Monochromatic paths and path squares in infinite graphs

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Freiburg, 2014

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Theorem (Erdős, Rado, (published in 1987))

Let $r \in \omega$. Suppose that the edges of the countable complete graph K_ω is coloured with r colors. Then there are r disjoint monochromatic paths with different colours which cover all vertices of K_ω.

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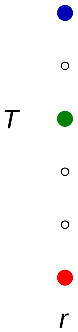

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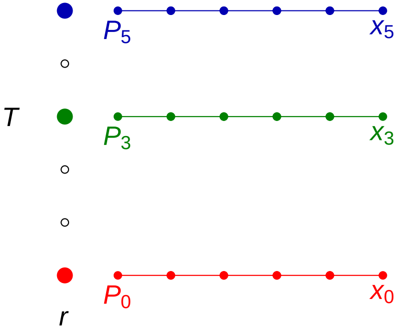

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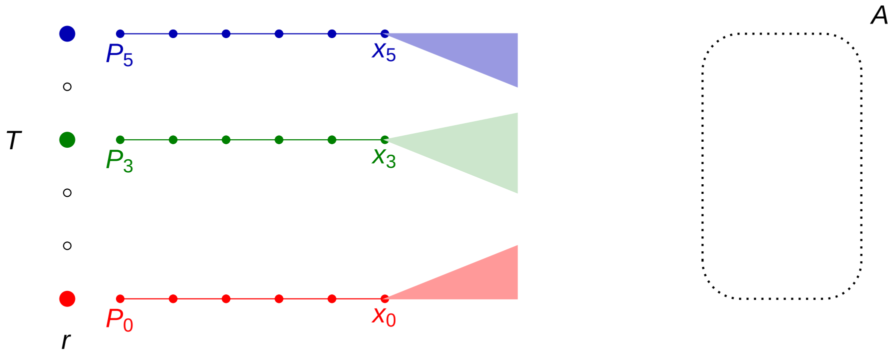

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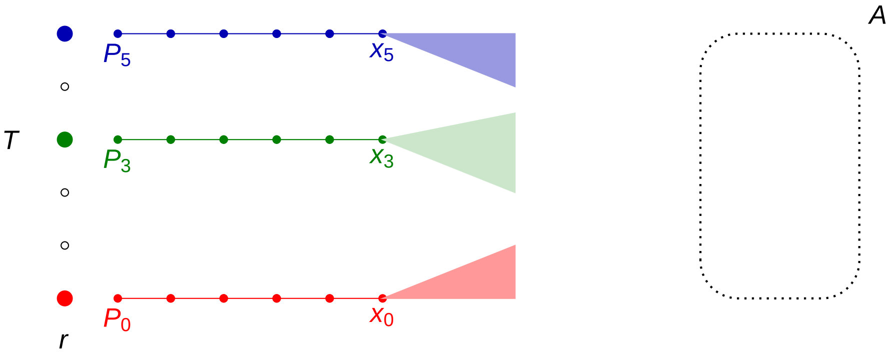

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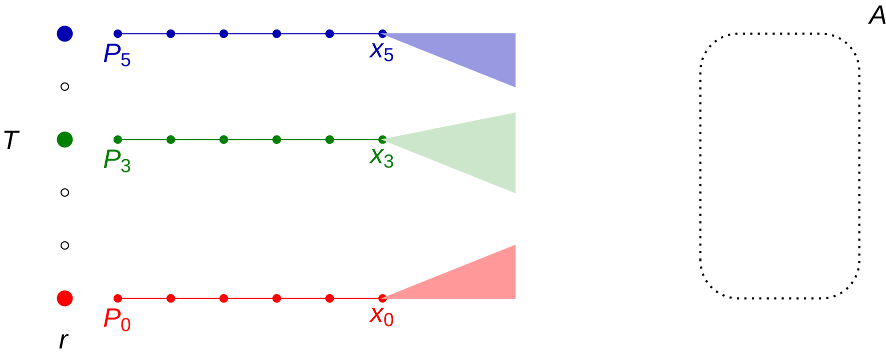

