Annals of Pure and Applied Logic 165 (2014) 573–608

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Many countable support iterations of proper forcings preserve Souslin trees [☆]

Heike Mildenberger [a,*], Saharon Shelah [b]

[a] *Einstein Institute of Mathematics, The Hebrew University, Edmond Safra Campus Givat Ram, Jerusalem 91904, Israel*
[b] *Abteilung für Mathematische Logik, Mathematisches Institut, Universität Freiburg, Eckerstr. 1, 79104 Freiburg im Breisgau, Germany*

A R T I C L E I N F O

Article history:
Received 16 November 2010
Accepted 20 August 2013
Available online 5 September 2013

MSC:
03E05
03E17
03E35
03E15

Keywords:
Games played on forcing orders
Creature forcing
Non-elementary proper forcing
Preservation theorems for trees on $\aleph_1$

A B S T R A C T

We show that many countable support iterations of proper forcings preserve Souslin trees. We establish sufficient conditions in terms of games and we draw connections to other preservation properties. We present a proof of preservation properties in countable support iterations in the so-called Case A that does not need a division into forcings that add reals and those who do not.

© 2013 Elsevier B.V. All rights reserved.

0. Introduction

This work is related to Juhász' question [14]: "Does Ostaszewski's club principle imply the existence of a Souslin tree?" We recall the club principle (also written ♣): There is a sequence $\langle A_\alpha\colon \alpha \text{ a limit ordinal } < \omega_1\rangle$ with the following properties: For every countable limit ordinal α, A_α is cofinal in α and for any uncountable $X \subseteq \omega_1$ there are stationarily many α with $A_\alpha \subseteq X$. Such a sequence is called a ♣-sequence. The club principle was introduced in [15].

Partial positive answers are known: Let $\mathcal{M}$ denote the ideal of meager sets. In every model of the club principle and $\mathrm{cov}(\mathcal{M}) > \aleph_1$ by Miyamoto [5, Section 4] there are Souslin trees. Brendle showed [5, Theorem 6]:

☆ The first author acknowledges Marie Curie grant PIEF-2008-219292 of the European Union. The second author's research was partially supported by the United States–Israel Binational Science Foundation (Grant no. 2002323). This is the second author's publication no. 973.

* Corresponding author.
E-mail addresses: heike.mildenberger@math.uni-freiburg.de (H. Mildenberger), shelah@math.huji.ac.il (S. Shelah).

0168-0072/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.apal.2013.08.002

In every model of the club principle and $\mathrm{cof}(\mathcal{M}) = \aleph_1$ there are Souslin trees. In this paper we give examples of models satisfying the club principle, the existence of Souslin trees, $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathrm{cof}(\mathcal{M}) = \aleph_2$ (i.e., neither of the sufficient conditions mentioned above holds).

Assume that we start with a ground model satisfying $\diamondsuit_{\omega_1}$ and that we force with a proper countable support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta\colon \alpha \leqslant \kappa,\ \beta < \omega_2\rangle$ of length ω_2. For this scenario in [12] we showed: If the single step forcings are suitable forcings from [16] (with finite or countable $\mathrm{H}(n)$, see Section 2.1), then the final model will satisfy the club principle. Note that the assumption of the diamond in the ground model is actually not necessary, since after ω_1 iteration steps of any forcing with two incompatible conditions with countable support $\diamondsuit_{\omega_1}$ holds anyway [11, Chapter 7, Theorem 8.3] and the length of our iterations is ω_2.

Let us look at the countable support iteration of length ω_2 of Miller forcing: According to the mentioned result, after ω_1 many steps we get $\diamondsuit_{\omega_1}$ and therefore a Souslin tree in the intermediate extension. Theorem 2.1 together with the results in Section 4 show that any countable support iteration of Miller forcing preserves Souslin trees. Hence after ω_2 many iteration steps there is a Souslin tree. Moreover by [12] the club principle holds. It is known that in the Miller model $\mathfrak{d} = \aleph_2$ (and hence $\mathrm{cof}(\mathcal{M}) = \aleph_2$) and $\mathrm{cov}(\mathcal{M}) = \aleph_1$. A countable support iteration of length ω_2 of Blass–Shelah forcing gives another model of $\mathfrak{d} = \aleph_2$ and $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and the club principle. Blass–Shelah forcing is not ω-Cohen preserving (see Definition 3.1) and increases the splitting number (see [3, Proposition 3.1]). Besides these two particular examples, the main technical work in this paper is a study of the preservation of Souslin trees.

We refer the reader to [2] for the definitions of cardinal characteristics, and to [12] for reading about the club principle. For background about properness we refer the reader to [22] and the more detailed introductions in [6,1]. In forcing notions, $q > p$ means that q is stronger than p. The paper is organised as follows:

In Section 1 we give some conditions on a forcing in terms of games that imply that the forcing is $(T, Y, \mathcal{S})$-preserving. A special case of $(T, Y, \mathcal{S})$-preserving is preserving the Souslinity of an ω_1-tree.

In Section 2 we show that for some tree-creature forcings from [16] the player COM has a winning strategy in one of the games from Section 1. Hence these forcings preserve Souslin trees. Without the games, we show that some linear creature forcings from [16] are $(T, Y, \mathcal{S})$-preserving. There are non-Cohen preserving examples.

For the wider class of non-elementary proper forcings we show in Section 3 that ω-Cohen preserving for certain candidates implies $(T, Y, \mathcal{S})$-preserving.

In Section 4 we give a less general but hopefully more easily readable presentation of a result from [22, Chapter 18, §3]: If all iterands in a countable support iteration are proper and $(T, Y, \mathcal{S})$-preserving, then also the iteration is $(T, Y, \mathcal{S})$-preserving. This is a presentation of the so-called Case A in which a division in forcings that add reals and those who do not is not needed.

1. A sufficient condition for $(T, Y, \mathcal{S})$-preserving

We introduce two games $\Game^\iota(\mathbb{P}, p)$, $\iota = 1, 2$, that are games about the completeness of the notion of forcing $\mathbb{P}$ above p. Similar games appear in [17,19,18]. We let $\mathbf{G}_\mathbb{P} = \{(\check{p}, p)\colon p \in \mathbb{P}\}$ be the standard name for a $\mathbb{P}$-generic filter. If it is clear which $\mathbb{P}$ is meant we write just $\underset{\sim}{\mathbf{G}}$.

Definition 1.1. Let $\mathbb{P}$ be a notion of forcing and $p \in \mathbb{P}$. We define the games $\Game^\iota(\mathbb{P}, p)$, $\iota = 1, 2$. The moves look the same for both games, and only in the winning conditions they are different.

(1) The game $\Game^1(\mathbb{P}, p)$ is played in ω rounds. In round n, player COM chooses an $\ell_n \in \omega \setminus \{0\}$ and a sequence $\langle p_{n,\ell}\colon \ell < \ell_n\rangle$ of conditions $p_{n,\ell} \in \mathbb{P}$ and then player INC plays $\langle q_{n,\ell}\colon n < \ell_n\rangle$ such that $p_{n,\ell} \leqslant q_{n,\ell}$. After ω rounds, COM wins the game iff there is $q \geqslant p$ such that for each n,

$$\{q_{n,\ell}\colon \ell < \ell_n\} \text{ is predense above } q.$$

(2) The game $\Game^2(\mathbb{P}, p)$ is played in ω rounds that look exactly like the rounds in $\Game^1(\mathbb{P}, p)$. After ω rounds, COM wins the game iff for every infinite $u \subseteq \omega$ there is $q_u \geqslant p$ such that

$$q_u \Vdash \big(\exists^\infty n \in u\big)(\exists \ell < \ell_n)(q_{n,\ell} \in \underset{\sim}{\mathbf{G}}).$$

Definition 1.2. For $\iota = 1, 2$, we say $\mathbb{P}$ has property Pr^ι and write $\mathrm{Pr}^\iota(\mathbb{P})$ iff for every $p \in \mathbb{P}$, in the game $\Game^\iota(\mathbb{P}, p)$ the player COM has a winning strategy.

We fix a sufficiently large regular cardinal χ. We write $H(\chi)$ for the set of sets of hereditary cardinality less than χ, and let $\mathcal{H}(\chi) = (H(\chi), \in, <^*_\chi)$ with a well-order $<^*_\chi$ on $H(\chi)$.

Definition 1.3. Let $\alpha(*)$ be an uncountable ordinal. Let $\mathcal{S} \subseteq [\alpha(*)]^\omega$ be stationary, let $\iota = 1, 2$, and let $\mathbb{P}$ be a forcing. Then $\mathrm{Pr}^\iota_{\mathcal{S}}(\mathbb{P})$ denotes the following property: For every sufficiently large χ and every countable $N \prec \mathcal{H}(\chi)$ with $\mathbb{P} \in N$, and $N \cap \alpha(*) \in \mathcal{S}$, for every $p \in \mathbb{P} \cap N$ player COM has a winning strategy in the game $\Game^\iota(N, \mathbb{P}, p)$. The game $\Game^\iota(N, \mathbb{P}, p)$ is defined like the $\Game^\iota(\mathbb{P}, p)$ except that we require that every initial segment of a play is in N.

$\mathrm{Pr}^\iota(\mathbb{P})$ implies $\mathrm{Pr}^\iota_{\mathcal{S}}(\mathbb{P})$ for any $\mathcal{S}$, and $\mathrm{Pr}^1(\mathbb{P})$ implies $\mathrm{Pr}^2(\mathbb{P})$.

Lemma 1.4. *In all the games any winning strategy for COM can be modified by playing at each stage a larger number ℓ_n and stronger conditions, that is, the resulting function is a winning strategy for COM as well.*

Definition 1.5. Let $\mathcal{S} \subseteq [\alpha(*)]^\omega$ be stationary. $\mathbb{P}$ is $\mathcal{S}$-proper if for any $N \prec \mathcal{H}(\chi)$ such that $N \cap \alpha(*) \in \mathcal{S}$, for any $p \in \mathbb{P} \cap N$ there is $q \geqslant p$ that is $(N, \mathbb{P})$-generic. q is $(N, \mathbb{P})$-generic means: For any $D \in N$, if D is dense in $\mathbb{P}$ then $q \Vdash \underset{\sim}{\mathbf{G}} \cap D \neq \emptyset$.

Definition 1.6.

(1) A forcing $\mathbb{P}$ is *${}^\omega\omega$-bounding* if for every $\mathbb{P}$-name $\underset{\sim}{f}$ for a function from ω to ω and for any p, there are $g \in {}^\omega\omega$ and $q \geqslant p$, $q \in \mathbb{P}$, such that $q \Vdash \forall n\, \underset{\sim}{f}(n) \leqslant g(n)$.

(2) A forcing $\mathbb{P}$ is *almost ${}^\omega\omega$-bounding* if for every name $\underset{\sim}{f}$ for a function from ω to ω, for any $A \subseteq \omega$ and any $p \in \mathbb{P}$ there are $q \geqslant p$ and $g \in {}^\omega\omega$ such that $q \Vdash (\exists^\infty n \in A)(\underset{\sim}{f}(n) \leqslant g(n))$.

Lemma 1.7.

(1) *If $\mathrm{Pr}^1_{\mathcal{S}}(\mathbb{P})$, then $\mathbb{P}$ is $\mathcal{S}$-proper, and $\mathbb{P}$ is ${}^\omega\omega$-bounding.*

(2) *If $\mathrm{Pr}^2_{\mathcal{S}}(\mathbb{P})$, then $\mathbb{P}$ is $\mathcal{S}$-proper, and $\mathbb{P}$ is almost ${}^\omega\omega$-bounding.*

Remark 1.8. The reverse implications do not hold: The NNR forcing from [22, Chapter IV] is a counterexample to both, as Theorem 1.17 will show.

