Descriptive set theory and the density point property

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The measure algebra.

(X, d, μ) a **Polish measure space**, i.e. (X, d) is an uncountable Polish space, and μ a σ-finite Borel measure on X which is **nonsingular** (i.e. $\mu(\{x\}) = 0$ for all $x \in X$) and non-null (i.e. $\mu(X) \neq 0$).
$\textsc{Meas}$ the μ-measurable sets, $\textsc{Null}$ the ideal of measure 0 sets. Sets that are neither null nor co-null are called **nontrivial.**

$$\textsc{Meas}/\textsc{Null} \cong \textsc{Bor}/\textsc{Null} = \textsc{Malg}$$

$\textsc{Malg}$ is unique up to isomorphism, that is it does not depend on μ or X. $\textsc{Malg}$ is a Polish space with distance

$$\delta([A],[B]) = \mu\,(A \triangle B)$$

$A \subseteq_\mu B \Leftrightarrow A \setminus B \in \textsc{Null}$, and $A =_\mu B \Leftrightarrow A \subseteq_\mu B \subseteq_\mu A$.

The Lebesgue density theorem

(X,d) a metric space, μ a **Radon measure**, i.e. Borel measure such that $0 < \mu(\mathrm{B}(x;\varepsilon)) < \infty$ for all $x \in X$ and all sufficiently small ε.

$x \in X$ has **density** $r \in [0;1]$ **in** $A \in \textsc{Meas}$ if

$$\mathscr{D}_A(x) \stackrel{\text{def}}{=} \lim_{\varepsilon \downarrow 0} \frac{\mu\left(A \cap \mathrm{B}(x;\varepsilon)\right)}{\mu\left(\mathrm{B}(x;\varepsilon)\right)} = r.$$

Definition

(X,d,μ) has the **Density Point Property** (DPP) iff $\forall A \in \textsc{Meas}\, \forall x \in X\ (\mathscr{D}_A(x) = \chi_A(x)$ almost everywhere)

equivalently iff

$\Phi(A) \stackrel{\text{def}}{=} \{x \in X \mid x$ has density 1 in $A\}$ is in $\textsc{Meas}$ and $\mu(A \triangle \Phi(A)) = 0$.

χ_A is the characteristic function of A.

The Lebesgue Density Theorem

Theorem (Lebesgue)

If $X = [0;1]$ or $X = \mathbb{R}$, d is the usual distance and μ is the Lebesgue measure, then (X, d, μ) has the DPP.

...this is what we teach in first year calculus: the Fundamental Theorem of Calculus says that if f is continuous then

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(t)\mathrm{d}t \to f(x)$$

The Lebesgue Density Theorem generalizes this to the characteristic function $f = \chi_A$ of a measurable set A.

Examples of DPP spaces

The Lebesgue Density Theorem remains true when the space X is

- the Euclidean space $\mathbb{R}^n$ with the ℓ_p distance, and any Radon measure,
- the Cantor space ${}^{\omega}2$ with the coin-tossing measure,

$$\mu_{\mathrm{C}}(\{x \in {}^{\omega}2 \mid s \subseteq x\}) = 2^{-\operatorname{lh} s}$$

  and the usual distance

$$d_{\mathrm{C}}(x, y) = 2^{-n} \text{ if } n \text{ is least such that } x(n) \neq y(n),$$

- any Polish **ultrametric** space [Mil08].

The metric plays a central role in these questions!

Properties of Φ for a general (X, d, μ)

1. $A \subseteq_\mu B \Rightarrow \Phi(A) \subseteq \Phi(B)$, hence $A =_\mu B \Rightarrow \Phi(A) = \Phi(B)$. The map $\mathrm{MALG}_\mu \to \mathscr{P}(X)$, $[A] \mapsto \Phi(A)$, is well-defined.
2. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$, $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$.
3. $\Phi(\complement A) \subseteq \complement\Phi(A)$ and $\Phi(A \cup B) \supseteq \Phi(A) \cup \Phi(B)$. More generally,
4. $\Phi(\bigcup_{i \in I} A_i) \supseteq \bigcup_{i \in I} \Phi(A_i)$, provided $\bigcup_{i \in I} A_i \in \mathrm{MEAS}_\mu$.
5. $\Phi(U) \supseteq U$, for U open, and $\Phi(C) \subseteq C$, for C closed.
6. $\Phi(C_1 \cup C_2) = \Phi(C_1) \cup \Phi(C_2)$, if C_1, C_2 are disjoint closed sets.
7. If X is separable, then $\Phi(A)$ is Borel.