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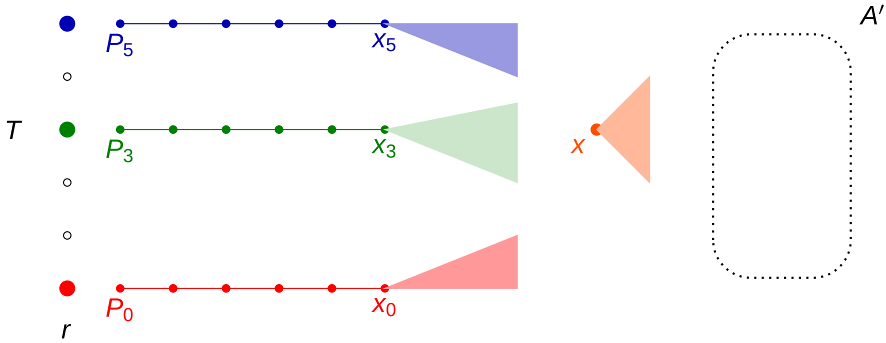

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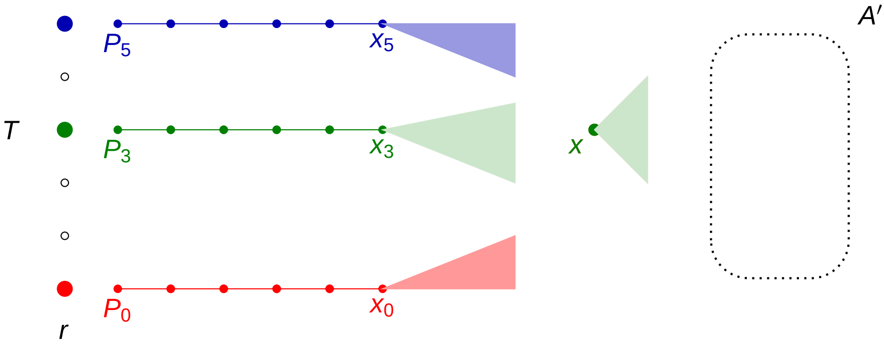

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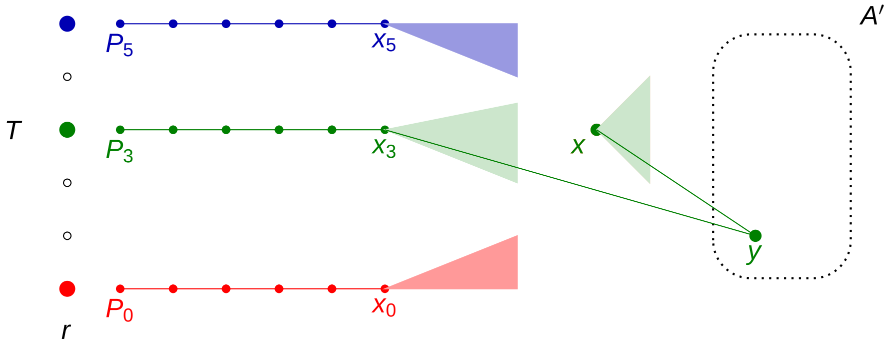

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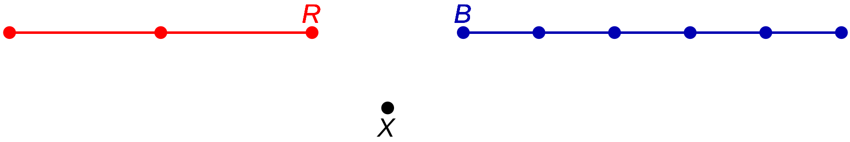

