operatorname{cf}(\delta) > \omega$, then $M^\beta_\delta \cap \beta$ is ω-closed.
- If $M^\beta_\delta \cap \kappa = M^\gamma_\eta \cap \kappa$, and $M^\beta_\delta, M^\gamma_\eta$ are both cofinal in α, then $M^\beta_\delta \cap \alpha = M^\gamma_\eta \cap \alpha$.

For $\alpha \in \kappa^+ \cap \operatorname{cof}^V(\kappa)$, let $\mathcal{D}_\alpha = \{D_\alpha\}$. And for $\beta \in \kappa^+ \cap \operatorname{cof}^V(< \kappa)$, let $\mathcal{D}_\beta = \mathcal{C}_\beta \cup \{D_\alpha \cap \beta \mid \exists \alpha \in \kappa^+ \cap \operatorname{cof}^V(\kappa), \beta \in \lim(D_\alpha)\}$. Then $\langle \mathcal{D}_\alpha \mid \alpha < \kappa^+\rangle$ is a $\square_{\kappa,\lambda}$-sequence. □

It is consistent for κ to be supercompact while $\kappa^+ \cap \operatorname{cof}(\kappa)$ carries a partial square. This because we have the following poset, due to Baumgartner:

Definition 6. Let $\mathbb{S}$ be the poset consisting of functions p such that:

(1) $\sup \operatorname{dom} p < \kappa^+$;
(2) $(\sup \operatorname{dom} p + 1) \cap (\kappa^+ \cap \operatorname{cof}(\kappa)) \subset \operatorname{dom} p$;
(3) $\forall \alpha \in \operatorname{dom} p$, $p(\alpha)$ is a club in α;
(4) $\forall \alpha \in \operatorname{dom} p, \forall \beta \in \lim p(\alpha)$, $p(\alpha) \cap \beta = p(\beta)$.

Fact 15. *$\mathbb{S}$ is κ-directed closed and adds a partial square sequence on $\kappa^+ \cap \operatorname{cof}(\kappa)$.*

If κ is indestructibly supercompact, then in $V^{\mathbb{S}}$, κ remains supercompact and there is a partial square sequence on $\kappa^+ \cap \operatorname{cof}(\kappa)$.

As a final remark here, note the following lemma from [9] about the state of pcf regardless of the type of ground model.

Lemma 16. *If V is an inner model of W, κ is regular in V, $(\operatorname{cf} \kappa)^W = \lambda < \kappa$, and $(\kappa^+)^V = (\kappa^+)^W$, then in W there is a scale $\langle f_\alpha : \alpha < \kappa^+\rangle$ such that all points α such that $\operatorname{cf} \alpha > \lambda$ are very good.*

4. A More Compact Magidor Model

In this section we prove Theorem 1. Specifically, we show that under appropriate preparation of the ground model, after singularizing a regular cardinal κ to uncountable cofinality while preserving cardinals, we can still have failure of weaker squares at κ. This is in contrast to the case of countable cofinality.

4.1. A few important notions.

As above $\mathbb{M}$ is Magidor's forcing for singularizing κ to have cofinality λ. As usual, if $p = (f, A), q = (g, B) \in \mathbb{M}$, then $p \leq^* q$ (that is, p is a *direct extension* of q) if $p \leq q$ and $f = g$. Observe that the ordering $\leq^*$ is at least λ^+-closed, and κ-closed for conditions with empty stems. If φ is a formula and $p \in \mathbb{M}$, there is a direct extension $q \leq^* p$ such that $q \| \varphi$.

This section will require us to make ample use of the diagonalization properties of $\mathbb{M}$. For these purpose, we employ the following notation: if $p = (f, A) \in \mathbb{M}$ and $\vec{\nu}$ is a function with domain $\lambda^{<\omega} \setminus \operatorname{dom} f$ such that $\vec{\nu}(\xi) \in A(\xi)$, then $p^\frown\vec{\nu}$ is the weakest possible extension of p such that $\operatorname{stem}(p^\frown\vec{\nu}) = \operatorname{stem}(p)^\frown\vec{\nu}$. Let $E(p)$ be the set of possible such $\vec{\nu}$ with regard to p. We have the following diagonal lemma:

Lemma 17. *(Diagonal lemma) If $p \in \mathbb{M}$, let $\langle q_{\vec{\nu}} : \vec{\nu} \in E(p)\rangle$ be such that $q_{\vec{\nu}} \leq^* p^\frown\vec{\nu}$. Then there is a direct extension $p' \leq^* p$ such that for all $\vec{\nu} \in E(p')$, $p'^\frown\vec{\nu} \leq q_{\vec{\nu}}$.*

We refer to p' above as the *diagonalization* over $\langle q_{\vec{\nu}} : \vec{\nu} \in E(p)\rangle$.

We also generalize the notion of an n-step extension in the original Prikry forcing: Suppose $p = (f, A) \in \mathbb{M}$ and $a \in \lambda^{<\omega} \setminus f$. If $q \leq p$ and $\operatorname{stem}(q) = \operatorname{stem}(p)^\frown\vec{\nu}$ where $\operatorname{dom}(\vec{\nu}) = a$, then we say that q is an *a-step extension* of p. Also let $E_a(p)$ be the set of all possible such $\vec{\nu}$ with domain a.