Proof. We prove (2). Item (1) is proved similarly. Let $\underset{\sim}{f}$ be a $\mathbb{P}$-name for a function from ω to ω. Fix a winning strategy $\mathbf{st}$ for COM in $\Game^2(\mathbb{P}, p)$. Let $\mathbb{P}, \underset{\sim}{f}, \mathbf{st} \in N \prec \mathcal{H}(\chi)$, $N \cap \alpha(*) \in \mathcal{S}$, $p \in \mathbb{P} \cap N$. Let $\langle \underset{\sim}{\tau}_k\colon k < \omega\rangle$ be a list of the $\mathbb{P}$-names in N of ordinals. In round n, INC plays such that for every $\ell < \ell_n$, $q_{n,\ell}$ forces a value to $\underset{\sim}{f}(i)$ for $i \leqslant n$ and a value to $\underset{\sim}{\tau}_i$ for $i < n$. Let $g(n)$ be the maximum of the values forced to $\underset{\sim}{f}(n)$ by $q_{n,\ell}$, $\ell < \ell_n$. Fix an infinite $u \subseteq \omega$ and let q_u witness that COM wins. Then q_u is N-generic: Let τ_k be a $\mathbb{P}$-name in N for an ordinal. Then q_u forces that here are infinitely many $k' > k$, $k' \in u$ and $\ell \in \omega$ that $q_{k',\ell} \in G$. This $q_{k',\ell}$ decides $(\tau_m)_{m<k'}$ in N and forces $f(k')$ to be some value less than $g(k')$. □

Remark 1.9. Sacks forcing satisfies Pr^1. COM can fix $\ell_n = 2^n \cdot \ell_{n-1}$ and play the restrictions of $q_{n-1,i}$, $i < \ell_{n-1}$, to the members of its n-th splitting front as $p_{n,i}$, $i < \ell_n$.

The following versions of the games that work for all starting points in a countable model simultaneously are interesting for themselves. However, 1.10 and 1.11 will not be used in the sequel so that a reader who is mainly interested in preserving Souslin trees can skip them.

Definition 1.10. Let $N \prec \mathcal{H}(\chi)$. We define a game $\eth^\iota(N,\mathbb{P})$: The moves are as in $\eth^\iota(N,\mathbb{P},p)$. The winning conditions read for $\iota = 1$: For every $p \in \mathbb{P} \cap N$ there is a $q \geqslant p$ such that for all but finitely many n, $\{q_{n,\ell}\colon\ \ell < \ell_n\}$ is predense above q. For $\iota = 2$: For every $p \in \mathbb{P} \cap N$ and infinite u there is a $q_u \geqslant p$ as in $\eth^2(\mathbb{P},p)$.

Lemma 1.11. *If $\mathrm{Pr}^\iota_{\mathcal{S}}(\mathbb{P})$ and $N \cap \alpha(*) \in \mathcal{S}$, then COM has a winning strategy in $\eth^\iota(N,\mathbb{P})$.*

Proof. Let $N \cap \mathbb{P} = \{p_j\colon\ j < \omega\}$. Let $\mathbf{st}_j$ be a strategy for COM in $\eth^\iota(N,\mathbb{P},p_j)$.

Let in the n-th move strategy $\mathbf{st}_j$ tell COM to choose $\overline{p_{j,n}} = \langle p_{j,n,\ell}\colon\ \ell < \ell_{j,n}\rangle$. Then COM moves in $\eth^\iota(N,\mathbb{P})$ by letting $\ell_n = \sum_{j\leqslant n} \ell_{j,n-j}$ and $\overline{p_n} = \overline{p_{0,n}}{}^\frown \dots {}^\frown \overline{p_{j,n-j}}{}^\frown \dots {}^\frown \overline{p_{n,n-n}}$. □

Now we describe Souslin trees.

Definition 1.12.

(1) An ω_1-tree is a tree of size ω_1 with at most countable levels and height ω_1.
(2) Let $(T,<_T)$ be a tree. We let $T_{<_T s} = \{t \in T\colon\ t < s\}$, and let $T_{\leqslant_T s}$, $T_{>_T s}$ be defined analogously.
(3) Let $(T,<_T)$ be a tree. Then we write T_α for $\{s \in T\colon\ (T_{<_T s},<_T) \cong \alpha\}$ and call T_α the α-th level of T.
(4) Moreover, we require that the trees are normal, i.e., for every node t on level $\alpha < \omega_1$ for every $\omega_1 > \beta > \alpha$ there are $t'' \neq t' >_T t$ on level β.

Definition 1.13. A Souslin tree is an ω_1-tree that has no uncountable chains and no uncountable antichains.

A notion of forcing $\mathbb{P}$ preserves any Souslin tree if it preserves any normal Souslin tree. This is seen as follows: Let T be a Souslin tree. We let $A = \{t \in T\colon\ T_{\geqslant_T t}$ is at most countable and t is minimal with the property$\}$. Since T is a Souslin tree, A is at most countable. We let $T' = T \setminus A$. T' is a Souslin tree in $\mathbf{V}^{\mathbb{P}}$ iff T is a Souslin tree in $\mathbf{V}^{\mathbb{P}}$.

Let T be a normal ω_1-tree. Let b be a cofinal branch. By normality, there are cofinally many $\alpha < \omega_1$ such that there are $t_\alpha \in b \cap T_\alpha$ and $t'_\alpha >_T t_\alpha$, $t'_\alpha \notin b$. Then these t'_α form an antichain. So T is Souslin iff it does not have any uncountable antichain.

Definition 1.14. We conceive a normal ω_1-tree $(T,<_T)$ without cofinal branches as a forcing notion. A stronger condition is higher up in the tree. For $\delta \in \omega_1$, we let $Y(\delta) \subseteq T_\delta$. Let $Y = \bigcup\{Y(\delta)\colon\ \delta \in \omega_1\}$ and reversely, given $Y \subseteq T$ we let $Y(\delta) = \{t \in Y\colon\ t \in T_\delta\}$. Some of the $Y(\delta)$ may be empty. Let $\mathcal{S} \subseteq [\omega_1]^\omega$ be stationary.

We say T is $(Y,\mathcal{S})$-*proper* iff $Y \subseteq T$ and for every sufficiently large χ for every countable $N \prec \mathcal{H}(\chi)$ with $\{T,\mathcal{S}\} \subset N$ and $N \cap \omega_1 \in \mathcal{S}$, $\delta = N \cap \omega_1$, for $t \in Y(\delta)$, $T_{<_T t} := \{s\colon\ s <_T t\}$ is (N,T)-generic.

The definition gets stronger the larger Y is. For characterising Souslin trees Y is taken to be a union of stationarily many levels of the tree. Another application is gotten by taking $Y(\delta) \neq \emptyset$ for stationarily many δ. The following known lemma, which is [22, Claim 3.9 B], characterises normal Souslin trees.

Lemma 1.15. *Let $(T, <_T)$ be a normal ω_1-tree. The following are equivalent*:

(1) *T is Souslin.*
(2) *T is $(Y, \mathcal{S})$-proper for every stationary $\mathcal{S} \subseteq [\omega_1]^\omega$ and for every Y of the form $\bigcup_{\delta \in W} T_\delta$, such that $W \subseteq \{\sup(a)\colon a \in \mathcal{S}\}$ stationary.*
(3) *T is $(Y, \mathcal{S})$-proper for some stationary $\mathcal{S} \subseteq [\omega_1]^\omega$ and for some Y of the form $\bigcup_{\delta \in W} T_\delta$, such that $W \subseteq \{\sup(a)\colon a \in \mathcal{S}\}$ stationary.*

Proof. (1) implies (2). Let T be a Souslin tree and let $N \prec \mathcal{H}(\chi)$ with $T \in N$, $N \in \mathcal{S}$. Let $\delta = N \cap \omega_1 \in W$. We show that every node t on level δ is (N, T)-generic: Let $I \in N$ be dense in T. Now let in N, $I' \subset I$ be a maximal antichain in T. $N \models$ "I' is countable", so $I' \subseteq N$. Now $\{s \in T\colon (\exists r \in I')(r \leqslant_T s)\} \cap \{s \in T\colon s <_T t\} \neq \emptyset$, since otherwise $I' \cup \{t\}$ is an antichain, in contradiction to the fact that by $N \prec \mathcal{H}(\chi)$ the set $I' \in N$ is also a maximal antichain in T in the sense of $\mathcal{H}(\chi)$ and in the sense of $\mathbf{V}$.

(3) implies (1). We fix $\mathcal{S}$ and Y as in (3). We consider the case that $A \subseteq T$ is an uncountable maximal antichain and take $N \prec \mathcal{H}(\chi)$ with $T, A \in N$, $N \in \mathcal{S}$, $\delta = N \cap \omega_1 \in W = \{\delta \in \omega_1\colon Y(\delta) = T_\delta\}$. Then A is dense in T in N. However, since A is uncountable, there is $t' \in A \setminus N$. Let $t = t' \restriction \delta \in T_\delta$. The node t is incompatible with every $a \in A \cap N$, so t cannot lie above an $a \in A \cap N$, so $T_{<_T t}$ is not (N, T)-generic. □

Definition 1.16. We say $\mathbb{P}$ is $(T, Y, \mathcal{S})$-*preserving* iff the following holds: Let $\mathcal{S} \subseteq \omega_1$ be stationary. There is $x \in H(\chi)$, for every countable $N \prec \mathcal{H}(\chi)$ with $\{x, Y, T, \mathbb{P}, \mathcal{S}\} \subseteq N$ and $p \in \mathbb{P} \cap N$: if $N \cap \omega_1 = \delta$, $N \cap \omega_1 \in \mathcal{S}$, and for every $t \in Y(\delta)$, $\{s\colon s <_T t\}$ is (N, T)-generic, then there is $q \geqslant_{\mathbb{P}} p$ such that q is $(N, \mathbb{P})$-generic and

$$q \Vdash_{\mathbb{P}} \big(\forall t \in Y(\delta)\big)\big(\{s\colon s <_T t\} \text{ is } \big(N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T\big)\text{-generic}\big).$$

We remark that the quantifier "for every countable $N \prec \mathcal{H}(\chi)$ with $\{x, Y, T, \mathbb{P}, \mathcal{S}\} \subseteq N$ and $p \in \mathbb{P} \cap N$" can be weakened and that the particular choice of $x \in H(\chi)$ is not essential, see [1, Theorem 2.13].

In Section 4 we show that "T is $(Y, \mathcal{S})$-proper" is preserved by countable support iterations of proper iterands if each iterand preserves it. Since we are mainly interested in countable support iterations (because of the club principle), we can focus onto the question: Which iterands preserve "T is $(Y, \mathcal{S})$-proper"?

A sufficient criterion is given by $\mathrm{Pr}^2_{\mathcal{S}}(\mathbb{P})$.

Theorem 1.17. *Assume $\alpha(*) = \omega_1$ and $\mathcal{S} \subseteq \omega_1$ is stationary. Let T be an ω_1-tree and $Y \subseteq T$. If $\mathrm{Pr}^2_{\mathcal{S}}(\mathbb{P})$, then $\mathbb{P}$ is $(T, Y, \mathcal{S})$-preserving.*

Proof. Assume $N \prec H(\chi)$, $N \cap \omega_1 \in \mathcal{S}$, $N \cap \omega_1 = \delta$, and $\mathbb{P} \in N$, $p \in N \cap \mathbb{P}$, and assume for every $t \in Y(\delta)$, $\{s\colon s <_T t\}$ is $(N, \mathbb{P}, p)$-generic. Let $x = \mathbf{st}$ for a winning strategy $\mathbf{st}$ for player COM in $\eth^2(N, \mathbb{P}, p)$. We show that there is a q as required in the previous definition.

Let $Y = \{t^\delta_k\colon k < \gamma_\delta, \delta \in W\}$ for suitable $\gamma_\delta \leqslant \omega$. Let $\{\underset{\sim}{\mathcal{I}}_n\colon n \in \omega\}$ list the $\mathbb{P}$-names of open dense sets in the forcing T that are in N and let $\{\mathcal{J}_n\colon n \in \omega\}$ list the open dense sets in $\mathbb{P}$ in N. Now we take a play $\langle(\overline{p_n}, \overline{q_n})\colon n \in \omega\rangle$ in which COM plays according to $\mathbf{st}$. INC plays in every round n in every $i < \ell_n$ the condition $q_{n,i}$ so strong that $q_{n,i} \in \bigcap_{r<n} \mathcal{J}_r$ and such that for every $k < n$ there is $t_{i,n,k} <_T t^\delta_k$ such that

$$q_{n,i} \Vdash_{\mathbb{P}} t_{i,n,k} \in \bigcap_{k'<n} \underset{\sim}{\mathcal{I}}_{k'}.$$

Why can INC play like this? Given $i < \ell_n$ and a starting point q', for $k < n$ he can strengthen $q_{n,i} \geqslant q'$ so that $q_{n,i} \Vdash_{\mathbb{P}} t_{i,n,k} \in \bigcap_{k'<n} \underset{\sim}{\mathcal{I}}_{k'}$ for a suitable $t_{i,n,k} <_T t^\delta_k$. Since $\{s\colon s <_T t^\delta_k\}$ is (N,T)-generic, there is such a $t_{i,n,k} <_T t^\delta_k$, $t_{i,n,k} \in \mathcal{J}$. Now he repeats this for each $k < n$. Since $\bigcap_{k'<n} \underset{\sim}{\mathcal{I}}_{k'}$ is (forced by the weakest conditions to be) open dense in the forcing T, the set $\mathcal{J} = \{s \in T \cap N\colon q \nVdash_{\mathbb{P}} s \notin \bigcap_{k'<n} \underset{\sim}{\mathcal{I}}_{k'}\}$ is dense in T in the ground model (before forcing with $\mathbb{P}$).