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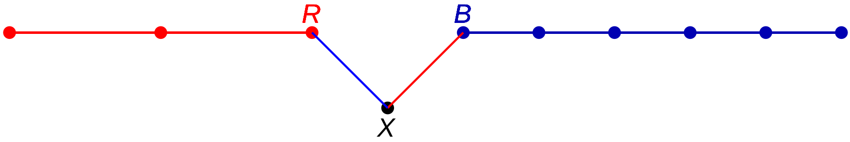

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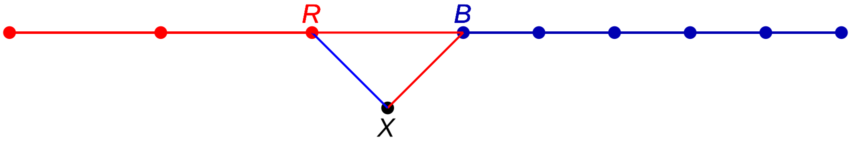

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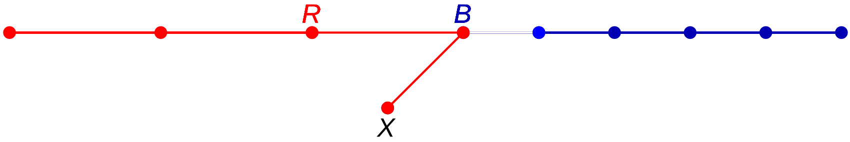

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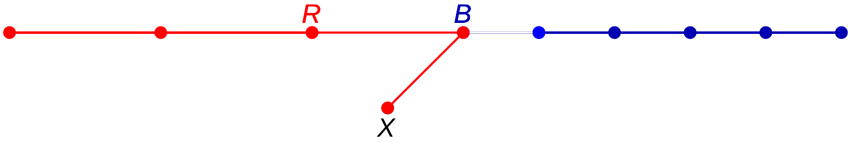

More colors? Cycles instead paths?

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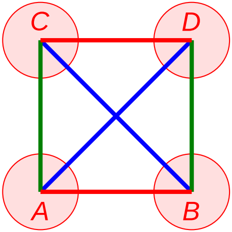

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- The "path" conjecture holds for $r = 3$, but the "cycle" conjecture fails for $r \geq 3$ (Pokrovskiy, 2012).

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Need: ultrafilter argument

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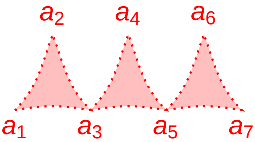

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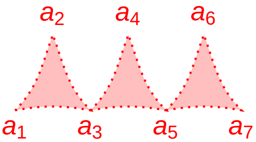

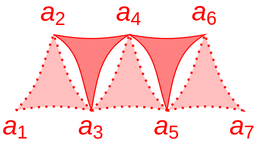

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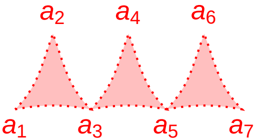

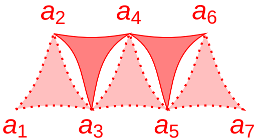

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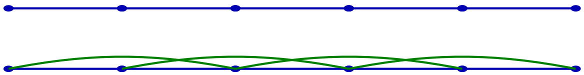

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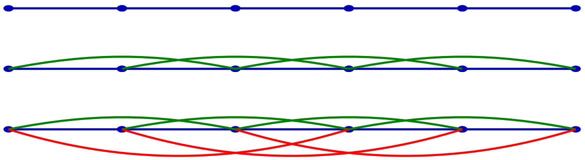

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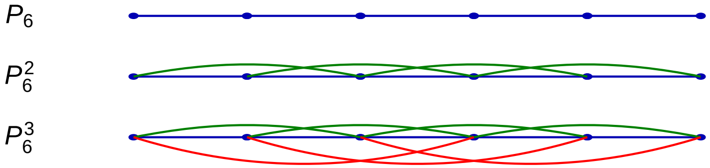