Using this definition, there is also a Prikry Density Lemma for $\mathbb{M}$:

Fact 18. *If p is a condition and $D \subset \mathbb{M}$ is an open dense subset, there is a direct extension $q \leq^* p$ and $a \in \lambda^{<\omega}$, such that* every *a-step extension of q is in D.*

4.2. **Setting up the construction.** Let κ be indestructibly supercompact and let $\mu > \kappa$ be Mahlo. We denote $\mathbb{C} = \operatorname{Col}(\kappa, < \mu)$. For $\gamma < \mu$, let $\mathbb{C}_\gamma = \operatorname{Col}(\kappa, < \gamma)$. Note that if γ is inaccessible, then $\mathbb{C}_\gamma$ has the γ-chain condition. Given a $\mathbb{C}$-generic H, let H_γ be the induced generic on $\mathbb{C}_\gamma$. Because κ remains supercompact in $V^{\mathbb{C}}$, it is possible to define $\mathbb{M}$, the Magidor forcing in $V^{\mathbb{C}}$ to singularize κ to have cofinality $\lambda \in (\omega, \kappa)$. We shall argue that $\square_{\kappa,\tau}$ fails for all $\tau < \kappa$ in $V^{\mathbb{C}*\mathbb{M}}$.

For $\xi < \lambda$, use the mixing principle to find $\mathbb{C}$-names $\dot{U}_\xi$ for normal measures on κ and $\mathbb{C}$-names $\dot{r}^\eta_\xi$ for $\xi < \eta < \lambda$ such that if $\xi < \eta < \lambda$, then $\Vdash_{\mathbb{C}}$ "$\dot{U}_\xi \trianglelefteq \dot{U}_\eta$" and $\Vdash_{\mathbb{C}}$ "$\dot{r}^\eta_\xi$ represents $\dot{U}_\xi$ in the ultrapower of V by $\dot{U}_\eta$".

Proposition 19. *There is a stationary set of inaccessible cardinals $I \subset \mu$ such that for all $\delta \in I$,*

1. *for all $\xi < \lambda$, $V[H_\delta] \models$ "$(\dot{U}_\xi)_H \cap V[H_\delta]$ is a normal ultrafilter on κ" and,*
2. *for all $\xi < \eta < \lambda$, $((\dot{U}_\xi)_H \cap V[H_\delta]) \trianglelefteq ((\dot{U}_\eta)_H \cap V[H_\delta])$.*

Proof. We define a function $F : \mu \to \mu$. For each nice $\mathbb{C}_\alpha$-name $\dot{X}$ for a subset of κ, let $\rho(\dot{X})$ be the supremum of $\beta < \mu$ such that there exists $c \in C_\beta$ such that for all $\xi < \eta < \lambda$, (1) $c \| \dot{X} \in \dot{U}_\xi$ and such that (2) if $c \Vdash \dot{X} \in \dot{U}_\xi$ then $c \Vdash \{\alpha < \kappa : \dot{X} \cap \alpha \in \dot{r}^\eta_\xi(\alpha)\} \in \dot{U}_\eta$. By the μ-chain condition of $\mathbb{C}$ and the fact that $|\operatorname{dom}(c)| < \kappa$ for all $c \in \mathbb{C}$, $\rho(X) < \mu$. Let $F(\alpha)$ be the supremum of $\rho(\dot{X})$ over all nice $\mathbb{C}_\alpha$-names $\dot{X}$ for subsets of κ. Let I be the set of inaccessible closure points of F, which is stationary because μ is Mahlo. If δ is inaccessible closure point of F and $X \in V^{\mathbb{C}_\delta}$ is a subset of κ, then there is some $\gamma < \delta$ such that $X \in V^{\mathbb{C}_\gamma}$, and so $\rho(\dot{X}) < \delta$, and hence all necessary information is decided in $V^{\mathbb{C}_\delta}$. □

Let U^δ_ξ be $(\dot{U}_\xi)_H \cap V[H_\delta]$ for $\delta \in I$. For these δ, we can define $\mathbb{M}_\delta$, the Magidor forcing defined in terms of U^δ_ξ, $\xi < \lambda$.

Proposition 20. *There is a dense set $D(\mathbb{C}_\delta * \mathbb{M}_\delta)$ of conditions of the form $(c, (\check{f}, \dot{A}))$ where $c \in \mathbb{C}_\delta$ and $c \| \dot{A}(\xi)$ for all $\xi < \max(\operatorname{dom} f)$. Similarly, there is such a dense subset of $\mathbb{C} * \mathbb{M}$.*

This of course follows from the κ-closure of $\mathbb{C}_\delta$, and of $\mathbb{C}$ respectively. We will abuse notation slightly and assume that all conditions from $\mathbb{C}_\delta * \mathbb{M}_\delta$ or $\mathbb{C} * \mathbb{M}$ are found from these dense sets.

Lemma 21. *For $\delta \in I$, there is a complete embedding $\iota_\delta : \mathbb{C}_\delta * \mathbb{M}_\delta \to \mathbb{C} * \mathbb{M}$.*

Proof. We define ι_δ as inclusion in both coordinates—note that $\dot{A}$, as a $\mathbb{C}_\delta$-name for a subset of κ, can be included naturally in $\mathbb{M}$. That ι_δ is order-preserving is straightforward, and it preserves incompatibility because incompatibility in both $\mathbb{M}_\delta$ and $\mathbb{M}$ is determined by the stems.