COM wins the play because he played according to the strategy. So for every u, in particular for $u = \omega$, there is $q_u \geqslant p$ such that

$$q_u \Vdash \left(\exists^\infty n \in u\right)(\exists \ell < \ell_n)(q_{n,\ell} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}). \tag{1.1}$$

Let $k \in \omega$ and $q' \geqslant q_u$ be given. Then there is $q'' \geqslant q'$ and $n \geqslant k$ such that $q'' \Vdash n \in u$. So there is $i < \ell_n$, $q'' \Vdash q_{n,i} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ and hence

$$q'' \Vdash_{\mathbb{P}} t_{i,n,k} \in \bigcap_{k'<n} \underset{\sim}{\mathcal{I}}_{k'} \wedge t_{i,n,k} \leqslant_T t^\delta_k. \tag{1.2}$$

Now we unfreeze k and combine Eqs. (1.1) and (1.2) and thus get

$$q_u \Vdash (\forall k < \omega)\left(T_{<_T t^\delta_k} \text{ is } \left(N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T\right)\text{-generic}\right).$$

From $q_{n,i} \in \bigcap_{r<n} \mathcal{J}_r$ we also get that q_u is $(N,\mathbb{P})$-generic. $\square$

Corollary 1.18. *If T is a Souslin tree, $\mathcal{S}$ is stationary, and $\mathbb{P}$ is a notion of forcing with $\mathrm{Pr}^2_{\mathcal{S}}(\mathbb{P})$, then T is Souslin in $\mathbf{V}^{\mathbb{P}}$.*

Proof. We let $Y = \bigcup\{T_\delta\colon \delta \in \mathcal{S}\}$. By Lemma 1.15 we have: T is $(Y,\mathcal{S})$-proper iff it is a Souslin tree. Now Theorem 1.17 shows the preservation of "T is $(Y,\mathcal{S})$-proper". $\square$

Historical remarks. Our notion of $\mathcal{S}$-properness this is called $(\{\mathcal{S}\},\emptyset,\emptyset)$-properness in [22, Definition IV, 2.2.]. The notions of ${}^\omega\omega$-bounding and almost ${}^\omega\omega$-bounding appeared in [21]. A general study of preservation of these and related properties in iterations is in [22, Chapter VI]. A even more extensive study of preservation properties is carried out in Chapter XVIII of [22]. In [22, XVIII 3.9 D] a variant of our definition of $(T,Y,\mathcal{S})$-preserving is mentioned.

2. Many creature forcings $\mathbb{P}$ preserve Souslin trees

We begin this section with a proof that Miller forcing hat Pr^2. Then we look at other creature forcings. We give a short self-contained introduction to creatures in general. In Section 2.3 we consider tree creatures and give sufficient conditions for Pr^1 and Pr^2. In Section 2.4 we are concerned with linear creatures. For these forcings we have not found strategies in our games. However, some forcings of this kind preserve Souslin trees for other reasons.

2.1. A game on the Miller forcing

Conditions in the Miller order are superperfect trees $p \subseteq \omega^{<\omega}$. A tree is called superperfect iff for any node $\eta \in p$ there is $\varrho \trianglerighteq \eta$ that has infinitely many immediate successors in p. Here $\trianglerighteq$ denotes the end extension of finite sequences. Stronger conditions are perfect subtrees.

Miller forcing answers our question about the consistency of the club principle together with the existence of a Souslin tree and $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathrm{cof}(\mathcal{M}) = \aleph_2$, since it is well-known [4,24,20] that in the Miller model $\mathrm{cov}(\mathcal{M}) \leqslant \mathfrak{u} = \aleph_1$ and $\mathrm{cof}(\mathcal{M}) \geqslant \mathfrak{d} = \aleph_2$. The "Miller model" means any countable support iteration of Miller forcing over a ground model of CH.

The Miller conditions such that each splitting node has infinitely many immediate successors are dense in the Miller order. From now on we work only with such conditions. For $r \in \mathbb{P}$ let $\mathrm{rt}(r)$ be the trunk, that is the shortest η such that $\mathrm{succ}_r(\eta) = \{\eta^\frown n\colon \eta^\frown n \in r\}$ is infinite.

Theorem 2.1. *Miller $\mathbb{P}$ forcing has* $\mathrm{Pr}^2(\mathbb{P})$.

Proof. We assume that all moves of both players in the game $\eth^2(\mathbb{P}, p)$ below have infinite splitting in each splitting node. Let $v \subseteq {}^{\omega>}\omega$. We let $\mathrm{dcl}(v) = \{\eta \restriction k\colon \eta \in v, k < \lg(\eta)\}$ be the downwards closure of v. A set v is a tree iff $v = \mathrm{dcl}(v)$.

We describe a strategy **st** for COM in $\eth^2(\mathbb{P}, p)$. On the side after the n-th move COM chooses a finite set of nodes v_n that are among the splitting nodes of INC's previously chosen conditions. COM plays so that the sequence $\langle \bar{p}_n, \bar{q}_n, v_n\colon n \in \omega\rangle$ has the following properties:

(0) $\ell_0 = 1$, $p_{0,0} = p$, $q_{0,0} \geqslant p_{0,0}$, $v_0 = \{\mathrm{tr}(q_{0,0})\}$.

(1) For $n \geqslant 1$, given v_{n-1}, COM chooses $\ell_n = |v_{n-1}|$ and for $\eta \in v_{n-1}$, $\eta = \mathrm{rt}(q_{n',\ell})$ for some $n' < n$, $\ell < \ell_{n'}$ he lets

$$m(\eta, n) = \min\bigl\{k\colon \eta^\frown k \in q_{n',\ell} \setminus \mathrm{dcl}(v_{n-1})\bigr\}.$$

Since the $q_{n',\ell}$ is a Miller condition and each $\eta \in v_{n-1}$ is a splitting node of $q_{n',\ell}$ for some $n' < n$ and $\ell < \ell_{n'}$ and v_{n-1} is finite, for each $\eta \in v_{n-1}$, $m(\eta, n)$ is defined. Let $\{\eta^n_\ell\colon \ell < \ell_n\}$ enumerate v_{n-1} and let $\eta^n_\ell = \mathrm{rt}(q_{n',\ell'})$. Now COM chooses $p_{n,\ell} = q_{n',\ell'}^{[\eta^n_\ell{}^\frown m(\eta_\ell, n)]}$.

(2) INC plays $q_{n,\ell} \geqslant p_{n,\ell}$.

(3) Now COM chooses his new helper: $v_n = v_{n-1} \cup \{\mathrm{rt}(q_{n,\ell})\colon \ell < \ell_n\}$, and the round is finished. Indeed $\ell_{n+1} = 2\ell_n$ and $\ell_0 = 1$, but this is not important.

The strategy **st** is a winning strategy for COM: Let $u \subseteq \omega$ be infinite. By induction on $n \in u$ we choose $s_n \subseteq v_n \setminus v_{n-1}$. If $n = \min(u)$, then $s_n \subseteq v_n \setminus v_{n-1}$ is a singleton. For $n > \min(u)$, let

$$s_n = s_{\max(u \cap n)} \cup \bigl\{\eta \in v_n \setminus v_{n-1}\colon \nu = \triangleleft\text{-}\max\{\varrho \in v_n\colon \varrho \trianglelefteq \eta\} \in s_{\max(u\cap n)}\bigr\}.$$

Lastly we let

$$q_u = \bigl\{\varrho\colon (\exists n \in u)(\exists \eta \in s_n)(\varrho \trianglelefteq \eta)\bigr\}.$$

By definition, q_u is a tree. It is a Miller tree, since for every $n \in u$, for every $\eta \in s_n$, η is a splitting node in q_u since $\eta = \mathrm{rt}(q_{n,\ell}) \in v_n \setminus v_{n-1}$. We show that for this pair (n, ℓ), an infinite subset of $\mathrm{succ}_{q_{n,\ell}}(\eta)$ is a subset of $\mathrm{succ}_{q_u}(\eta)$: For any $k > n$ and there is $\nu \trianglerighteq \eta^\frown m(\eta, k)$, such that $\nu \in v_k \setminus v_{k-1}$, by the choice of $\langle v_k\colon k \in \omega\rangle$. For such a ν we have $\max_\triangleleft\{\varrho \in v_k\colon \varrho \trianglelefteq \nu\} = \eta \in s_n \subseteq s_{\max(u\cap k)}$. If $k \in u$, then $\nu \in s_k$.

Moreover $q_u \Vdash_{\mathbb{P}} (\exists^\infty n \in u)(\exists \ell < \ell_n)(q_{n,\ell} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}})$: Suppose that not. Let $r \geqslant_{\mathbb{P}} q_u$ be a Miller condition such that $r \Vdash \forall n \in u\ (n \geqslant k \to (\forall \ell < \ell_k)(q_{k,\ell} \notin \underset{\sim}{\mathbf{G}}_{\mathbb{P}}))$. By strengthening r, we may assume that $\mathrm{rt}(r) \in s_n$ for some $n \geqslant k$, so $r \geqslant q_{n,\ell}$ for some $\ell < \ell_n$. This is a contradiction. $\square$

The properties Pr^1 and Pr^2 hold for tree-creature forcings with the lim norm or the lim-sup norm. We explain this now.

2.2. Forcings with (tree) creatures

Now we give more examples. In order to describe the relevant properties we give a brief review of the definitions to forcings with creatures. This concept is explained in the book [16], and it is divided into two main streams: one kind of creature forcing is forcing with creatures such that the conditions are written in an ω-sequence like Blass–Shelah forcing [3]. Another example of a forcing with ω-sequences of creatures is the (historically first) creature forcing in [21] that forces $\mathfrak{b} < \mathfrak{s}$. The other stream is forcing with tree creatures. For historical reasons the first kind is often just called "creature forcing" and the second kind is called "tree creature forcing". In this subsection we give a very short introduction to the main concepts.

Let $\mathrm{H} : \omega \to \mathcal{H}(\omega_1)$ be a function such that $(\forall n)(|\mathrm{H}(n)| \geqslant 2 \wedge 0 \in \mathrm{H}(n))$.

Definition 2.2. Let χ be a regular cardinal. A triple $t = (\mathrm{nor}[t], \mathrm{val}[t], \mathrm{dis}[t])$ is a *weak creature for* H*,* χ if the following hold:

(a) $\mathrm{nor}[t] \in \mathbb{R}^{\geqslant 0} \cup \{\infty\}$,
(b) $\mathrm{val}[t]$ is a non-empty subset of

$$\left\{(x, y) \in \bigcup_{m_0 < m_1 < \omega} \prod_{i < m_0} \mathrm{H}(i) \times \prod_{i < m_1} \mathrm{H}(i) \colon\ x \vartriangleleft y\right\},$$

(c) $\mathrm{dis}[t] \in \mathcal{H}(\chi)$.

The family of weak creatures for H and χ is denoted by WCR[H].

We omit the parameter χ since in the following $\mathrm{dis}[t]$ is constant or empty.

Definition 2.3. We say H is *finitary* if $\mathrm{H}(n)$ is finite for each n, we say H is of *countable character* if $\mathrm{H}(n)$ is at most countable for every n. We say $K \subseteq \mathrm{WCR}[\mathrm{H}]$ is *finitary* if H is finitary and for every $t \in K$, $\mathrm{val}[t]$ is finite.

By our choice of $\mathrm{H} : \omega \to \mathcal{H}(\omega_1)$ all the creatures in the following will be of countable character.

Definition 2.4. Let $K \subseteq \mathrm{WCR}[\mathrm{H}]$.