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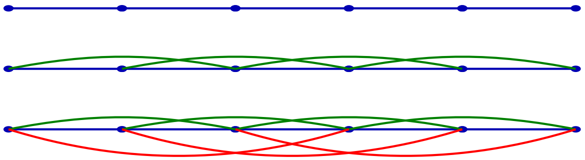

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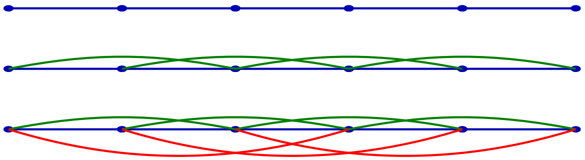

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For $k = r = 2$ we have a partition into ***5*** *monochromatic squares of paths.*

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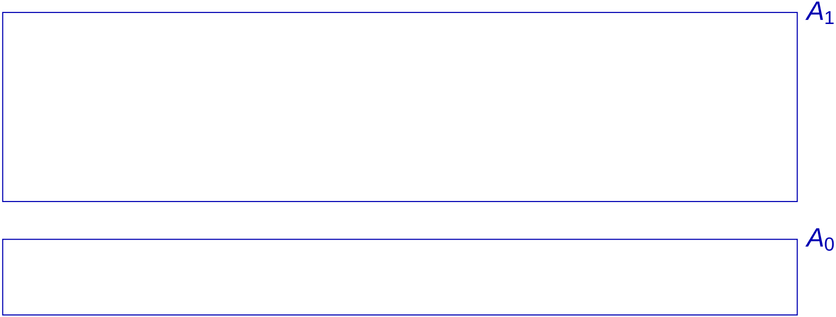

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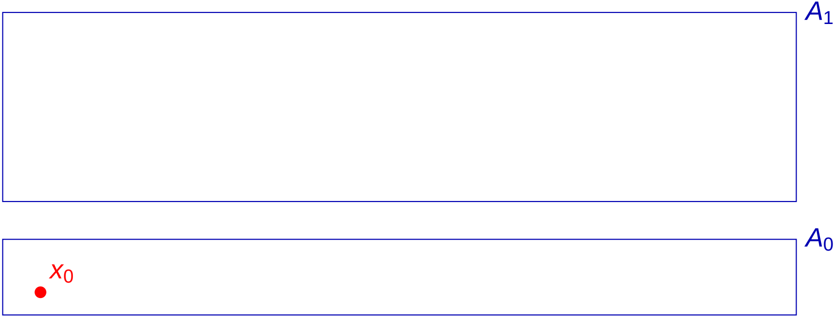

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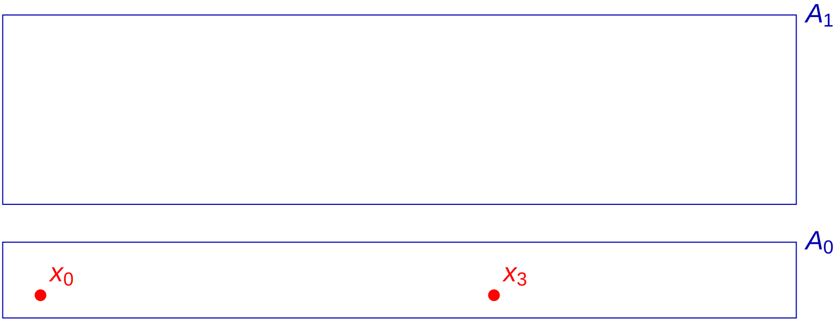

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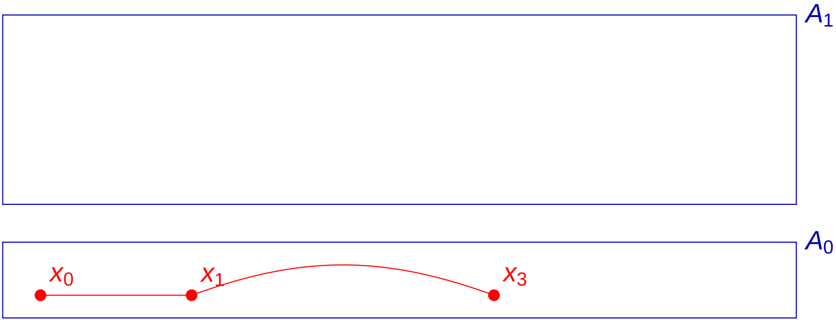