To show completeness, consider a condition of the form $q = (c, (\check{f}, \dot{A}))$ in $\mathbb{C} * \mathbb{M}$, strengthening if necessary so that all information about $(\check{f}, \dot{A})$ up to $\max(\operatorname{dom} f)$ is decided by c. Then let $p = (c \restriction \delta, (\check{f}, \dot{B}))$ where $\dot{B}$ is chosen to be anything producing a valid condition. Then for any $r \leq p$, $\iota_\delta(r)$ is compatible with q. $\square$

We will generally suppress the notation for ι_δ when indicating quotients. If G is $\mathbb{M}$-generic, let G_δ be the induced generic on $\mathbb{M}_\delta$. The core of the technical difficulty is represented by the following lemma:

Lemma 22. *If δ is inaccessible and $\mathcal{C}$ is a $\square_{\kappa,\tau}$-sequence in $V^{\mathbb{C}_\delta * \mathbb{M}_\delta}$, for $\tau < \kappa$, then the extension $\mathbb{C} * \mathbb{M}/(\mathbb{C}_\delta * \mathbb{M}_\delta)$ does not thread $\mathcal{C}$.*

Once we have the lemma, we can prove the theorem:

Proof of the Theorem. Suppose for contradiction that for some $\tau < \kappa$, there is a $\square_{\kappa,\tau}$-sequence $\mathcal{C} = \langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ in $V^{\mathbb{C}*\mathbb{M}}$. Note that $\mu = (\kappa^+)^{V^{\mathbb{C}*\mathbb{M}}}$. Note that $\mathbb{C} * \mathbb{M}$ itself has the μ-chain condition, and so there is a club of $\gamma < \mu$ such that for all $\beta < \gamma$, $\mathcal{C}_\beta \in V[H_\gamma * G_\gamma]$. Since μ is Mahlo, there is an inaccessible δ in this club. Then $\langle \mathcal{C}_\alpha : \alpha < \delta \rangle$ is a $\square_{\kappa,\tau}$-sequence in $V^{\mathbb{C}_\delta * \mathbb{M}_\delta}$ which is threaded by any element of $\mathcal{C}_\delta$. (It is important here that δ has uncountable cofinality in $V^{\mathbb{C}*\mathbb{M}}$.) However, the existence of this thread contradicts Lemma 22. $\square$

For the rest of the section, we work towards the proof of Lemma 22.

4.3. **The main lemma.**

First we will prove the lemma for the case when τ is less than the first point on the Magidor sequence. The advantage in that case is that all of the normal measures are $(2^\tau)^+$-complete.

Lemma 23. *Suppose δ is inaccessible, $H_\delta * G_\delta$ is $\mathbb{C}_\delta * \mathbb{M}_\delta$-generic and τ is below the first point on the Magidor sequence added by G_δ. If $\mathcal{C}$ is a $\square_{\kappa,\tau}$-sequence in $V[H_\delta * G_\delta]$, then the extension $\mathbb{C} * \mathbb{M}/(H_\delta * G_\delta)$ does not thread $\mathcal{C}$.*

For such δ and work in $V[H_\delta]$. In a slight abuse of notation, we let $\mathbb{C}$ denote $\mathrm{Col}(\kappa, < \mu)/H_\delta$. Let $\dot{\mathbb{Q}}$ denote the $\mathbb{M}_\delta$-name for the quotient $(\mathbb{C} * \mathbb{M})/\mathbb{M}_\delta$. Note that since $\mathbb{C}$ is κ-closed, there is a dense subset of conditions of the form $(r, (c, \dot{p})) \in \mathbb{M}_\delta * \dot{\mathbb{Q}}$, such that c decides the stem of $\dot{p}$, so we exclusively consider conditions of this form.

Proposition 24. *Suppose that $r = (f, A) \in \mathbb{M}_\delta$, $c \in \mathbb{C}$ and $\dot{p} = (g, \dot{B})$ is a $\mathbb{C}$-name for a condition in $\mathbb{M}$. Then $r \Vdash (c, \dot{p}) \notin \dot{\mathbb{Q}}$ iff at least one of the following hold:*

1. *f, g are incompatible (i.e. not equal and neither one extends the other),*
2. *$f \supset g$ and $c \Vdash \exists \xi \in \operatorname{dom}(f) \setminus \operatorname{dom}(g)(f(\xi) \notin \dot{B}(\xi))$,*
3. *$g \supset f$ and for some $\xi \in \operatorname{dom}(g) \setminus \operatorname{dom}(f)$, $g(\xi) \notin A(\xi)$.*

Proof. The proposition follows from the fact that $r \Vdash (c, \dot{p}) \notin \dot{\mathbb{Q}}$ iff for any $\mathbb{C}$-generic H with $c \in H$, we have that r and $\dot{p}_H$ are incompatible in $\mathbb{M}$. □

Proposition 25. *Suppose that $r = (f, A) \in \mathbb{M}_\delta$, $c \in \mathbb{C}$ and $\dot{p} = (g, \dot{B})$ is a $\mathbb{C}$-name for a condition in $\mathbb{M}$ and*

- *$f = g$, or*
- *$f \supset g$, and $c \not\Vdash \exists \xi \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)(f(\xi) \notin \dot{B}(\xi))$.*

Then there is an extension $r' \leq^* r$, and $r' \Vdash (c, \dot{p}) \in \dot{\mathbb{Q}}$.

Proof. By the Prikry Property, let $r' \leq^* r$ decide "$(c, \dot{p}) \in \dot{\mathbb{Q}}$". Since none of the conditions of Proposition 24 hold, r' cannot force that $(c, \dot{p})$ is not in the quotient. So, r' is as desired. □

Proof of Lemma 23. Suppose for contradiction that $\dot{d}$ is a name for such a thread.