$\Phi \colon \mathrm{MEAS} \to \mathrm{MEAS}$ is not a homomorphism.

Properties of Φ on $^{\omega}2$ [AC13]

1. $\operatorname{ran}\Phi \subseteq \mathbf{\Pi}^0_3$, (also true when the ambient space is $\mathbb{R}^n$ with the Lebesgue measure and usual distance),
2. $\Phi\colon \mathrm{Bor} \to \mathbf{\Pi}^0_3$ is Borel-in-the-codes,
3. $\Phi(U)$ can attain *any* complexity in the Wadge hierarchy below $\mathbf{\Pi}^0_3$, for U an open set or a closed set,
4. for all $B \in \mathbf{\Pi}^0_3$ with $\emptyset \neq B \neq {}^{\omega}2$, the set $\{[A] \mid \Phi(A) \equiv_{\mathrm{W}} B\}$ is dense in the Polish space MALG,
5. $\{[A] \mid \Phi(A) \text{ is complete } \mathbf{\Pi}^0_3\}$ is comeager in the Polish space MALG,
6. $\{[A] \mid \Phi(A) \text{ is complete } \mathbf{\Pi}^0_3\}$ is dense in the complete Boolean algebra MALG.

Key result:

Proposition

If $K \subseteq {}^{\omega}2$ is compact, non-null, with empty interior, then $\Phi(K)$ is $\mathbf{\Pi}^0_3$-complete.

G. Carotenuto has shown that the Proposition is still true when the Cantor measure μ_C is replaced by a μ such that there is $0 < p \leq 1/2$ such that $p \leq \mu(\boldsymbol{N}_{s^\frown\langle i\rangle})/\mu(\boldsymbol{N}_s) \leq 1 - p$, with $i = 0, 1$. In particular it holds for all Bernoulli measures.

Question

Is this result true for all nonsingular probability measures on ${}^{\omega}2$?

What about $\mathbb{R}$?

A Cantor set $K \subseteq \mathbb{R}$ is of the form $[a;b] \setminus \bigcup_{s \in {}^{<\omega}s} I_s$ with I_s open interval. It is **uniform** if at stage n the intervals I_s with $\operatorname{lh} s = n$ have all the same length; it is **centered** if each I_s is centered in the interval from which it is removed.

Theorem (G. Carotenuto)

If $K \subseteq \mathbb{R}$ is centered and uniform Cantor set of positive measure, then $\Phi(K)$ is $\mathbf{\Pi}^0_3$-complete.

If the result can be extended to *all* Cantor sets of positive measure, then $\{[A] \mid \Phi(A) \text{ is complete } \mathbf{\Pi}^0_3\}$ is comeager in the Polish space $\textsc{Malg}$.

Failure of DPP

Theorem (Käenmäki-Rajala-Suomala [KRS])

There is a complete metric δ on ${}^{\omega}2$ compatible with the standard topology, a Borel finite measure ν, and a compact C of positive measure such that $\Phi(C) = \emptyset$.

Thus the density point property fails in $({}^{\omega}2, \delta, \nu)$. By [Mil08] δ cannot be an ultrametric. In fact the DPP can fail in *any* Polish space (if you perturb the metric).

Lemma

If ν is a Borel measure on ${}^{\omega}2$ and μ is a Radon measure on X Polish such that $\mu(X) > \nu({}^{\omega}2)$, then there is a continuous, injective, measure preserving map $F\colon {}^{\omega}2 \to X$.

When does the DPP hold?