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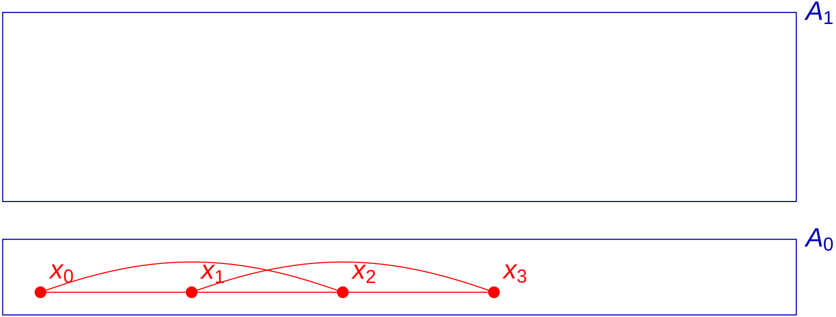

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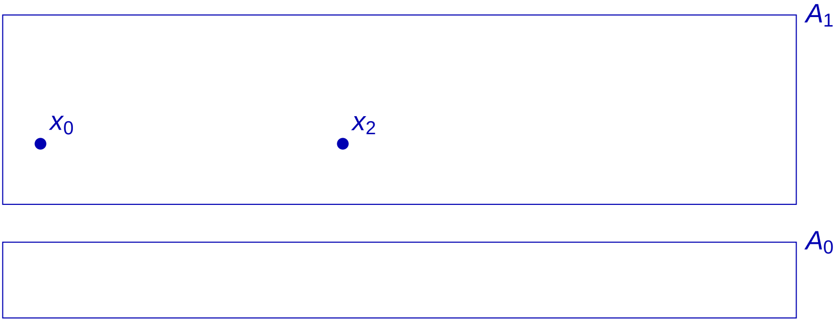

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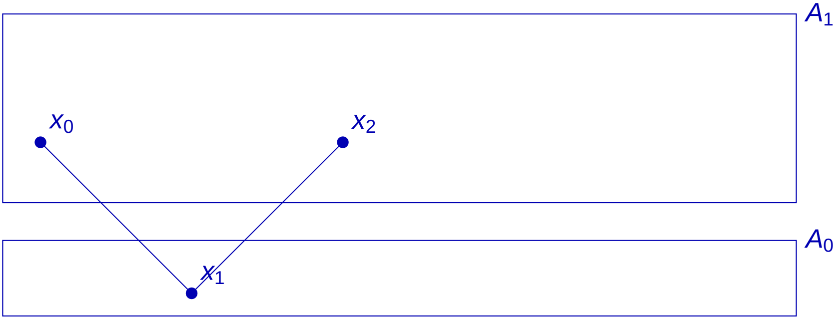

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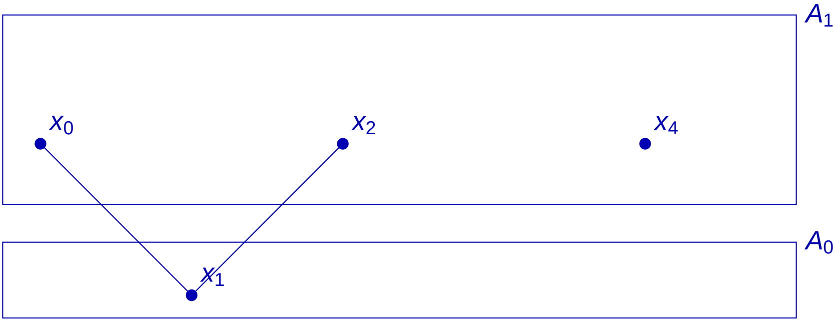