The key to the argument is the following Splitting Lemma:

Lemma 26. *(Splitting lemma) There is $r \in \mathbb{M}_\delta, c \in \mathbb{C}, \bar{\gamma} < \delta, a \in \lambda^{<\omega}$, such that for all $r' \leq^* r$, there is $r" \leq^* r'$, such that each a-step extension of $r"$ is in the set $D_{\bar{\gamma},c} := \{q \in \mathbb{M}_\delta \mid \forall c' \leq c \forall \gamma' > \bar{\gamma} \exists \gamma > \gamma', \exists c_0, c_1 \leq c', \exists \dot{p}_0, \dot{p}_1$ s.t. for $i \in 2$,*

- *$q \Vdash (c_i, \dot{p}_i) \in \dot{\mathbb{Q}}$,*
- *$(c_i, \dot{p}_i) \Vdash \dot{d} \cap (\gamma', \gamma) \neq \emptyset, \dot{d} \cap \gamma = e_i$, for some $\mathbb{M}_\delta$ name e_i,*
- *$q \Vdash e_0 \neq e_1$,*
- *$q, \dot{p_0}, \dot{p_1}$ have the same stem.$\}$*

Proof. Suppose otherwise. Then we can find a $\leq^*$-decreasing sequence (according to some enumeration) $\langle r^a \mid a \in \lambda^{<\omega} \rangle$, of conditions in $\mathbb{M}_\delta$, $\leq$-decreasing conditions $\langle c^a \mid a \in \lambda^{<\omega} \rangle$ in $\mathbb{C}$, and points $\langle \gamma^a \mid a \in \lambda^{<\omega} \rangle$ in δ, such that for each a, for all $r" \leq^* r^a$, there is an a-step extension of $r"$ not in the set D_{γ^*, c^*}, as witnessed by c_a, γ_a, where $\gamma^* \geq \gamma_{a'}$ and $c^* \leq c_{a'}$ for each a' coming before a in the fixed enumeration.

More precisely, there is an a-step extension of $r"$, q, for which are no $\gamma > \gamma^a, c_0, c_1 \leq c^a, \dot{p}_0, \dot{p}_1$ with the same stem as q, such that the q forces that $(c_0, \dot{p}_0)$, $(c_1, \dot{p}_1)$ are in $\dot{\mathbb{Q}}$ and decide $\dot{d} \cap \gamma$ in contradictory ways while forcing $\dot{d} \cap (\gamma_a, \gamma) \neq \emptyset$.

Now let r be a direct extension of each r^a, $c \leq c^a$ $\gamma > \gamma^a$ for each a. Let $D := \{q \in \mathbb{M}_\delta \mid (\exists \gamma' \geq \gamma, \exists c_0, c_1 \leq c, \dot{p}_0, \dot{p}_1$ with the same stem as $q) q \Vdash$ "$(c_i, \dot{p}_i) \in \dot{\mathbb{Q}}, (c_i, \dot{p}_i) \Vdash \dot{d} \cap (\gamma, \gamma') \neq \emptyset, \dot{d} \cap \gamma' = e_i$, for $i = 0, 1, e_0 \neq e_1$"$\}$.

Claim. *D is dense.*

Proof. Let $q \in \mathbb{M}_\delta$, and let G_δ be $\mathbb{M}_\delta$-generic containing q. In $V[H_\delta][G_\delta]$, let $(c_0, \dot{p}_0), (c_1, \dot{p}_1) \in \mathbb{Q}$ force contradictory information about $\dot{d} \cap \gamma'$ and also that $\dot{d} \cap (\gamma, \gamma') \neq \emptyset$, for some $\gamma < \gamma' < \delta$. For $i = 0, 1$, denote $\dot{p}_i = (g_i, \dot{H}_i)$. Note that $g_0 \cup g_1$ is a finite segment of the G_δ- Magidor generic sequence.

Now let $q' \leq q$ in G_δ force that $(c_0, \dot{p}_0), (c_1, \dot{p}_1)$ are as above. By extending q' we may assume that $q' = (g, H)$ where $g \supset g_0 \cup g_1$.

Work again in $V[H_\delta]$. For $i = 0, 1$, let ϕ_i be the sentence $\bigwedge_{\xi \in \mathrm{dom}(g) \setminus \mathrm{dom}(g_i)} g(\xi) \in \dot{H}_i(\xi)$. Since q' forces that $(c_i, \dot{p}_i)$ is in $\dot{\mathbb{Q}}$, by Fact 24 it follows that $c_i \not\Vdash \neg \phi_i$. Let $c_i' \leq c_i$ be such that $c_i' \Vdash \phi_i$. Let $\dot{p}_i' = (g, \dot{H}_i')$ be a name for a condition in $\mathbb{M}$, such that for each $\xi \notin \mathrm{dom}(g)$, if $c_i' \Vdash \dot{H}_i'(\xi) = \dot{H}_i(\xi) \cap g(\xi')$, where $\xi' = \min(\mathrm{dom}(g) \setminus \xi)$ if such exists and $c_i' \Vdash \dot{H}_i'(\xi) = \dot{H}_i(\xi)$ otherwise. Then $c_i' \Vdash \dot{p}_i' \leq \dot{p}_i$, for $i = 0, 1$.