Let $({}^{\omega}2, \delta, \nu)$ be the Finnish counterexample, let X be Polish and μ a Borel measure on X, with $\mu(X) > \nu({}^{\omega}2)$. Let $K \subseteq X$ be compact and $F \colon {}^{\omega}2 \to K$ be measure preserving homeomorphism. Copy the distance δ onto K. Thus $(K, \delta, \mu \restriction K)$ is not DPP and there is a closed set $C \subseteq K$ such that $\mu(C) > 0$ and and $\mathscr{D}_C^{K,\delta}(x) \neq 1$ for all $x \in K$. By a theorem of Hausdorff from 1932 [BP75], δ can be extended to a complete metric $\overline{\delta}$ on X compatible with the topology. Since $\mu(\mathrm{B}^{X,\overline{\delta}}(x;r)) \geq \mu(\mathrm{B}^{K,\delta}(x;r))$, then $\mathscr{D}_C^{X,\overline{\delta}}(x) \neq 1$ for all $x \in X$.

Theorem

For any Polish measure space (X, d, μ) there is a compatible metric δ such that (X, δ, μ) does not have the DPP.

Proof of the Lemma

Since every Polish space is the continuous injective image of a closed subset on the Baire space ${}^{\omega}\omega$, we may assume that μ, ν are measures on $X = {}^{\omega}\omega$ and ${}^{\omega}2$ respectively, and that $\mu({}^{\omega}\omega) > \nu({}^{\omega}2)$. Write $\nu(s)$ for $\nu(\boldsymbol{N}_s)$ with $s \in {}^{<\omega}2$, and similarly for μ. Choose $\varepsilon > 0$ such that $\mu({}^{\omega}\omega) > \nu({}^{\omega}2) + \varepsilon$.
We want a continuous $F \colon {}^{\omega}2 \to {}^{\omega}\omega$ that is measure preserving.
Pick incompatible $t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+m} \in {}^{<\omega}\omega$ such that

$$\sum_{i=1}^{n} \mu(t_i) > \nu(0) + \varepsilon/2, \qquad \sum_{i=n+1}^{n+m} \mu(t_i) > \nu(1) + \varepsilon/2$$

and declare that $F(\boldsymbol{N}_0) \subseteq \bigcup_{i=1}^{n} \boldsymbol{N}_{t_i}$ and $F(\boldsymbol{N}_1) \subseteq \bigcup_{i=1+n}^{n+m} \boldsymbol{N}_{t_i}$. This construction preserves the measure, but does not guarantee that the diameters shrink to 0.

(continued)

Proof of the Lemma

Choose a maximal finite antichain $\mathcal{A}$ below $\langle 0 \rangle$ so that $\mathcal{A} = \bigsqcup_{i=1}^{n} \mathcal{A}_i$ and

$$\sum_{s \in \mathcal{A}_i} \nu(s) < \mu(t_i) \quad (i = 1, \ldots, n).$$

Similarly choose $\mathcal{A}'$ a maximal finite antichain below $\langle 1 \rangle$ so that $\mathcal{A}' = \bigsqcup_{i=n+1}^{n+m} \mathcal{A}_i$ and $\sum_{s \in \mathcal{A}_i} \nu(s) < \mu(t_i)$ for $i = n+1, \ldots, n+m$. Declare that

$$F(\mathbf{N}_s) \subseteq \mathbf{N}_{t_i} \quad \text{for } s \in \mathcal{A}_i, i = 1, \ldots, n+m.$$

This guarantees that the diameters get smaller. We can now repeat the argument as before.

Complexity of DPP

$(\mathbb{U}, d_{\mathbb{U}})$ Urysohn metric space,
$\mathbf{P}(\mathbb{U})$ Polish space of all probability Borel measures on $\mathbb{U}$.
Let (X, d, μ) be a Polish measure space with $\mu(X) = 1$. There is an isometric copy of (X, d) in $(\mathbb{U}, d_{\mathbb{U}})$, so we can extend μ to measure on $\mathbb{U}$.
Conversely, given any $\mu \in \mathbf{P}(\mathbb{U})$, consider

$$\operatorname{supp}(\mu) = \bigcap \{C \subseteq X \mid C \text{ closed and } \mu(C) = 1\}.$$

Then $(\operatorname{supp}(\mu), d_{\mathbb{U}}, \mu)$ is a Polish measure space.
Therefore $\mathbf{P}(\mathbb{U})$ can be see as the Polish space of all Polish measure spaces.