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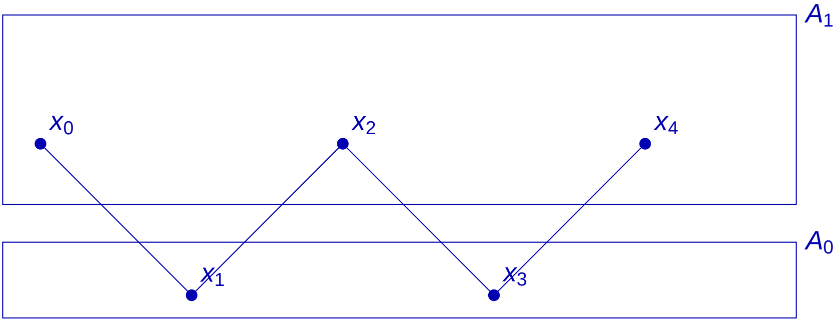

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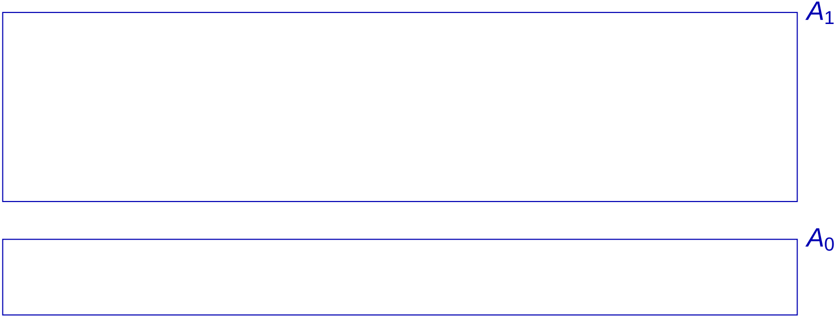

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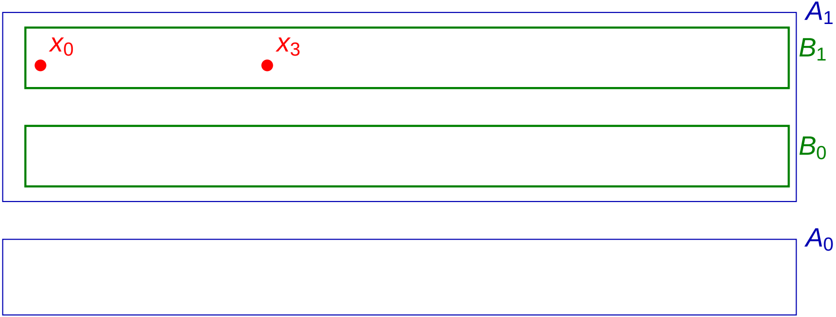

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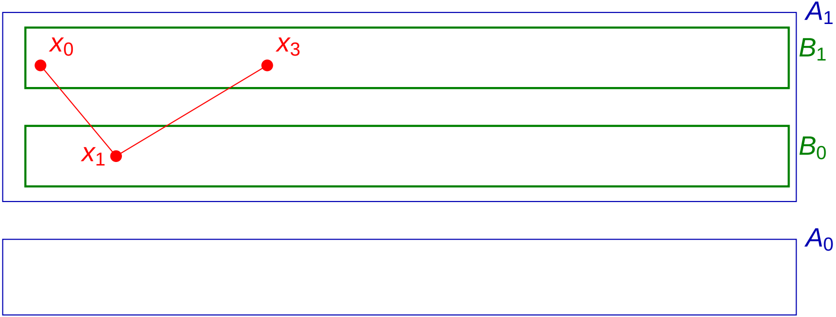