By Fact 25 there is a direct extension $q" \leq^* q'$ forcing that for $i = 0, 1$. $(c'_i, \dot{p}'_i) \in \dot{\mathbb{Q}}$. Then $q" \in D$. $\square$

Then by the Prikry lemma, there is $r" \leq^* r$ and $a \in \lambda^{<\omega}$, such that every a step extension of $r"$ is in D. Contradiction with the choice of r^a, c^a, γ^a and the fact that $r" \leq^* r^a, c \leq c^a, \gamma > \gamma^a$. $\square$

Remark 1. *Instead of requiring the Prikry conditions have the same stem in the definition of $D_{\gamma,c}$, we can just require that for each $i = 0, 1$, $g := stem(q)$ extends $g_i := stem(\dot{p}_i)$ and for all $\xi \in \mathrm{dom}(g) \setminus \mathrm{dom}(g_i)$, $c_i \Vdash g(\xi) \in H^{\dot{p}_i}(\xi)$. That is a weaker statement but will suffice in the arguments that follow.*

Fix $\bar{r}, \bar{c}, \bar{\gamma}, a$ as in the conclusion of the Splitting lemma. Let $E = E_a(\bar{r})$, i.e. all possible $\vec{\nu}$ corresponding to a-step extensions of $\bar{r}$. For every $\vec{\nu} \in E$, recall that $\bar{r}^\frown\vec{\nu}$ is the weakest possible a-step extension of $\bar{r}$ with that stem.

Build $\langle r_{|\sigma|}, c^{\vec{\nu}}_\sigma, \dot{p}^{\vec{\nu}}_{\sigma^\frown i}, \gamma^{\vec{\nu}}_\sigma \mid \sigma \in 2^{<\tau}, i \in 2, \vec{\nu} \in E\rangle$ by induction on $\mathrm{ot}(\sigma)$, such that:

1. each $r_{|\sigma|} \leq^* \bar{r}$, $c^{\vec{\nu}}_\sigma \leq \bar{c}$, $\gamma^{\vec{\nu}}_\sigma \geq \bar{\gamma}$
2. for every $\vec{\nu}$, and $\sigma \sqsubset \sigma'$, $c^{\vec{\nu}}_{\sigma'} \leq c^{\vec{\nu}}_\sigma$,
3. $\dot{p}^{\vec{\nu}}_{\sigma^\frown i}$ has the same stem as $\bar{r}^\frown\vec{\nu}$,
4. $r_{|\sigma|}{}^\frown\vec{\nu}$ forces that for $i = 0, 1$, $(c^{\vec{\nu}}_{\sigma^\frown i}, \dot{p}^{\vec{\nu}}_{\sigma^\frown i})$ are in $\dot{\mathbb{Q}}$, and decide contradictory values for $\dot{d} \cap \gamma^{\vec{\nu}}_\sigma$.
5. for $i = 0, 1$, $(c^{\vec{\nu}}_{\sigma^\frown i}, \dot{p}^{\vec{\nu}}_{\sigma^\frown i}) \Vdash \dot{d} \cap (\gamma^*, \gamma^{\vec{\nu}}_\sigma) \neq \emptyset$, where $\gamma^* = \sup_{\bar{\sigma}, \mathrm{ot}(\bar{\sigma}) < \mathrm{ot}(\sigma)} \gamma^{\vec{\nu}}_{\bar{\sigma}}$.

To construct such a sequence, suppose we have defined $c^{\vec{\nu}}_\sigma$ for all $\vec{\nu}$. Let $\gamma^* = \sup_{\vec{\nu}, \bar{\sigma}, o.t.(\bar{\sigma}) < o.t.(\sigma)} \gamma^{\vec{\nu}}_{\bar{\sigma}}$. Let $r_{|\sigma|} \leq^* \bar{r}$ be given by the Splitting lemma, i.e. every a-step extension of $r_{|\sigma|}$ is in $D_{\bar{\gamma},\bar{c}}$.

Then for every $\vec{\nu}$, $r_{|\sigma|}{}^\frown\vec{\nu}$ is in that set, and so there is $\gamma^{\vec{\nu}}_\sigma > \gamma^*$ and conditions $c^{\vec{\nu}}_{\sigma^\frown i} \leq c^{\vec{\nu}}_\sigma$ and $\dot{p}^{\vec{\nu}}_{\sigma^\frown i}$ for $i = 0, 1$, such that items (3), (4), and (5) above hold.

For the limit stages, we just have to worry about $c^{\vec{\nu}}_\sigma$. For a fixed ν, and $\sigma \in 2^\eta$ for limit η, let $c^{\vec{\nu}}_\sigma$ be a lower bound of each $c^{\vec{\nu}}_{\sigma\restriction\xi}$ for $\xi < \eta$. Here we use the closure of $\mathbb{C}$.

Let $r \leq^* r_\eta$, for all $\eta < \tau$, and let $\beta = \sup_{\sigma,\vec{\nu}} \gamma^{\vec{\nu}}_\sigma$. Note that by construction, for any $\vec{\nu}$, $\beta = \sup_\sigma \gamma^{\vec{\nu}}_\sigma$.

For all $\vec{\nu}, f \in 2^\tau$, let $c^{\vec{\nu}}_f \leq c^{\vec{\nu}}_{f\restriction\eta}$ for all $\eta < \tau$. Also let $\dot{p}^{\vec{\nu}}_f$ be a name for a condition in $\mathbb{M}$ with stem the same as the stem of $\bar{r}^\frown\vec{\nu}$, such that $c^{\vec{\nu}}_f$ forces it to be a lower bound of all $\dot{p}^{\vec{\nu}}_{f\restriction\eta}$ for all $\eta < \tau$. To do that we just have to take canonical name for the intersection of measure one sets. (Actually we can get $1_{\mathbb{C}}$ to force that it is a lower bound).