(1) A function $\Sigma : [K]^{\leqslant\omega} \to \mathcal{P}(K)$ is called a *sub-composition operation on* K if the following holds:
   (a) (Transitivity) If $\mathcal{S} \in [K]^{\leqslant\omega}$ and for each $s \in \mathcal{S}$ we have $s \in \Sigma(\mathcal{S}_s)$ for some $\mathcal{S}_s \in \mathrm{dom}(\Sigma)$, then $\Sigma(\mathcal{S}) \subseteq \Sigma(\bigcup_{s \in \mathcal{S}} \mathcal{S}_s)$.
   (b) We write $\Sigma(r)$ for $\Sigma(\{r\})$. $r \in \Sigma(r)$ for each $r \in K$ and $\Sigma(\emptyset) = \emptyset$.
(2) In the situation described above (K, Σ) is called a *weak creating pair.*

Definition 2.5. Let (K, Σ) be a weak creating pair for H.

(1) For a weak creature $t \in K$ we define its *basis* with respect to (K, Σ) as

$$\mathit{basis}(t) = \left\{w \in \bigcup_{m < \omega} \prod_{i < m} \mathrm{H}(i) \colon\ \big(\exists s \in \Sigma(t)\big)(\exists u)\big(\langle w, u\rangle \in \mathrm{val}[s]\big)\right\}.$$

(2) For $w \in \bigcup_{m < \omega} \prod_{i < m} \mathrm{H}(i)$ and $\mathcal{S} \in [K]^{\leqslant\omega}$ we define the set $\mathrm{pos}(w, \mathcal{S})$ of *possible extensions of* w *from the point of view of* $\mathcal{S}$ as

$$\mathrm{pos}^*(w,\mathcal{S}) = \big\{u\colon \exists s \in \Sigma(\mathcal{S})\big(\langle w,u\rangle \in \mathrm{val}[s]\big)\big\},$$

$$\begin{aligned}\mathrm{pos}(w,\mathcal{S}) = \Big\{u\colon\ & \text{there are } m \in \omega \text{ and disjoint sets } \mathcal{S}_i \text{ for } i<m, \bigcup_{i<m}\mathcal{S}_i = \mathcal{S},\\ & \text{and a sequence } 0<\ell_1<\cdots<\ell_{m-1}<\lg(u) \text{ such that}\\ & u \restriction \ell_1 \in \mathrm{pos}^*(w,\mathcal{S}_0) \text{ and}\\ & u \restriction \ell_2 \in \mathrm{pos}^*(u\restriction \ell_1,\mathcal{S}_1),\ldots,u \in \mathrm{pos}^*(u\restriction \ell_{m-1},\mathcal{S}_{m-1})\Big\}.\end{aligned}$$

2.3. Tree creatures

From now on we specialise on tree creatures. They have the special property that val$[t]$ has just one root.

Definition 2.6.

(1) A *quasi tree* $(T, \lhd_T)$ is a set of finite sequences, ordered by initial segment, and there is a $\lhd_T$ smallest element $\mathrm{rt}(T)$, called the root of T.
(2) A quasi tree is called a *tree* if it is closed under initial segments. If T is a quasi tree we denote its closure under initial segments by $\mathrm{dcl}(T)$. (This is the smallest tree containing T.)
(3) We define the *set of immediate successors of η in T*, the *restriction of T to η*, the *splitting points of T* and the *maximal points of T* by

$$\begin{aligned}\mathrm{succ}_T(\eta) &= \big\{\nu \in T\colon \eta \lhd_T \nu \wedge \neg(\exists \rho \in T)(\eta \lhd_T \rho \lhd_T \nu)\big\},\\ T^{[\eta]} &= \{\nu \in T\colon \eta \trianglelefteq_T \nu\},\\ \mathrm{split}(T) &= \big\{\eta \in T\colon \left|\mathrm{succ}_T(\eta)\right| \geqslant 2\big\},\\ \max(T) &= \big\{\nu \in T\colon \neg(\exists \rho \in T)(\nu \lhd_T \rho)\big\}.\end{aligned}$$

(4) The *n-th level* of T is

$$T_n = \{\eta \in T\colon \eta \text{ has } n \lhd_T\text{-predecessors}\}.$$

(5) A *branch of T* is a maximal subset of T that is linearly ordered by $\lhd_T$. The *set of infinite branches through T* is

$$\lim(T) = \Big\{\eta\colon \eta \text{ is an } \omega\text{-sequence and } \bigwedge\big(\exists^\infty k\big)(\eta \restriction k \in T)\Big\}.$$

A quasi tree is *well-founded* if there are no infinite branches through it.
(6) A subset F of a quasi tree T is called a *front of T* if every infinite branch of T and every finite branch of T passes through this set, and the set consists of $\lhd_T$-incomparable elements.

Definition 2.7.

(1) A weak creature $t \in \mathrm{WCR}[\mathrm{H}]$ is a *tree creature* if $\mathrm{dom}(\mathrm{val}[t])$ is a singleton $\{\eta\}$ and no two distinct elements of $\mathrm{range}(\mathrm{val}[t])$ are $\lhd$-comparable (so also not compatible as finite partial functions since every $\eta \in \mathrm{range}(\mathrm{val}[t])$ has as a domain some $n \in \omega$). $\mathrm{TCR}[\mathrm{H}]$ is the family of all tree creatures for H.
(2) $\mathrm{TCR}_\eta[\mathrm{H}] = \{t \in \mathrm{TCR}[\mathrm{H}]\colon \mathrm{dom}(\mathrm{val}[t]) = \{\eta\}\}$.
(3) A sub-composition operation Σ on $K \subseteq \mathrm{TCR}[\mathrm{H}]$ is a *tree-composition* (and then (K,Σ) is called a tree-creating pair for H) if the following holds:

(∗) If $\mathcal{S} \in [K]^{\leqslant\omega}$ and $\Sigma(\mathcal{S}) \neq \emptyset$, and $\mathcal{S} = \{s_\nu\colon \nu \in \hat{T}\}$ for some well-founded quasi tree $\hat{T} \subseteq \bigcup_{n<\omega} \prod_{i<n} \mathrm{H}(i)$ and if for each finite sequence $\nu \in \hat{T}$, $s_\nu \in \mathrm{TCR}_\nu[\mathrm{H}]$ and for $\nu \in \hat{T} \setminus \max(\hat{T})$, $\mathrm{range}(\mathrm{val}[s_\nu]) = \mathrm{succ}_{\hat{T}}(\nu)$ and if $t \in \Sigma(\{s_\nu\colon \nu \in \hat{T}\})$ then $t \in \mathrm{TCR}_{\mathrm{rt}(\hat{T})}[\mathrm{H}]$ and

$$\mathrm{range}\bigl(\mathrm{val}[t]\bigr) \subseteq \bigcup\bigl\{\mathrm{range}\bigl(\mathrm{val}[s_\nu]\bigr)\colon \nu \in \max(\hat{T})\bigr\}.$$

We write $\Sigma(s_\nu\colon \nu \in \hat{T})$ instead of $\Sigma(\{s_\nu\colon \nu \in \hat{T}\})$. If $\hat{T} = \{\mathrm{rt}(\hat{T})\}$, $t = s_{\mathrm{rt}(\hat{T})} \in \mathrm{TCR}_{\mathrm{rt}(\hat{T})}[\mathrm{H}]$ then we will write $\Sigma(t)$ instead of $\Sigma(s_\nu\colon \nu \in \hat{T})$.

So for a tree-creating pair, if $t \in \mathrm{TCR}_\eta[\mathrm{H}]$, then *basis*$(t) = \{\eta\}$ and $\mathrm{pos}^*(\eta, \{t\}) = \mathrm{succ}_t(\eta) = \mathrm{range}(\mathrm{val}[t])$. We write only $\mathrm{pos}^*(t)$ for $\mathrm{pos}^*(\eta, \{t\})$.

The next definition introduces requirements on the norms of the creatures in a condition. We focus on the limsup condition and the lim condition. To speak in a uniform way about both variants, we introduce a parameter e, and $e = 0$ stands for the limsup case, and $e = 1$ stands for the lim case.

Definition 2.8. Let (K, Σ) be a tree-creating pair for H, such that there are $t \in \mathrm{TCR}_\eta[\mathrm{H}] \cap K$ of arbitrary high norm.

(1) We define the forcing notion $\mathbb{Q}_e^{\mathrm{tree}}(K, \Sigma)$ for $e = 0, 1$ by letting $p = \langle t_\eta\colon \eta \in T\rangle \in \mathbb{Q}_e^{\mathrm{tree}}(K, \Sigma)$:
  (a) $T \subseteq \bigcup_{n<\omega} \prod_{i<n} \mathrm{H}(i)$ is a non-empty quasi tree with $\max(T) = \emptyset$, and
  (b) $t_\eta \in \mathrm{TCR}_\eta[\mathrm{H}] \cap K$ and $\mathrm{pos}^*(t_\eta) = \mathrm{succ}_T(\eta)$, and
  (c) in the lim case ($e = 1$) we require for $\eta \in \lim(T)$,

$$\lim\bigl\langle \mathrm{nor}[t_{\eta\restriction k}]\colon k < \omega, \eta \restriction k \in T\bigr\rangle = \infty.$$

  In the limsup case ($e = 0$) we require for $\eta \in \lim(T)$ the sequence

$$\limsup\bigl\langle \mathrm{nor}[t_{\eta\restriction k}]\colon k < \omega, \eta \restriction k \in T\bigr\rangle = \infty.$$

  We define the forcing order $\leqslant = \leqslant_{\mathbb{Q}_e^{\mathrm{tree}}}$ by $\langle t^1_\eta\colon \eta \in T^1\rangle \leqslant \langle t^2_\eta\colon \eta \in T^2\rangle$ iff $T^2 \subseteq T^1$ and for each $\eta \in T^2$ there is a quasi tree $\hat{T}_{0,\eta} \subseteq (T^1)^{[\eta]}$ such that $\mathrm{dcl}(\hat{T}_{0,\eta})$ is well-founded and $t^2_\eta \in \Sigma(\{t^1_\nu\colon \nu \in \hat{T}_{0,\eta}\})$. If $t = \langle t_\eta\colon \eta \in T\rangle$ then we write $\mathrm{rt}(p) = \mathrm{rt}(T)$ and $T^p = T$ and $t^p_\eta = t_\eta$, etc.

(2) If $p \in \mathbb{Q}_e^{\mathrm{tree}}(K, \Sigma)$ then we let $p^{[\eta]} = \langle t^p_\nu\colon \nu \in (T^p)^{[\eta]}\rangle$ for $\eta \in T^p$.

We write $\mathrm{succ}_p(\eta)$ for $\mathrm{succ}_{T^p}(\eta)$.

The prominent real added by $\mathbb{Q}_e^{\mathrm{tree}}(K, \Sigma)$, $e = 0, 1$, is $\underset{\sim}{W}$ with

$$\Vdash_{\mathbb{Q}_e^{\mathrm{tree}}(K,\Sigma)} \underset{\sim}{W} = \bigcup\bigl\{\mathrm{rt}(p)\colon p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_e^{\mathrm{tree}}(K,\Sigma)}\bigr\}.$$

Usually the conditions on the norm imply that $\underset{\sim}{W}$ is forced to be not in $\mathbf{V}$.

Definition 2.9. Let (K, Σ) be a tree-creating pair $p \in \mathbb{Q}_e^{\mathrm{tree}}(K, \Sigma)$, $e = 0, 1$. A set $A \subseteq T^p$ is called *an e-thick antichain* if it is an antichain in $(T^p, \lhd)$ and for every condition $q \geqslant p$ the intersection $A \cap \mathrm{dcl}(T^q)$ is not empty.

Proposition 2.10.

(1) *Let $e = 0, 1$. $\mathbb{Q}_e^{\mathrm{tree}}$ is a partial order. Each e-thick antichain A in T^p gives a maximal antichain $\{p^{[\eta]}\colon \eta \in A\}$ in $\mathbb{Q}_e^{\mathrm{tree}}$ above p. Every front of T^p is an e-thick antichain in T^p.*

(2) *We define*

$$F_n^m(p) = \{\eta \in T^p\colon \operatorname{nor}[t_\eta^p] > n \text{ and } |\{\eta' \in T^p\colon \eta' \lhd \eta \wedge \operatorname{nor}[t_{\eta'}^p] > n\}| = m\}.$$

Each $F_n^m(p)$ is a front of T^p and an e-thick antichain of T^p for $e = 0, 1$.