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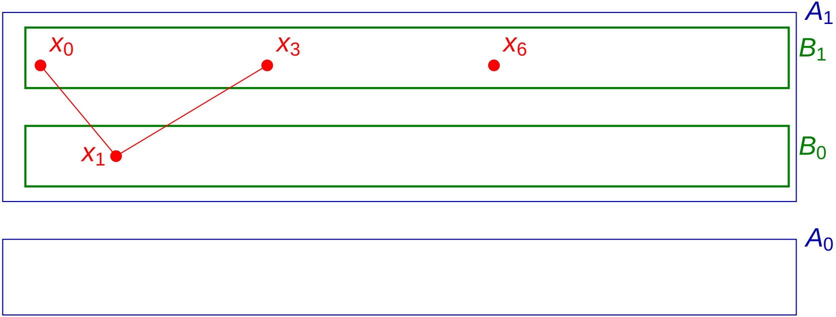

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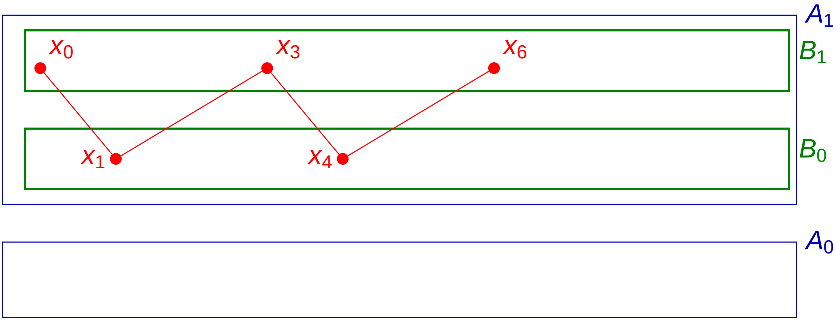

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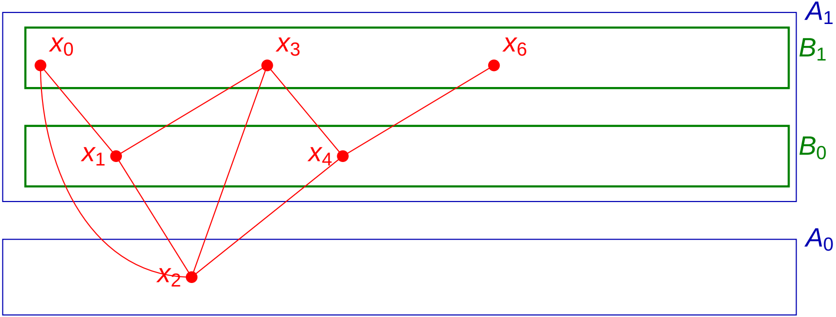

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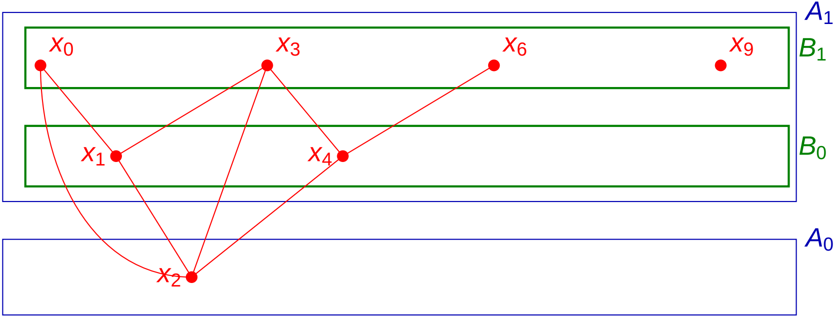

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Theorem (D. Soukup)

If G is an infinite complete graph and $r \in \omega$, then for every r-edge colouring of G we can partition the vertices into finitely many monochromatic paths.

$$K_\kappa \sqsubset (Path)_{r,<\omega}.$$

Problems

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Infinitely many colors????

Thank you!