Then by the last item of our construction, for each f, $\vec{\nu}$, $(c^{\vec{\nu}}_f, \dot{p}^{\vec{\nu}}_f) \Vdash \beta \in \lim(\dot{d})$.

We do not necessarily have that $r^\frown\vec{\nu}$ forces that $(c^{\vec{\nu}}_f, \dot{p}^{\vec{\nu}}_f)$ is in the quotient. However, since the Prikry conditions have the same stem, for each $\vec{\nu}$, applying Fact 25 inductively 2^τ-many times, there is a direct extension $r^{\vec{\nu}} \leq^* r^\frown\vec{\nu}$, such that for every $f \in 2^\tau$, $r^{\vec{\nu}} \Vdash (c^{\vec{\nu}}_f, \dot{p}^{\vec{\nu}}_f) \in \dot{\mathbb{Q}}$. Here we use that since 2^τ is below the first Prikry point, the direct extension order $\leq^*_{\mathbb{M}_\delta}$ is more than 2^τ-closed.

Diagonalize $\langle r^{\vec{\nu}} \mid \vec{\nu} \in E\rangle$ to get $r' \leq^* r$, such that for all $\vec{\nu}$, $r'^\frown\vec{\nu} \leq r^{\vec{\nu}}$.

Let G_δ be $\mathbb{M}_\delta$-generic with $r' \in G_\delta$. Let $\vec{\nu} \in E$ be the unique, such that $r'^\frown\vec{\nu} \in G_\delta$. Work in $V[H_\delta][G_\delta]$. By choice of r', $r^{\vec{\nu}} \in G_\delta$. So for all$f \in 2^\tau$, $(c^{\vec{\nu}}_f, \dot{p}^{\vec{\nu}}_f) \in \mathbb{Q}$.

For all f, let $(c_f, \dot{p}_f) \leq_{\mathbb{Q}} (c_f^{\vec{\nu}}, \dot{p}_f^{\vec{\nu}})$, be such that that for some e_f, $(c_f, \dot{p}_f) \Vdash_{\mathbb{Q}} \dot{d} \cap \beta = e_f$. Note that the Prikry conditions no longer have a fixed stem.

Claim. *$f \neq g$ implies $e_f \neq e_g$.*

Proof. Let $\sigma \in 2^{<\tau}$ be such that $\sigma^\frown 0 \subseteq f, \sigma^\frown 1 \subseteq g$. By construction, and since $r_{|\sigma|}^\frown \vec{\nu} \in G_\delta$, we have that: for $i = 0, 1$, $(c_{\sigma^\frown i}^{\vec{\nu}}, \dot{p}_{\sigma^\frown i}^{\vec{\nu}})$ are in $\mathbb{Q}$ and decide contradictory values for $\dot{d} \cap \gamma_\sigma^{\vec{\nu}}$. Let e_0, e_1 be these values.

Since, $(c_f, \dot{p}_f) \leq (c_{\sigma^\frown 0}^{\vec{\nu}}, \dot{p}_{\sigma^\frown 0}^{\vec{\nu}})$, we have $e_0 \sqsubset e_f$. Similarly, $e_1 \sqsubset e_g$. So $e_f \neq e_g$. □

Contradiction, since each e_f is in $\mathcal{C}_\beta$, $|\mathcal{C}_\beta| \leq \tau$, and the number of f's is 2^τ. □

This concludes the proof that $\square_{\kappa,\tau}$ fails in $V[H * G]$ for any τ below the first Magidor point. Since the latter is an inaccessible, in particular that means that:

Corollary 27. *We can singularize a regular κ to uncountable cofinality λ in a cardinal preserving way and have failure of $\square_{\kappa,\lambda}$ in the outer model.*

Next we show the failure of $\square_{\kappa,\tau}$ for all $\tau < \kappa$. The main difficulty in this general case is that not all of the measures used in the Magidor forcing have enough closure. However we can split the generic Magidor sequence into an upper part where the closure is more than τ, and a lower part, where the cardinality of possible conditions is less than τ. Then the argument will be similar to the first version, but now when building our splitting tree, we have to consider all lower parts as well as all a-step extensions. For that we will need a stronger version of the Splitting Lemma, where all the Magidor conditions witnessing the splitting have not only the same stem, but the same lower part.

More precisely, working in $V[H_\delta][G_\delta]$, let $\langle \kappa_\xi \mid \xi < \lambda \rangle$ be the generic Magidor sequence through κ. Let $\xi < \lambda$ be such that $\kappa_\xi < \tau < 2^\tau < \kappa_{\xi+1}$. (Without loss of generality, τ will not be in the generic sequence.) Now fix a condition in G_δ, deciding the values of $\kappa_\xi, \kappa_{\xi+1}$ (namely, $\xi, \xi+1$ are in the domain of its stem) and for the rest of the proof of the lemma work below it. In other words, any conditions in $\mathbb{M}$ or $\mathbb{M}_\delta$ below are assumed to be of the form (g, H), where $\xi, \xi+1 \in \mathrm{dom}(g)$ and $\kappa_\xi = g(\xi) < 2^\tau < g(\xi+1)$.