Question

What is the complexity of the collection of DPP spaces?

Solid and dualistic sets

Work in a DPP space (X, d, μ).

Definition

The **oscillation** of x in A is

$$\mathscr{O}_A(x) = \limsup_{\varepsilon \downarrow 0} \frac{\mu(A \cap \mathrm{B}(x; \varepsilon))}{\mu(\mathrm{B}(x; \varepsilon))} - \liminf_{\varepsilon \downarrow 0} \frac{\mu(A \cap \mathrm{B}(x; \varepsilon))}{\mu(\mathrm{B}(x; \varepsilon))}$$

Thus $\mathscr{D}_A(x)$ exists iff $\mathscr{O}_A(x) = 0$.
The set A is

- **solid** iff $\mathscr{O}_A(x) = 0$ for all $x \in X$,
- **quasi-dualistic** iff $\forall x\,(\mathscr{O}_A(x) = 0 \Rightarrow \mathscr{D}_A(x) \in \{0, 1\})$, i.e. $\operatorname{ran} \mathscr{D}_A \subseteq \{0, 1\}$,
- **dualistic** iff it is quasi-dualistic and solid iff $\forall x \in X\,(\mathscr{D}_A(x) \in \{0, 1\})$.

Exceptional points in $\mathbb{R}$

Work in $\mathbb{R}$ with the usual metric and the Lebesgue measure λ.

Definition

Let $0 \leq \delta \leq 1/2$. A point $x \in \mathbb{R}$ is **δ-exceptional for A** iff

$$\delta \leq \liminf_{\varepsilon \downarrow 0} \frac{\lambda(A \cap (x-\varepsilon; x+\varepsilon))}{2\varepsilon} \leq \limsup_{\varepsilon \downarrow 0} \frac{\lambda(A \cap (x-\varepsilon; x+\varepsilon))}{2\varepsilon} \leq 1-\delta.$$

In other words x is δ-exceptional for A if

- either $\mathscr{D}_A(x)$ exists and belongs $[\delta; 1-\delta]$,
- or else $0 < \mathscr{O}_A(x) \leq 1 - 2\delta$.

If A is an interval, the the endpoints are $1/2$-exceptional, while the others are 0-exceptional.

A surprising result

Let $\mathcal{H}(\delta)$ be the statement:

$$\forall A \text{ nontrivial } \exists x\, (x \text{ is } \delta\text{-exceptional for } A).$$

If $\delta_1 > \delta_2$ then $\mathcal{H}(\delta_1) \Rightarrow \mathcal{H}(\delta_2)$, hence define

$$\delta_{\mathcal{H}} = \sup \{\delta \mid \mathcal{H}(\delta) \text{ holds}\}.$$

It seems reasonable to conjecture that if this constant is nonzero, then it should be 0.5.
V. Kolyada [Kol83] showed that $1/4 \leq \delta_{\mathcal{H}} \leq (\sqrt{17}-3)/4$. These bounds were successively improved in [Sze11, CGO12], and in [Kur12] O. Kurka obtained the exact value:

$$\delta_{\mathcal{H}} = \text{ the unique real solution to } 8x^3+8x^2+x-1 \approx 0.268486\ldots.$$

A surprising result

Corollary

There are nontrivial sets $A \subset \mathbb{R}$ such that $\operatorname{ran}(\mathscr{D}_A) \cap (\delta_{\mathcal{H}}; 1 - \delta_{\mathcal{H}}) = \emptyset$. In other words, for any real x either $\mathscr{O}_A(x) > 1 - 2\delta_{\mathcal{H}}$ or else $\mathscr{D}_A(x) \in [0; \delta_{\mathcal{H}}] \cup [1 - \delta_{\mathcal{H}}; 1]$.
In particular:

- *there is a set A that does not have points of density $1/2$*
- *there are no nontrivial dualistic sets.*

Question

Is there a solid $A \subseteq \mathbb{R}$ such that $\mathscr{D}_A(x) \in [0; \delta_{\mathcal{H}}] \cup [1 - \delta_{\mathcal{H}}; 1]$?