(3) *If K is finitary and $\operatorname{dcl}(T^p)$ is well-founded, then every front of T^p is finite.*

(4) *$p \leqslant p^{[\eta]} \in \mathbb{Q}_e^{\text{tree}}$ and $\operatorname{rt}(p^{[\eta]}) = \eta$.*

Proof. See [16, Proposition 1.3.7]. □

Now the forcings notions with the normed trees let us define strengthenings of the forcing order $\leqslant$ that are natural candidates for Axiom A (a definition can be found, e.g., in [2, Definition 7.1.1]).

Definition 2.11. Let (K, Σ) be a tree-creating pair and let $p, q \in \mathbb{Q}_0^{\text{tree}}(K, \Sigma)$.

(1) For the limsup case, we define $\leqslant_n^0$ for $n < \omega$ by $p \leqslant_0^0 q$ if $p \leqslant q$ and $\operatorname{rt}(p) = \operatorname{rt}(q)$, $p \leqslant_{n+1}^0 q$ if $p \leqslant_0^0 q$ and if $\eta \in F_n^0(p)$ and $\nu \in T^p$ and $\nu \trianglelefteq \eta$ then $\nu \in T^q$ and $t_\nu^q = t_\nu^p$.

(2) For the lim case we define $\leqslant_n^1$ for $n < \omega$ by $p \leqslant_0^1 q$ if $p \leqslant q$ and $\operatorname{rt}(p) = \operatorname{rt}(q)$, $p \leqslant_{n+1}^1 q$ if $p \leqslant_0^1 q$ and if $\eta \in F_n^0(p)$ and $\nu \in T^p$ and $\nu \trianglelefteq \eta$ then $\nu \in T^q$ and $t_\nu^q = t_\nu^p$ and

$$\{(\eta, t_\eta^q)\colon \eta \in T^q \wedge \operatorname{nor}[t_\eta^q] \leqslant n\} \subseteq \{(\eta, t_\eta^p)\colon \eta \in T^p\}.$$

Note that $t_\nu^p = t_\nu^q$ means also that the immediate successors of ν in p coincide with the immediate successors of ν in q.

Proposition 2.12. *Suppose that $e = 0, 1$, $p_n \in \mathbb{Q}_e^{\text{tree}}(K, \Sigma)$ and for $n \in \omega$, $p_n \leqslant_{n+1}^e p_{n+1}$. Then the limit condition $p = \lim_{n\to\omega} p_n$ is defined by $T^p = \bigcap_{n<\omega} T^{p_n}$ and for $\eta \in \bigcap_{n<\omega} T^{p_n}$ we take the creature $t_\eta^p = \bigcap_{n\in\omega} t_\eta^{p_n}$ (note that this is actually a finite intersection since the descending sequence $t_\eta^{p_n}$, $n \in \omega$ eventually becomes constant) into p. Then $p \in \mathbb{Q}_e^{\text{tree}}(K, \Sigma)$ and $p \geqslant_{n+1}^e p_n$ for each n.*

Proposition 2.13. *Let $p \in \mathbb{Q}_e^{\text{tree}}(K, \Sigma)$, $n < \omega$, and let $A \subseteq T^p$ be an antichain in T^p such that $(\forall \eta \in A)(\exists \nu \in F_n^0(p))(\nu \lhd \eta)$. Assume that for each $\eta \in A$ we have a condition $q_\eta \in \mathbb{Q}_e^{\text{tree}}(K, \Sigma)$ such that $p^{[\eta]} \leqslant_0^e q_\eta$ and*

$$\text{if } e = 1 \quad \text{then } (\forall \eta \in A)(\forall \nu \in T^{q_\eta})\big((\nu \trianglerighteq \eta \wedge \operatorname{nor}(t_\nu^{q_\eta}) \leqslant n) \to t_\nu^{q_\eta} = t_\nu^p\big).$$

Then there exists $q \in \mathbb{Q}_e^{\text{tree}}(K, \Sigma)$ such that $p \leqslant_{n+1}^e q$, $A \subseteq T^q$, $q^{[\eta]} = q_\eta$ for $\eta \in A$ and if $\nu \in T^p$ is such that there is no $\eta \in A$ with $\eta \trianglelefteq \nu$ then $\nu \in T^q$ and $t_\nu^q = t_\nu^p$.

Since we repeatedly use the construction from Proposition 2.13 in a re-ordered setting for the lim case, we name it:

Definition 2.14. We call the q constructed from p, A and q_η, $\eta \in A$, as in the previous proposition:

$$q = p \restriction \{\nu \in T^p\colon \forall \eta \in A \nu \not\trianglerighteq_p \eta\}^\frown \sum_{\eta \in A} q_\eta.$$

When we use this expression we assume that the conditions on p, A, $q_\eta, \eta \in A$, as given in the proposition are fulfilled.

Definition 2.15. A tree-creating pair (K, Σ) is *t-omittory* if for each system $\langle s_\nu \colon \nu \in \hat{T}\rangle$ such that $\mathrm{dcl}(\hat{T})$ is a well-founded tree and $\mathrm{rt}(s_\nu) = \nu$ and $\mathrm{pos}^*(s_\nu) = \mathrm{succ}_{\hat{T}}(s_\nu)$ for $\nu \in \hat{T} \setminus \max(\hat{T})$ and for every $\nu_0 \in \hat{T}$ such that $\mathrm{pos}^*(s_{\nu_0}) \subseteq \bigcup\{\mathrm{range}(\mathrm{val}[s_\nu])\colon \nu \in \max(\hat{T})\}$ there is $s \in \Sigma(s_\nu \colon \nu \in \hat{T})$ such that

$$\mathrm{nor}[s] \geqslant \mathrm{nor}[s_{\nu_0}] - 1 \quad \text{and} \quad \mathrm{pos}^*(s) \subseteq \mathrm{pos}^*(s_{\nu_0}).$$

Note that t-omittoriness implies that the domain of Σ contains $(s_\nu \colon \nu \in \hat{T})$ for all well-founded subtrees $\hat{T}$. A suitable equivalent formulation of Miller forcing is t-omittory.

Now there is an important construction we want to recall and use in the proofs of Lemma 2.17 and of Proposition 2.18 and Theorem 2.19.

Lemma 2.16. *Let $e = 0, 1$ and let (K, Σ) be t-omittory. If $p \leqslant q$ then there is $r \in \mathbb{Q}_e^{\mathrm{tree}}(K, \Sigma)$ such that $p \leqslant_0^e r$ and $\mathrm{dcl}(T^r) \subseteq \mathrm{dcl}(T^q)$ and $t_\nu^r = t_\nu^q$ for $\nu \in T^r \setminus \{\mathrm{rt}(T^r)\}$ and $\mathrm{rt}(q) \in \mathrm{dcl}(T^r)$ and $\mathrm{nor}(t^r_{\mathrm{rt}(r)}) \geqslant \mathrm{nor}(t^q_{\mathrm{rt}(q)}) - 1$.*

Proof. We let $\eta = \mathrm{rt}(q)$ and let T^* be a well-founded quasi tree such that $(\forall \nu \in T^*)(\mathrm{succ}_{T^*}(\nu) = \mathrm{pos}^*(t_\nu^p))$ and $\mathrm{rt}(T^*) = \eta$ and $t_\eta^q \in \Sigma(t_\nu^p \colon \nu \in T^*)$. We let $T^- = \{\mathrm{rt}(p)\} \cup \{\nu \in T^p \colon \nu \lhd \eta\} \cup \{\eta\}$. T^- is a well-founded quasi tree and we may apply t-omitting to $\langle t_\nu^p \colon \nu \lhd \eta \colon \nu \in T^p\rangle ^\frown \langle t_\eta^q\rangle$ and η. Thus we get $t^r_{\mathrm{rt}(p)} \in \Sigma(\{t_\nu^p \colon \nu \lhd \eta, \nu \in T^p\} \cup \{t_\eta^q\})$ such that $\mathrm{pos}^*(t^r_{\mathrm{rt}(p)}) \subseteq \mathrm{pos}^*(t_\eta^q)$ and $\mathrm{nor}(t^r_{\mathrm{rt}(r)}) \geqslant \mathrm{nor}(t^q_{\mathrm{rt}(q)}) - 1$. Note that by transitivity of Σ, $t^r_{\mathrm{rt}(p)} \in \Sigma(t_\nu^p \colon \nu \in T^- \cup T^*)$. For $\nu \in T^q$ such that $(\exists \nu' \in \mathrm{pos}^*(t^r_{\mathrm{rt}(p)}))(\nu \trianglerighteq \nu')$ let $t_\nu^r = t_\nu^q$. □

Lemma 2.17. *Suppose that (K, Σ) is a finitary t-omittory tree-creating pair. Then $\mathbb{Q}_1^{\mathrm{tree}}(K, \Sigma)$ is dense in $\mathbb{Q}_0^{\mathrm{tree}}(K, \Sigma)$.*

Proof. Given a $p \in \mathbb{Q}_0^{\mathrm{tree}}(K, \Sigma)$, we repeatedly use Lemma 2.16 to change it into a stronger condition in $\mathbb{Q}_1^{\mathrm{tree}}(K, \Sigma)$. □

Proposition 2.18.

(1) *Suppose that (K, Σ) is a finitary t-omittory tree-creating pair. Then player COM has a winning strategy in $\Game^1(\mathbb{Q}_1^{\mathrm{tree}}(K, \Sigma), p)$.*
(2) *Suppose that (K, Σ) is a finitary creating pair that is t-omittory. Then player COM has as winning strategy in $\Game^1(\mathbb{Q}_0^{\mathrm{tree}}(K, \Sigma), p)$.*

Proof. By Lemma 2.17 we need to prove only (2). We describe a strategy **st** for COM in $\Game^1(\mathbb{P}, p)$. The play will be $\langle \bar{p}_n, \bar{q}_n, \colon n \in \omega\rangle$. This time we let $v_n = \{\mathrm{rt}(q_{n,\ell}) \colon \ell < \ell_n\}$. COM plays so that the play has the following properties:

(0) $\ell_0 = 1$, $p_{0,0} = p$, $q_{0,0} \geqslant p_{0,0}$, $v_0 = \{\mathrm{rt}(q_{0,0})\}$.
(1) $v_n \setminus v_{n-1}$ is a subset of some nodes $\eta_{n,\ell}$ of $q_{n,\ell}$, $\ell < \ell_n$. In contrast to the proof of Theorem 2.1, now $v_m \cap v_n = \emptyset$ and we only need to look at the $q_{n,\ell}$ from the previous round.
(2) COM lets for $\eta \in v_{n-1}$, $\eta = \eta_\ell^n = \mathrm{rt}(q_{n-1,\ell})$ for some $\ell < \ell_{n-1}$,

$$F(n, \eta) = \text{a front in } \bigl\{\zeta \rhd \mathrm{rt}(q_{n-1,\ell}) \colon \zeta \in T^{q_{n-1,\ell}} \setminus \mathrm{dcl}(v_{n-1}), \mathrm{nor}\bigl(t_\zeta^{q_{n-1,\ell}}\bigr) > n\bigr\}.$$

Since the $q_{n-1,\ell}$, $\ell < \ell_{n-1}$, are tree conditions with pairwise incomparable roots and each $\eta \in v_n$ is a node of $q'_{n-1,\ell}$ for some $\ell < \ell_{n-1}$ and v_{n-1} is finite, for each $\eta \in v_{n-1}$, (η, n) is defined. Now COM chooses for each $\eta \in v_{n-1}$, with $\eta = \mathrm{rt}(q_{n-1,\ell})$, for each $\zeta \in F(\eta, n)$, and for each $\rho \in \mathrm{range}(\mathrm{val}[t_\zeta^{q_{n-1,\ell}}])$,

$$p_{n,\eta,\rho} = (q_{n-1,\ell'})^{[\rho]}.$$

COM lets ℓ be the sum of these finite cardinalities of the fronts $F(n,\eta)$ for all $\eta \in v_{n-1}$, of all ρ's for all $\zeta \in F(n,\eta)$ and rearranges his move as

$$\{p_{n,\ell}\colon \ell < \ell_n\} = \bigl\{p_{n,\eta,\rho}\colon \eta = \eta^n_\ell \in v_{n-1}, \ell < \ell_{n-1}, \zeta \in F(\eta,n),\ \rho \in \operatorname{range}\bigl(\operatorname{val}\bigl[t^{q_{n-1,\ell}}_\zeta\bigr]\bigr)\bigr\}.$$

(3) INC plays $q_{n,\ell} \geqslant p_{n,\ell}$.