We say a condition r in $\mathbb{M}_\delta$ or $\mathbb{M}$ has *lower part b* to mean that $r \restriction \xi = b$. (If $r \in \mathbb{M}$, that means that it is forced to have lower part b.) As before, by closure of $\mathbb{C}$, there is a dense subset of conditions of the form $(r, (c, \dot{p})) = ((f, A), (c, (\check{g}, \dot{B}))) \in \mathbb{M}_\delta * \dot{\mathbb{Q}}$, such that c decides the lower part of $\dot{p}$, so we only consider conditions of this form.

Proposition 28. *Suppose that $r = (f, A) \in \mathbb{M}_\delta$, $c \in \mathbb{C}$ and $\dot{p} = (g, \dot{B})$ is a $\mathbb{C}$-name for a condition in $\mathbb{M}$, such that $c \Vdash r \restriction \xi = \dot{p} \restriction \xi$ and*

- *$f = g$, or*
- *$f \supset g$, and $c \not\Vdash \exists \xi \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)(f(\xi) \notin \dot{B}(\xi))$.*

Then there is an extension $r' \leq^* r$ with $r' \restriction \xi = r \restriction \xi$, such that $r' \Vdash (c, \dot{p}) \in \dot{\mathbb{Q}}$.

Proof. By the Prikry Property, let $r' = (f, A') \leq^* (f, A)$ be such that $(f, A')_\xi = (f, A)_\xi$ and if $q \leq r'$ decides "$(c, \dot{p}) \in \dot{\mathbb{Q}}$", then $q \restriction \xi^\frown (f, A')^\xi$ decides it the same way. We claim that r' is as desired.

Otherwise, there is some $q \leq r'$ such that $q \Vdash (c, \dot{p}) \notin \dot{\mathbb{Q}}$. Then $q' := q \restriction \xi^\frown (f, A')^\xi \Vdash (c, \dot{p}) \notin \dot{\mathbb{Q}}$. So c forces that the stem of q' is incompatible with the measure one sets of $\dot{p}$ as in Proposition 24. Then by item (1) and (2) of the assumption, c has to force that $stem(q \restriction \xi)$ is incompatible with the measure one sets of $\dot{p} \restriction \xi$. Contradiction with the assumption that c forces that $r \restriction \xi = \dot{p} \restriction \xi$. $\square$

Proof of Lemma 22.

Lemma 29. *(Strong splitting lemma) There is $r \in \mathbb{M}_\delta, c \in \mathbb{C}, \bar{\gamma} < \delta, a \in \lambda^{<\omega}$, and a dense set of lower parts L* [1]*, such that for all $r' \leq^* r$, with a lower part $b \in L$, there is $r" \leq^* r'$, with lower part b, such that each a-step extension of $r"$ is in the set $D'_{\bar{\gamma},c} :=$.*

$\{q \in \mathbb{M}_\delta \mid \forall c' \leq c \forall \gamma' > \bar{\gamma} \exists \gamma > \gamma', \exists c_0, c_1 \leq c', \exists \dot{p}_0, \dot{p}_1$ s.t. for $i \in 2$,

- *$q \Vdash (c_i, \dot{p}_i) \in \dot{\mathbb{Q}}$,*
- *$(c_i, \dot{p}_i) \Vdash \dot{d} \cap (\gamma', \gamma) \neq \emptyset, \dot{d} \cap \gamma = e_i$, for some $\mathbb{M}_\delta$ name e_i,*
- *$q \Vdash e_0 \neq e_1$,*
- *$q, \dot{p}_0, \dot{p}_1$ have the same stem and the same lower part.$\}$*

Proof. Let $\bar{r}, \bar{c}, \bar{\gamma}, a$ be as in the conclusion of the splitting lemma but with the added restriction that the Magidor conditions to have the same lower part. More precisely, for every $r' \leq^* r$, there is $r" \leq^* r'$, such that each a-step extension of $r"$ is in $D'_{\bar{\gamma},c}$.

Now suppose the lower part of $\bar{r}$ is $\bar{b}$, and all $b \leq \bar{b}$ do not have the property in the statement of the corollary. Then there is a sequence $\langle r^b \mid b \leq \bar{b} \rangle$, such that each $r^b \leq^* \bar{r}$, $r^b \restriction \xi = b$, and for all $r" \leq^* r^b$ with lower part b, there is an a-step extension $r"^\frown \vec{\nu}$ not in $D'_{\bar{\gamma},c}$. Let $r \leq^* \bar{r}$ be such that for all b, $b^\frown r \restriction [\xi, \kappa) \leq^* r^b$.

But then by choice of $\bar{r}$, there is $r" \leq^* r$, such that each a-step extension of $r"$ is in $D'_{\bar{\gamma},c}$. Let $b = r" \restriction \xi$; contradiction with choice of r^b. $\square$

Fix $\bar{r}, \bar{c}, \bar{\gamma}, a$ and L as in the conclusion of the above corollary. Note that $|L| < \tau$. As before, let $E = E_a(\bar{r})$.

For every $b \in L$, let $\bar{r}^\frown b$ denote $b^\frown \bar{r} \restriction [\xi, \kappa)$ and for $\vec{\nu} \in E, b \in L$, let $\bar{r}^\frown b, \vec{\nu}$ denote the weakest possible a-step extension of $\bar{r}^\frown b$ with that stem, provided that $\vec{\nu}$ is compatible with b.