The Kolyada-Kurka theorem does not seem to generalize to $\mathbb{R}^n$.

Intermezzo: the density topology

Given (X, d, μ) a Polish metric space, let

$$\mathfrak{T} = \{A \subseteq X \mid A \subseteq \Phi(A)\}.$$

If (X, d, μ) is DPP then $\mathfrak{T}$ is the **density topology** on X. But there are spaces which are not DPP, yet $\mathfrak{T}$ is a topology.

- $\mathfrak{T}$ is finer than the topology induced by d,
- $A = \Phi(A)$ iff A is regular open in $\mathfrak{T}$,
- A is $\mathfrak{T}$-clopen iff A is dualistic,
- $(\mathbb{R}, \mathfrak{T})$ is connected,
- there is a Borel measure on $\mathbb{R}^2$ such that $(\mathbb{R}^2, \mathfrak{T})$ is not connected.

Definition

Given (X, d, μ) we say that μ is **continuous** if the map

$$X \times \mathbb{R}_+ \to \mathbb{R}_+, \qquad (x, r) \mapsto \mu(\mathrm{B}(x; r))$$

is continuous.

The Lebesgue measure on $\mathbb{R}^n$ and on 2^ω is continuous.

Proposition

Suppose μ is continuous and A is solid. Then $\mathscr{D}_A : X \to \mathbb{R}_+$ is Baire class 1, and $\Phi(A) \in \boldsymbol{\Pi}^0_2$.
Moreover if 1 is an isolated value of $\mathscr{D}_A$, that is to say $\operatorname{ran} \mathscr{D}_A \subseteq [0; r] \cup \{1\}$ for some $r < 1$, then $\Phi(A) \in \boldsymbol{\Delta}^0_2$.

The influence of the metric

Two equivalent metrics d_1 and d_2 on X can give rise to very different density notions.

1. both metrics give rise to DPP-spaces, and $\mathscr{D}^{d_1} = \mathscr{D}^{d_2}$, $\mathscr{O}^{d_1} = \mathscr{O}^{d_2}$, and $\Phi^{d_1} = \Phi^{d_2}$,
2. both metrics give rise to DPP-spaces, but $\mathscr{D}_A^{d_1} \neq \mathscr{D}_A^{d_2}$ for some A, although $\Phi^{d_1}(A) = \Phi^{d_2}(A)$,
3. both metrics give rise to DPP-spaces, $\Phi^{d_1}(A) \neq \Phi^{d_2}(A)$ for some A (hence $\mathscr{D}_A^{d_1} \neq \mathscr{D}_A^{d_2}$), yet the density topologies are the homeomorphic,
4. both metrics give rise to DPP-spaces, but the density topologies are not homeomorphic,
5. one of the two metrics gives rise to DPP-space, the other does not.

Theorem

For every $S \subseteq [0;1]$ in $\mathbf{\Sigma}^1_1$ then there is a set $A \subseteq {}^\omega 2$ such that $\operatorname{ran}(\mathscr{D}_A) = S \cup \{0,1\}$.
Moreover A can be taken to be open or closed.
Furthermore A can be taken to be solid.

Corollary

1. $\{K \in \mathbf{K}({}^\omega 2) \mid \operatorname{ran}(\mathscr{D}_K) = \{0,1\}\}$ *and* $\{K \in \mathbf{K}({}^\omega 2) \mid K$ *solid and* $\operatorname{ran}(\mathscr{D}_K) = \{0,1\}\}$ *are* $\mathbf{\Pi}^1_1$*-complete.*
2. $\{K \in \mathbf{K}({}^\omega 2) \mid \operatorname{ran}(\mathscr{D}_K) = [0;1]\}$ *and* $\{K \in \mathbf{K}({}^\omega 2) \mid K$ *solid and* $\operatorname{ran}(\mathscr{D}_K) = [0;1]\}$ *are* $\mathbf{\Pi}^1_2$*-complete.*

Proposition

There is a set $A \subseteq [0;1]$ which is quasi-dualistic, i.e.
$\forall x \in \mathbb{R}\,(\mathscr{O}_A(x) = 0 \Rightarrow \mathscr{D}_A(x) \in \{0,1\})$.
Moreover A can be taken to be open or closed.