Now we prove $\mathrm{Pr}^1(\mathbb{P})$. We let

$$q = \bigl\langle \varrho\colon \varrho \in \operatorname{range}\bigl(\operatorname{val}\bigl[t^{q_{n-1,\ell}}_\zeta\bigr]\bigr)\colon n \in \omega, \ell < \ell_{n-1}, \zeta \in F(\eta_\ell, n)\bigr\rangle.$$

By definition, q is a quasi tree. It is a condition since we have by Proposition 2.13 that there is a sequence $\langle q_n\colon n < \omega\rangle$ such that $q_0 = q$, $q_{n+1} = q_n \restriction \{\eta\colon \forall \nu \rhd \eta \nu \notin \bigcup_{\ell<\ell_n} F_{n,\eta^n_\ell}\}^\frown \sum_{\eta \in \cup_{\ell<\ell_n} F(n,\eta^n_\ell)} (q)^{[\zeta]}_{n,\ell}$. We consider $q' \geqslant q$ and assume q' forces that $\{q_{n,\ell}\colon \ell < \ell_n\}$ is not dense. By the tree omittoriness, we can find $q'_n \geqslant q', q_n$ as in Lemma 2.16 such that $\operatorname{pos}(t^{q'_n}_\eta) \subseteq \operatorname{pos}(t^{q_n}_\eta)$ for each $\eta \in T^{q'_n}$. Then $q'_n \Vdash_{\mathbb{P}} (\exists \ell < \ell_n)(q_{n,\ell} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}})$ holds since a subset of $\{\operatorname{rt}(q_{n,\ell})\colon \ell < \ell_n\}$ is a front in $T^{q'_n}$. $\square$

Theorem 2.19. *Suppose that $\bigcup_{m<\omega} \mathrm{H}(m)$ is countable and (K,Σ) is a tree-creating pair for* H *that is t-omittory. Then the forcing COM has a winning strategy in $\Game^2(\mathbb{Q}^{\mathrm{tree}}_0(K,\Sigma), p)$.*

Proof. We show that there is a strategy for COM in $\Game^2(\mathbb{P},p)$. This time he plays a tuple as a side move that is more complex than the one in Theorem 2.1. We let $F(n,\eta)$ and $\operatorname{val}[t^{q_{n-1,\ell}}_\zeta]$ be as in the proof of Proposition 2.18. Now both of them are infinite. So in order to visit them all, we organise the induction so that at stage n we visit all the finitely many previous stages again and add just one node $\zeta(n,\eta)$ of each previous $F(n',\eta)$, $n' < n$, $\eta = \eta^n_\ell = \operatorname{rt}(q_{n',\ell'}) \in v_n$ for a specific $n' < n$ and $\ell' < \ell_{n'}$, and we add one node $\rho(n,\eta)$ of $\operatorname{val}[t^{q_{n',\ell'}}_{\zeta(n,\eta)}]$. Again $\ell_n = 2^n$. Both kinds of tasks, the $F(n',\eta)$ and the $\operatorname{val}[t^{q_{n,\ell'}}_{\zeta(n,\eta)}]$ will now appear as the fronts $I_{\mathbf{x},\eta}$ in the construction below.

We describe a strategy **st** for COM in $\Game^2(\mathbb{P},p)$.

First, let $\langle \varrho^*_k\colon k < \omega\rangle$ list $\operatorname{dcl}(T^p)$.

We say $\mathbf{x} = (v_\mathbf{x}, \bar{p}_\mathbf{x}, \bar{I}_\mathbf{x})$ is an *expanded state in* $\Game^2(\mathbb{P},p)$ if $\mathbf{x}$ consists of

(a) $v = v_\mathbf{x}$ a finite, non-empty set of splitting nodes of p with sufficiently high norm that has a root, $\operatorname{rt}(v_\mathbf{x})$, such that $\varrho \in v_\mathbf{x} \to \operatorname{rt}(v_\mathbf{x}) \trianglelefteq \varrho$,
(b) a tuple of conditions $\bar{p}_\mathbf{x} = \langle p_{\mathbf{x},\eta}\colon \eta \in v_\mathbf{x}\rangle$ such that $p \leqslant p_{\mathbf{x},\eta}$, $\eta = \operatorname{rt}(p_{\mathbf{x},\eta})$ and $\operatorname{nor}(t^{p_{\mathbf{x},\eta}}_\eta) > 1$,
(c) a tuple $\bar{I} = \bar{I}_\mathbf{x} = \langle I_{\mathbf{x},\eta}\colon \eta \in v_\mathbf{x}\rangle$ of fronts such that $I_{\mathbf{x},\eta} \subseteq \operatorname{dcl}(T^{p_{\mathbf{x},\eta}}) \setminus \{\eta\}$ is a front in $p_{\mathbf{x},\eta}$, it can be taken the direct successors of η in $\operatorname{dcl}(T^{p_{\mathbf{x},\eta}})$,
(d) if $\eta \in v_\mathbf{x}$ and $\eta \vartriangleleft \nu \in I_{\mathbf{x},\eta}$ and $\eta \vartriangleleft \varrho \vartriangleleft \nu$ then $\varrho \notin v_\mathbf{x}$.

For two expanded states $\mathbf{x}$, $\mathbf{y}$, we say $\mathbf{y} \in \operatorname{succ}(\mathbf{x})$ if

(α) $v_\mathbf{x} \subseteq v_\mathbf{y}$ and $\operatorname{rt}(v_\mathbf{x}) = \operatorname{rt}(v_\mathbf{y})$, $(v_\mathbf{y} \setminus v_\mathbf{x}) \cap \operatorname{dcl}(v_\mathbf{x}) = \emptyset$,
(β) $\bar{p}_\mathbf{x} = \bar{p}_\mathbf{y} \restriction v_\mathbf{x}$,
(γ) $\bar{I}_\mathbf{x} = \bar{I}_\mathbf{y} \restriction v_\mathbf{x}$,
(δ) if $\eta \in v_\mathbf{x} \setminus v_\mathbf{y}$ then $\operatorname{nor}[t^{p_{\mathbf{y},\eta}}_\eta] \geqslant |v_\mathbf{x}|$,
(ε) if $\eta \in v_\mathbf{x}$ and $k < \omega$ is minimal such that $\varrho^*_k \in I_{\mathbf{x},\eta}$ and $(\neg\exists \varrho)(\varrho^*_k \trianglelefteq \varrho \in v_\mathbf{x})$ then $(\exists \varrho)(\varrho^*_k \trianglelefteq \varrho \in v_\mathbf{y})$.

COM chooses on the side after the n-th move $\mathbf{x}_n \geqslant \mathbf{x}_{n-1}$ such the play $\langle \bar{p}_n, \bar{q}_n, \mathbf{x}_n\colon n \in \omega\rangle$ has the following properties:

(0) $\ell_0 = 1$, $p_{0,0} = p$, $q_{0,0} \geqslant p$, $\mathrm{rt}(q_{0,0}) = \eta$, now COM chooses $\nu \in T^{q_{0,0}}$ such that $\mathrm{nor}[t_\nu^{q_{0,0}}] > 1$ and $v_{\mathbf{x}_0} = \{\nu\}$, $p_{\mathbf{x}_0,\nu} = q_{0,0}^{[\nu]}$, $I_{\mathbf{x}_0,\eta} = \mathrm{succ}_{\mathrm{dcl}(T^{q_{0,0}})}(\nu)$,

(1) In the n-th move COM first lets for $\eta \in v_{\mathbf{x}_{n-1}}$,

$$k_{n,\eta} = \min\bigl\{k \in \omega\colon\ \varrho_k^* \in I_{\mathbf{x}_{n-1},\eta} \wedge (\neg\exists\varrho)\bigl(\varrho_k^* \lhd \rho \in v_{\mathbf{x}_{n-1}}\bigr)\bigr\}.$$

COM makes the move $\bar{p}_n = \langle p_{\mathbf{x}_{n-1},\eta}^{[\varrho^*_{k_{n,\eta}}]}\colon\ \eta \in v_{\mathbf{x}_{n-1}}\rangle$. Then INC makes moves $\langle q_\eta^{*,n}\colon\ \eta \in v_{\mathbf{x}_{n-1}}\rangle$ so $p_{\mathbf{x}_{n-1},\eta} \leqslant q_\eta^{*,n}$ and $\varrho^*_{k_{n,\eta}} \trianglelefteq \mathrm{rt}(q_\eta^{*,n})$.

(2) Now on the side COM chooses $\langle \nu_\eta^n\colon\ \eta \in v_{\mathbf{x}_{n-1}}\rangle$ such that $(\nu_\eta^n \in T^{q_\eta^{*,n}}$ and $\mathrm{nor}(t_{\nu_\eta^n}[q_\eta^{*,n}]) > |v_{\mathbf{x}_{n-1}}| = 2^{n-1}$ and $\varrho^{*,n}_{k_{n,\eta}} \trianglelefteq \upsilon_\eta^n)$.

(3) COM defines $\mathbf{x}_n$ with the following properties:

$$v_{\mathbf{x}_n} = v_{\mathbf{x}_{n-1}} \cup \bigl\{\nu_\eta^n\colon\ \eta \in v_{\mathbf{x}_{n-1}}\bigr\},$$

$$p_{\mathbf{x}_n,\nu} = \begin{cases} p_{\mathbf{x}_{n-1},\nu} & \text{if } \nu \in v_{\mathbf{x}_{n-1}}; \\ (q_\eta^{*,n})^{[\nu]} & \text{if } \eta \in v_{\mathbf{x}_{n-1}} \wedge \nu = \nu_\eta^n, \end{cases}$$

$$I_{\mathbf{x}_n,\nu} = \begin{cases} I_{\mathbf{x}_{n-1},\nu} & \text{if } \nu \in v_{\mathbf{x}_{n-1}}; \\ \mathrm{succ}_{q_\eta^{*,n}}(\nu_\eta^n) & \text{if } \eta \in v_{\mathbf{x}_{n-1}} \wedge \nu = \nu_\eta^n. \end{cases}$$

Now the round is finished.

Now we prove $\mathrm{Pr}^2(\mathbb{P})$. Let $\langle \bar{p}_n, \bar{q}_n, \mathbf{x}_n\colon\ n < \omega\rangle$ be a play in which COM uses **st** and let $u \subseteq \omega$ be infinite. We show how to define q_u.

For $m_1 < m_2 < \omega$ we define a function f_{m_1,m_2} as follows

$$\mathrm{dom}(f_{m_1,m_2}) = \bigl\{\nu\colon\ (\exists\eta \in v_{\mathbf{x}_{m_1}})\bigl(\nu \in I_{\mathbf{x}_{m_1},\eta} \wedge (\exists\rho \in v_{\mathbf{x}_{m_2}})(\nu \trianglelefteq \rho)\bigr)\bigr\}$$

and for $\nu \in \mathrm{dom}(f_{m_1,m_2})$ we let $f_{m_1,m_2}(\nu) \in v_{\mathbf{x}_{m_2}}$ be such that $\nu \lhd f(\nu)$ and $(\neg\exists\rho)(f_{m_1,m_2}(\nu) \lhd \varrho \in v_{\mathbf{x}_{m_2}})$, that is, $f_{m_1,m_2}(\nu)$ is $\lhd$-maximal.

Next we choose $w_{u,n}$ by induction on $n \in u$ such that $w_{u,n} \subseteq v_{\mathbf{x}_n}$.

Case 1: $n = \min(u)$. We let $\eta \in v_{\mathbf{x}_n}$ be $\lhd$-maximal and let $w_{u,n} = \{\eta\}$.