Build $\langle r_{|\sigma|}, c^{b,\vec{\nu}}_\sigma, \dot{p}^{b,\vec{\nu}}_{\sigma^\frown i}, \gamma^{b,\vec{\nu}}_\sigma \mid \sigma \in 2^{<\tau}, i \in 2, b \in L, \vec{\nu} \in E \rangle$ by induction on $\mathrm{ot}(\sigma)$, such that:

1. each $r_\eta \leq^* \bar{r}$, $r_\eta \restriction \xi = \bar{r} \restriction \xi$, $c^{b,\vec{\nu}}_\sigma \leq \bar{c}$, $\gamma^{b,\vec{\nu}}_\sigma \geq \bar{\gamma}$
2. for every $b, \vec{\nu}$, and $\sigma \sqsubset \sigma'$, $c^{b,\vec{\nu}}_{\sigma'} \leq c^{b,\vec{\nu}}_\sigma$,
3. $\dot{p}^{b,\vec{\nu}}_{\sigma^\frown i}$ has the same stem and lower part as $\bar{r}^\frown b, \vec{\nu}$.
4. $r_{|\sigma|}{}^\frown b, \vec{\nu}$ forces that for $i = 0, 1$, $(c^{b,\vec{\nu}}_{\sigma^\frown i}, \dot{p}^{b,\vec{\nu}}_{\sigma^\frown i})$ are in $\dot{\mathbb{Q}}$, and decide contradictory values for $\dot{d} \cap \gamma^{b,\vec{\nu}}_\sigma$.
5. for $i = 0, 1$, $(c^{b,\vec{\nu}}_{\sigma^\frown i}, \dot{p}^{b,\vec{\nu}}_{\sigma^\frown i}) \Vdash \dot{d} \cap (\gamma^*, \gamma^{b,\vec{\nu}}_\sigma) \neq \emptyset$, where $\gamma^* = \sup_{\bar{\sigma}, \mathrm{ot}(\bar{\sigma}) < \mathrm{ot}(\sigma)} \gamma^{b\vec{\nu}}_{\bar{\sigma}}$.

We build as before, this time using Lemma 29 and the fact that the normal measures in the upper parts are sufficiently closed, so that we can index over every element in L.

Let $r \leq^* r_\eta$, for all $\eta < \tau$, and let $\beta = \sup_{\sigma, b, \vec{\nu}} \gamma^{b,\vec{\nu}}_\sigma$. For all $b, \vec{\nu}, f \in 2^\tau$, let $c^{b,\vec{\nu}}_f \leq c^{b,\vec{\nu}}_{f \restriction \eta}$ for all $\eta < \tau$. Also let $\dot{p}^{b,\vec{\nu}}_f$ be a name for a condition in $\mathbb{M}$ with the same stem and lower part as $\bar{r}^\frown b, \vec{\nu}$, such that it is forced to be a lower bound of all $\dot{p}^{b,\vec{\nu}}_{f \restriction \eta}$ for all $\eta < \tau$. Then, $(c^{b,\vec{\nu}}_f, \dot{p}^{b,\vec{\nu}}_f) \Vdash \beta \in \lim(\dot{d})$.

[1] I.e. a dense subset of $\{q \restriction \xi \mid q \leq^* r\} \subset \mathbb{M}_{\xi, g^r(\xi)}$

As before, we do not necessarily have that $r^\frown b, \vec{\nu}$ forces that $(c_f^{b,\vec{\nu}}, \dot{p}_f^{b,\vec{\nu}})$ is in the quotient. However, applying Proposition 25 inductively 2^τ-many times, there is a direct extension $r^{b,\vec{\nu}} \leq^* r^\frown b, \vec{\nu}$, such that $r^{b,\vec{\nu}}$ and $r^\frown b, \vec{\nu}$ have the same lower part and for every $f \in 2^\tau$, $r^{b,\vec{\nu}} \Vdash (c_f^{b,\vec{\nu}}, \dot{p}_f^{b,\vec{\nu}}) \in \dot{\mathbb{Q}}$.

Diagonalize $\langle r^{b,\vec{\nu}} \mid \vec{\nu} \in E \rangle$ to get $r' \leq^* r$, such that for all $b, \vec{\nu}$, $r'^\frown b, \vec{\nu} \leq r^{b,\vec{\nu}}$.

Let G_δ be $\mathbb{M}_\delta$-generic with $r' \in G_\delta$. Let $b \in L$ and $\vec{\nu} \in E$ be such that $r'^\frown b, \vec{\nu} \in G_\delta$. Work in $V[H_\delta][G_\delta]$. Then, by choice of r', $r^{b,\vec{\nu}} \in G_\delta$ and so for each $f \in 2^\tau$, $(c_f^{b,\vec{\nu}}, \dot{p}_f^{b,\vec{\nu}}) \in \mathbb{Q}$.

For each $f \in 2^\tau$ let $(c_f, \dot{p}_f) \leq_{\mathbb{Q}} (c_f^{b,\vec{\nu}}, \dot{p}_f^{b,\vec{\nu}})$, be such that for some e_f, $(c_f, \dot{p}_f) \Vdash_{\mathbb{Q}} \dot{d} \cap \beta = e_f$. As before, by the splitting construction, $f \neq g$ implies that $e_f \neq e_g$. Contradiction with $|\mathcal{C}_\beta| \leq \tau$. $\square$

Question 1. *Does the Magidor forcing at κ add a $\square_{\kappa,<\kappa}$ sequence. Alternatively, is there a version of Theorem 1 in which $\square_{\kappa,<\kappa}$ fails?*

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Universität Wien, Kurt Gödel Research Center for Mathematical Logic, Austria
E-mail address: maxwell.levine@univie.ac.at

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago IL 60613, USA
E-mail address: sinapova@uic.edu