Such set is highly non-solid...

Theorem

Work in $\mathbb{R}^n$ with the ℓ_p metric $(1 \leq p \leq \infty)$ and the Lebesgue measure.
If $A \subseteq \mathbb{R}^n$ is nontrivial and solid, then $\mathscr{D}_A(\mathbf{x}) = \frac{1}{2}$ for some $\mathbf{x} \in \mathbb{R}^n$.
In particular, there are no nontrivial dualistic sets.

Theorem

For every $n \geq 1$ there is a solid set $A \subseteq \mathbb{R}^n$ such that
$\operatorname{ran}(\mathscr{D}_A) = [0;1]$.
Moreover A can be taken to be open or closed.

What is $\operatorname{ran} \mathscr{D}_A$ when $A \subseteq \mathbb{R}$ is solid?

Theorem

Let $S \subseteq [0;1]$ be $\mathbf{F}_\sigma$. Then there is a solid set $A \subseteq \mathbb{R}$ such that $\operatorname{ran} \mathscr{D}_A = S \cup \{0, 1/2, 1\}$.
Moreover A can be taken to be open or closed.

Question

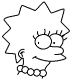
Is the result true if S is $\mathbf{G}_\delta$?

Theorem

For any $S \subseteq [0;1]$ which is $\mathbf{\Sigma}^1_1$ there is a (non-necessarily solid) set $A \subseteq \mathbb{R}$ such that $\operatorname{ran} \mathscr{D}_A = S \cup \{0, 1\}$.
As before, A can be taken to be open or closed.

Would you like to see a proof?

NO WAY MAN!



OK, just a sketch...[1]

For every $S \subseteq [0;1]$ in $\boldsymbol{\Sigma}^1_1$ then there is an open set $A \subseteq {}^\omega 2$ such that $\operatorname{ran}(\mathscr{D}_A) = S \cup \{0, 1\}$.

Let $F\colon {}^\omega 2 \twoheadrightarrow [0;1]$, $F(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}$, so that $F^{-1}[S]$ is the projection of a closed subset of ${}^\omega 2 \times {}^\omega \omega$. Since ${}^\omega \omega$ is homeomorphic to $\mathcal{N} = \{y \in {}^\omega 2 \mid \exists^\infty n\, y(n) = 1\}$, there is a pruned tree T on 2×2 such that

$$\forall x \in {}^\omega 2 \ (F(x) \in S \Leftrightarrow \exists y \in \mathcal{N}\, (x, y) \in [T])$$

[1] Homer and Lisa Simpsons appear by courtesy of Matt Groening and Raymond Chen's package `simpsons`

Sketch of a proof

Let $\oplus\colon {}^{\omega}2 \times {}^{\omega}2 \to {}^{\omega}2$ be some standard pairing function.
$C \stackrel{\text{def}}{=} \{z \in {}^{\omega}2 \mid \forall n\, z(3n+2) = 0\}$ is null and $z \mapsto \tilde{z}$ is the homeomorphism between ${}^{\omega}2$ and C.
We shall construct an open set $A \subseteq {}^{\omega}2 \setminus C$ such that

$$\begin{aligned} z \notin C &\Rightarrow \mathscr{D}_A(z) \in \{0,1\}\,, \\ x \in F^{-1}[S] &\Rightarrow \mathscr{D}_A(\widetilde{x \oplus y}) = F(x), \\ x \notin F^{-1}[S] &\Rightarrow \mathscr{O}_A(\widetilde{x \oplus y}) > 0. \end{aligned}$$

The idea is to add clopen sets U_s inside the neighborhoods of the form $N_{t^\frown 1}$ where $\operatorname{lh} t = 3k+1$ so that the requirements above are satisfied: if $z \in C$, then $z = \widetilde{x \oplus y}$ and

- either $(x,y) \in [T]$ and $\mathscr{D}_A(z) = F(x)$,
- or else $(x,y) \notin [T]$ and $\mathscr{D}_A(z)$ does not exist.

Thank-you for your attention!

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