Case 2: $n \in u$, $n > \min(u)$. $m = \max(u \cap n)$,

$$w_{u,n} = w_{u,m} \cup \bigl\{f_{m,n}(\nu)\colon\ (\exists\eta \in w_{u,m})\bigl(\nu \in I_{\mathbf{x}_m,\eta} \wedge (\neg\exists\varrho)(\nu \trianglelefteq \varrho \in v_{\mathbf{x}_m})\bigr)\bigr\}.$$

We define $q_u \in \mathbb{P}$ by induction on n such that by $T^{q_u} \subseteq \mathrm{dcl}(\bigcup_{m\in u} w_{u,m})$. For $\zeta \in T^{q_u}$ we let $f_{m,n}(\nu) = \zeta$ and set $\mathrm{pos}(t_\zeta^{q_u}) \subseteq \mathrm{pos}(t_\zeta^{p_{\mathbf{x}_n,\zeta}})$ and $t_\zeta^{q_u} = t_\zeta^{p_{\mathbf{x}_n,\zeta}}$. Note that $\mathrm{dcl}(\bigcup_{m\in u} w_{u,m})$ is a tree without maxima since $(\forall m < n)(m, n \in u \to (\forall\eta \in w_{u,m})(\exists\nu \in I_{\mathbf{x}_m,\eta})(\exists\varrho \in w_{u,n})(\rho = f(\nu)))$. We show:

$$q_u \Vdash \bigl(\exists^\infty n \in u\bigr)\bigl\{q_\eta^{*,n}\colon\ \eta \in w_{u,n}\bigr\} \text{ is predense.} \tag{$\odot$}$$

Since $w_{u,n} \subseteq v_{\mathbf{x}_n}$ from ($\odot$) we get $q_u \Vdash_{\mathbb{P}} (\exists^\infty n \in u)(\exists\ell < \ell_n)(q_{n,\ell} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}})$. Suppose that ($\odot$) is false. Let $r \geqslant_{\mathbb{P}} q_u$ be a condition such that $r \Vdash (\forall n \in u)(n \geqslant k \to (\forall\eta \in w_{u,k})(q_\eta^{*,k} \notin \underset{\sim}{\mathbf{G}}_{\mathbb{P}}))$. Now we use t-omittoriness. By strengthening r according to Lemma 2.16, we may assume that $\mathrm{rt}(r) \in w_{u,n}$ for some $n \geqslant k$, so, by our construction of q_u, $r \geqslant q_\eta^{*,n}$ for some $\eta \in w_{u,n}$. This is a contradiction. $\square$

Definition 2.20. Let (K, Σ) be a tree-creating pair for H, $k < \omega$.

(1) A tree creature $t \in K$ is called *k-big* if $\mathrm{nor}[t] > 1$ and for every function $h : \mathrm{pos}(t) \to k$ there is $s \in \Sigma(t)$ such that $h \restriction \mathrm{pos}(s)$ is constant an $\mathrm{nor}[s] \geqslant \mathrm{nor}[t] - 1$.
(2) We say (K, Σ) is *k-big* if every $t \in K$ with $\mathrm{nor}[t] > 1$ is k-big.

Although t-omittoriness does not literally imply bigness, it gives an analogue of bigness if the function h from Definition 2.20(1) colours the second but highest level of a tree of possibilities, since this level corresponds to the top level of the tree $\hat{T}$ in the definition of t-omittory. So every tree-creature forcing construction performed with bigness can also be done with t-omittoriness. This sheds some light on the conditions in [16, Lemma 2.3.6 and Theorem 2.3.7].

Remarks. The definitions are taken from [16, Chapters 1–3]. Lemma 2.19 is an analogue to the result that for linear creatures various kinds of limits of norms give the same notion of forcing under a suitable condition on finitariness and omittoriness [16, Proposition 2.1.3]. Proposition 2.18 adds an intermediate step to the implication that for finitary t-omittory pairs (K, Σ) the forcing notion $\mathbb{Q}_0^{\mathrm{tree}}(K, \Sigma)$ is ${}^{\omega}\omega$-bounding (see [16, Conclusion 3.1.1]). Theorem 2.19 is a strengthening of the implication: If $\bigcup_{m<\omega} \mathrm{H}(m)$ is countable and (K, Σ) is a tree-creating pair for H that is t-omittory, then the forcing $\mathbb{Q}_0^{\mathrm{tree}}(K, \Sigma)$ is almost ${}^{\omega}\omega$-bounding (which is [16, Theorem 4.3.9]).

2.4. The case of linear creature forcings

In this section we look at a second main kind of creature forcings, namely forcings with ω-sequences of creatures. This kind is sometimes just called creature forcing, for historic reasons. The best-known examples are Blass–Shelah forcing [3] and the forcing from [21].

Blass–Shelah forcing [3] fulfils the conditions of the next theorem. The assumptions resemble the assumptions made in Proposition 2.18 and Theorem 2.19. Under these conditions, the various limit conditions on the divergence of norms coincide: $\mathbb{Q}^*_{w,\infty}(K, \Sigma)$ and $\mathbb{Q}^*_{\infty}(K, \Sigma)$ and $\mathbb{Q}^*_{s,\infty}(K, \Sigma)$ are equivalent forcings by [16, Proposition 2.1.3]. We first recall these families of notions of forcing:

Definition 2.21. Suppose that (K, Σ) is a weak creating pair for $\mathbf{H}$ and $\mathcal{C}(\mathrm{nor})$ is a property of ω-sequences of weak creatures from K (i.e., $\mathcal{C}(\mathrm{nor})$ is a subset of K^{ω}). We define the forcing notion $\mathbb{Q}_{\mathcal{C}(\mathrm{nor})}(K, \Sigma)$. Conditions are pairs (w, T) such that for some $k_0 < \omega$,

(a) $w \in \prod_{i<k_0} \mathbf{H}(i)$;
(b) $T = \langle t_i \colon i < \omega \rangle$ where
 (i) $t_i \in K$ for each i;
 (ii) $w \in \mathit{basis}(t_i)$ for some $i < \omega$, and for each finite set $I_0 \subseteq \omega$ and $u \in \mathrm{pos}(w, \{t_i \colon i \in I_0\})$ there is $i \in \omega \setminus (\max(I_0) + 1)$ such that $u \in \mathit{basis}(t_i)$;
(c) the sequence $\langle t_i \colon i < \omega \rangle$ satisfies the conditions $\mathcal{C}(\mathrm{nor})$.

The order is given by $(w_1, T^1) \leqslant (w_2, T^2)$ if and only if for some disjoint sets $\mathcal{S}_0, \mathcal{S}_1, \ldots \subseteq \omega$ we have $w_2 \in \mathrm{pos}(w_1, \{t^1_\ell \colon \ell \in \mathcal{S}_0\})$ and $t^2_i \in \Sigma(\{t^1_\ell \colon \ell \in \mathcal{S}_{i+1}\})$ for each $i < \omega$ where $T^i = \langle t^i_i \colon i < \omega \rangle$.

If $p = (w, T)$ we let $w^p = w$ and $T^p = T$ and if $T^p = \langle t_i \colon i < \omega \rangle$ then we let $t^p_i = t_i$. We may write $(w, t_0, t_1 \ldots)$ instead of (w, T) when $T = \langle t_i \colon i < \omega \rangle$.

If (K, Σ) is a weak creating pair and $\mathcal{C}(\mathrm{nor})$ is a property of sequences of elements of K then $\mathbb{Q}_{\mathcal{C}}(\mathrm{nor})$ is a forcing notion. Now we explain what properties $\mathcal{C}(\mathrm{nor})$ are meant in (c) of the previous definition.

Definition 2.22. For a weak creature t let us denote

$$m^t_{\mathrm{dn}} = \min\bigl\{\lg(u)\colon\ u \in \mathrm{dom}\bigl(\mathrm{val}[t]\bigr)\bigr\}.$$

We introduce the following basic properties of sequences of weak creatures which may serve as $\mathcal{C}(\mathrm{nor})$

($s\infty$) A sequence $\langle t_i\colon\ i < \omega\rangle$ satisfies $\mathcal{C}^{s\infty}(\mathrm{nor})$ if and only if

$$(\forall i < \omega)\bigl(\mathrm{nor}(t_i) > \max\bigl\{i, m_{\mathrm{dn}}(t_i)\bigr\}\bigr).$$

(∞) A sequence $\langle t_i\colon\ i < \omega\rangle$ satisfies $\mathcal{C}^{\infty}(\mathrm{nor})$ if and only if

$$\lim_{i\to\infty} \mathrm{nor}(t_i) = \infty.$$

($w\infty$) A sequence $\langle t_i\colon\ i < \omega\rangle$ satisfies $\mathcal{C}^{w\infty}(\mathrm{nor})$ if and only if

$$\limsup_{i\to\infty} \mathrm{nor}(t_i) = \infty.$$

The forcing notions corresponding to the above properties for a weak creating pair (K, Σ) will be denoted by $\mathbb{Q}_{s\infty}(K, \Sigma)$, $\mathbb{Q}_{\infty}(K, \Sigma)$, $\mathbb{Q}_{w\infty}(K, \Sigma)$.

Adding more properties to a weak creature gives an **H**-creature:

Definition 2.23. Let t be a weak creature for **H**.

(1) If there is $m < \omega$ such that $\forall\langle u, v\rangle \in \mathrm{val}[t]$, $\lg(u) = m$, then this unique m is called m^t_{dn}.
(2) If there is $m < \omega$ such that $\forall\langle u, v\rangle \in \mathrm{val}[t]$, $\lg(v) = m$, then this unique m is called m^t_{up}.
(3) If both m^t_{dn} and m^t_{up} are defined then t is called an $(m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$-creature of just a creature.
(4) $\mathrm{CR}_{m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}}[\mathbf{H}] = \{t \in \mathrm{WCR}[\mathbf{H}]\colon\ m^t_{\mathrm{dn}} = m_{\mathrm{dn}}, m^t_{\mathrm{up}} = m_{\mathrm{up}}\}$. The set $\mathrm{CR}[\mathbf{H}] = \bigcup_{m_{\mathrm{dn}} < m_{\mathrm{up}} < \omega} \mathrm{CR}_{m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}}[\mathbf{H}]$ is called the set of **H**-creatures.

Definition 2.24. Suppose that $K \subseteq \mathrm{CR}[\mathbf{H}]$ and Σ is a sub-composition operation on K. We say that Σ is a *composition on* K and we say (K, Σ) is a *creating pair for* **H** if

(1) if $\mathcal{S} \in [K]^{\leqslant\omega}$ and $\Sigma(\mathcal{S}) \neq \emptyset$ then $\mathcal{S}$ is finite and for some enumeration $S = \{t_0, \ldots, t_{m-1}\}$ we have $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$ for $i < m - 1$, and
(2) for each $s \in \Sigma(t_0, \ldots, t_{m-1})$ we have $m^s_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$ and $m^s_{\mathrm{up}} = m^{t_{m-1}}_{\mathrm{up}}$.

Definition 2.25. Let (K, Σ) be a creating pair and $\mathcal{C}(\mathrm{nor})$ be a property of ω-sequences of creatures. The forcing notion $\mathbb{Q}^*_{\mathcal{C}(\mathrm{nor})}(K, \Sigma)$ is a suborder of $\mathbb{Q}_{\mathcal{C}(\mathrm{nor})}(K, \Sigma)$ consisting of these conditions $(w, t_0, t_1, \ldots)$ for which additionally $\forall i \in \omega$, $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$.

Definition 2.26. Let (K, Σ) be a weak creating pair for **H**.

(1) For $t \in K$, $m_0 \leqslant m^t_{\mathrm{dn}}$, $m^t_{\mathrm{up}} \leqslant m_1$ we define the creature $s = t \restriction [m_0, m_1)$ by

$$\mathrm{nor}[s] = \mathrm{nor}[t],$$

$$\mathrm{val}[s] = \Big\{ \langle w,u\rangle \in \prod_{i<m_0} \mathbf{H}(i) \times \prod_{i<m_1} \mathbf{H}(i)\colon\ \langle v \restriction m^t_{\mathrm{dn}}, u \restriction m^t_{\mathrm{up}}\rangle \in \mathrm{val}[t] \wedge$$

$$w \lhd u \wedge \big(\forall i \in \big[m_0, m^t_{\mathrm{dn}}\big) \cup \big[m^t_{\mathrm{up}}, m_1)\big)\big(u(i) = 0\big)\Big\}.$$

Note that $t \upharpoonright [m_0, m_1)$ is well-defined only if $\mathrm{val}[s] \neq \emptyset$ and then $m^s_{\mathrm{dn}} = m_0$ and $m^s_{\mathrm{up}} = m_1$.

(2) The creating pair (K, Σ) is *omittory* if it has the following properties:
  - (o$_1$) If $t \in K$ and $u \in \mathit{basis}(t)$ then $u^\frown 0_{[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})} \in \mathrm{pos}(u,t)$ but there is $v \in \mathrm{pos}(u,t)$ such that $v \restriction [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}) \neq 0_{[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})}$.
  - (o$_2$) For every $(t_0, \ldots, t_{n-1})$ sequence of (K, Σ)-creatures, if for every $i < n$, $m^{t_{i+1}}_{\mathrm{dn}} = m^{t_i}_{\mathrm{up}}$ then for every $i < n$, $t_i \upharpoonright [m^{t_0}_{\mathrm{dn}}, m^{t_{n-1}}_{\mathrm{up}}) \in \Sigma(t_0, \ldots, t_{n-1})$.
  - (o$_3$) If $t, t \upharpoonright [m_0, m_1) \in K$ then for every $u \in \mathit{basis}(t \upharpoonright [m_0, m_1))$ and $v \in \mathrm{pos}(u, t \upharpoonright [m_0, m_1))$ we have

$$v(n) \neq 0 \wedge n \in \big[\lg(u), \lg(v)\big) \to n \in \big[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})\big).$$

  Note that (o$_1$) implies that in the cases relevant for (o$_2$) the creature $t \upharpoonright [m^{t_0}_{\mathrm{dn}}, m^{t_{n-1}}_{\mathrm{up}})$ is well defined.

Definition 2.27. An omittory creating pair (K, Σ) is *omittory-big* if for every $k < \omega$ there is $m < \omega$ such that if $t \in K$, $\mathrm{nor}(t) > m$, $u \in \mathit{basis}(t)$, $c\colon \mathrm{pos}(u,t) \to \{0,1\}$ then there is $s \in \Sigma(t)$ such that $\mathrm{nor}(s) \geqslant k$ and $c \restriction \mathrm{pos}(u,s) \setminus \{0_{[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})}\}$ is constant. We call m an *omittory bigness witness for k*.

Definition 2.28. (K, Σ) is finitary, that means every $\mathbf{c}^p_i$ has a finite range and $\Sigma(\mathcal{S}) \neq \emptyset$ only for finite subsets $\mathcal{S} \subseteq K$ and also $\Sigma(\mathbf{c}_0, \ldots, \mathbf{c}_{n-1})$ is finite.

If $p = (\eta^p, \mathbf{c}^p_0, bc^p_1 \ldots)$ is a condition and $n \in \omega$, and $\nu \in \mathrm{pos}(\eta^p, \mathbf{c}^p_0, \ldots, c^p_{n-1})$ then we let $p^{[\nu]} = (\nu, \mathbf{c}^p_n, \mathbf{c}^p_{n+1}, \ldots)$.

Theorem 2.29. *Assume $\mathbb{P} = \mathbb{Q}^*_{w\infty}(K, \Sigma)$ is finitary and omittory and is omittory-big. Then $\mathbb{Q}$ is $(T, Y, \mathcal{S})$-preserving.*

Proof. Assume that $\chi \geqslant 2^{2^\omega}$ and $N \prec \mathcal{H}(\chi)$ is countable and that $\mathbb{P} \in N$, $p \in N \cap \mathbb{P}$, $T, \mathcal{S}, Y \in N$. Let $\delta = N \cap \omega_1$ and $N \cap \omega_1 \in \mathcal{S}$. Let T be $(Y, \mathcal{S})$-proper. Assume that $p = (\eta^p, \mathbf{c}^p_0, \mathbf{c}^p_1, \ldots)$.

We show that there is $q \geqslant p$ that is $(N, \mathbb{P})$-generic and such that for every $t \in Y(\delta)$,

$$q \Vdash T_{<_T t} \text{ is } \big(N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T\big)\text{-generic.}$$

Now we use the Axiom A structure: We enumerate all the $\mathbb{P}$-names $\underset{\sim}{\mathcal{I}} \in N$ for dense sets in T as $\{\underset{\sim}{\mathcal{I}}_n\colon n < \omega\}$, all the $\mathcal{J} \in N$ that are dense in $\mathbb{P}$ as $\{\mathcal{J}_n\colon n < \omega\}$ and all the $t \in Y(\delta)$ as $\{t_n\colon n < \omega\}$. We choose p_n by induction on $n \in [n_*, \omega)$ such that

(a) $p_n \in \mathbb{P} \cap N$,
(b) $p_n \leqslant p_{n+1}$,
(c) $p_{n^*} = p$,
(d) for some countable $\mathcal{J}^*_n \subseteq \mathcal{J}_n$ $\mathcal{J}^*_n$ is predense above p_{n+1},
(e) if $k > n \geqslant n_*$ then $\mathrm{nor}[\mathbf{c}^{p_n}_k] \geqslant n$,
(f) $(\eta^{p_n}, \mathbf{c}^{p_n}_0, \mathbf{c}^{p_n}_1, \ldots, \mathbf{c}^{p_n}_{n-1}) = (\eta^{p_{n+1}}, \mathbf{c}^{p_{n+1}}_0, \mathbf{c}^{p_{n+1}}_1, \ldots, \mathbf{c}^{p_{n+1}}_{n-1})$,
(g) if $\nu \in \mathrm{pos}(\eta^{p_n}, \mathbf{c}^{p_n}_0, \mathbf{c}^{p_n}_1, \ldots, \mathbf{c}^{p_n}_{n-1})$ and there are $q \geqslant p_n$ and $s \in T$ satisfying $(*)_{n,\nu,p,q,s}$ below, then $(*)_{n,\nu,p_n,p^{[\nu]}_{n+1},s}$.

Here we use

$$\begin{aligned}
&\nu \in \operatorname{pos}\bigl(\eta^p, \mathbf{c}_0^p, \mathbf{c}_1^p, \dots, \mathbf{c}_{n-1}^p\bigr) \wedge \\
&\eta^q = \nu \wedge \\
&q \geqslant \bigl(\nu, \mathbf{c}_n^p, \dots\bigr) \geqslant p \wedge \\
&s <_T t_n \wedge \\
&q \Vdash s \in \bigcap_{k<n} \underset{\sim}{\mathcal{I}}_k .
\end{aligned} \qquad (*)_{n,\nu,p,q,s}$$

There is no problem in carrying this induction as $\mathbb{P}$ is finitary and omittory.

In the end we let

$$p_\omega = \lim_{n\to\omega} p_n = \bigl(\eta^p, \mathbf{c}_0^{p_{n_*}}, \dots, \mathbf{c}_{n_*-1}^{p_{n_*}}, \mathbf{c}_{n_*}^{p_{n_*+1}}, \mathbf{c}_{n_*+1}^{p_{n_*+2}}, \dots\bigr).$$

By (f), p_ω is $(N, \mathbb{P})$-generic. Now we strengthen p_ω once more to get a condition $p_{\omega+1} \geqslant p_\omega$ that forces that every $t \in Y(\delta)$ is $(N[\mathbf{G}_{\mathbb{P}}], T)$-generic. This strengthening is carried out as follows:

Now for $n < \omega$ we let $C_n = \operatorname{pos}(\eta^{p_\omega}, \mathbf{c}_0^{p_\omega}, \dots, \mathbf{c}_{n-1}^{p_\omega})$. So $C = \bigcup_{n<\omega} C_n$ is a tree. We colour this tree in two colours: $\mathbf{c} : C \to \{\text{yes}, \text{“no”}\}$ for $\nu \in C$, $\mathbf{c}(\nu) = \text{“yes”}$, iff for some $s <_T t$, $(*)_{n,\nu,p_\omega,p_{\omega[\nu]},s}$ and no otherwise. If $n \leqslant n_1 < n_2$ and $\nu_i \in C_{n_i}$ and $\nu_1 \lhd \nu_2$ and $\mathbf{c}(\nu_1) = \text{yes}$, then $\mathbf{c}(\nu_2) = \text{yes}$, since $p_\omega^{[\nu_1]} \leqslant p_\omega^{[\nu_2]}$.

Now by [16, Theorem 2.2.6] we have the following consequence of omittory-big: There is $p_{\omega+1} \geqslant_0 p_\omega$ such that the following holds: If $\nu_i \in C_{n_i}$, $n_1 < n_2$, and $\nu_i \in \{\operatorname{rt}(q)\colon\ p_{\omega+1} \leqslant q\}$ and $\nu_1 \lhd \nu_2$ then $\mathbf{c}(\nu_1) = \mathbf{c}(\nu_2)$.

We check that the uniform colour is “yes”. Suppose for a contradiction that $(\forall \nu_* \rhd \eta^{p_{\omega+1}})(\mathbf{c}(\nu_*) = \text{no})$. We let $p_{\omega+2} = (\nu_*, \mathbf{c}_{m_*}^{p_{\omega+1}}, \mathbf{c}_{m_*+1}^{p_{\omega+1}}, \dots) \geqslant p_{\omega+1}$ for a suitable m_* with $m_* \in C$. So there are $s < t_{m_*}$ and $q \geqslant p_{\omega+2}$ with $q \Vdash s \in \bigcap_{k<m_*} \mathcal{I}_k$. As $\mathcal{I}_k$, $k < m_*$, are dense subsets of $(T, <_T)$ that have names in N there is such a pair (s, q). Now $\mathbf{c}(\operatorname{rt}(q)) = \text{yes}$. So the uniform colour cannot be “no”. □

We recall Blass–Shelah forcing in order to see that it fulfils the conditions of the previous theorem.

Definition 2.30. We define a depth function on $\{A \subseteq [\omega]^{<\omega}\colon\ 2 \leqslant |A| < \omega\}$ as follows:

$$\begin{aligned}
&\operatorname{dp}(A) \geqslant 0, && \text{always,}\\
&\operatorname{dp}(A) \geqslant 1, && \text{if } A \neq \emptyset,\\
&\operatorname{dp}(A) \geqslant n+2, && \text{if for every set } X \subseteq \omega \text{ one of the following conditions holds}\\
& && \operatorname{dp}\bigl(\{a \in A\colon\ a \subseteq X\}\bigr) \geqslant n+1,\\
& && \text{or } \operatorname{dp}\bigl(\{a \in A\colon\ a \subseteq \omega \setminus X\}\bigr) \geqslant n+1.
\end{aligned}$$

Definition 2.31. Blass–Shelah forcing is $\mathbb{Q}^*_{s\infty}(K, \Sigma)$ with the following creating pair (K, Σ): We let $H(m) = 2$ for $m \in \omega$. A creature $t \in \mathrm{CR}[H]$ is in K if $m^t_{\mathrm{dn}} + 2 < m^t_{\mathrm{up}}$ and there is a sequence $\langle A^t_u\colon\ u \in \prod_{i<m^t_{\mathrm{dn}}} H(i)\rangle$ such that for every $u \in \prod_{i<m^t_{\mathrm{dn}}} H(i)$ the following holds:

(α) A^t_u is a non-empty family of subsets of $[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$ such that each member of A^t_u has at least 2 elements,
(β) $\langle u, v\rangle \in \operatorname{val}[t]$ iff $u \lhd v$ and $\{i \in [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})\colon\ v(i) = 1\} \in A^t_u \cup \{\emptyset\}$,
(γ) $\operatorname{nor}(t) = \min\{\log_2(\operatorname{dp}(A^t_u))\colon\ u \in \prod_{i<m^t_{\mathrm{dn}}} H(i)\}$.

Suppose $t_0, \dots, t_n$ in K are such that $m^{t_{i+1}}_{\mathrm{dn}} = m^{t_i}_{\mathrm{up}}$ for $i < n$. Then $s \in \Sigma(t_0, \dots, t_n)$ iff $s \in K$ and $m^s_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$ and $m^s_{\mathrm{up}} = m^{t_n}_{\mathrm{up}}$ and for every $\langle u, v\rangle \in \operatorname{val}(s)$ for every $i \leqslant n$, $\langle v \restriction m^{t_i}_{\mathrm{dn}}, v \restriction m^{t_i}_{\mathrm{up}}\rangle \in \operatorname{val}[t_i]$.

Blass–Shelah forcing is finitary, omittory and omittory-big. So by Theorem 2.29 it is $(T, Y, \mathcal{S})$-preserving. There is a parallel result without the property "omittory" but with strong enough bigness and halving.

Theorem 2.32. *Assume that $\mathbb{P} = \mathbb{Q}_{\mathrm{w}}(K, \Sigma)$ is creature forcing with the following properties*:

(a) $p \in \mathbb{P}$ *has the form* $(f, \mathbf{c}_0, \mathbf{c}_1, \ldots) = (w^p, \mathbf{c}_0^p, \mathbf{c}_1^p, \ldots)$ *with* $\liminf\langle \mathrm{nor}(\mathbf{c}_n)\colon n < \omega\rangle = \infty$.
(b) (K, Σ) *is finitary.*
(c) *For some sufficiently fast increasing sequence* $\bar{k} = \langle k_i\colon i < \omega\rangle$ *we have the following strong versions of bigness and halving*: *First, we assume that there is a function* $i\colon K \to \omega$ *such that*
  – $\mathbf{c} \in \Sigma(\mathbf{c}_0, \ldots, \mathbf{c}_{n-1}) \to i(\mathbf{c}) \leqslant \max\{i(\mathbf{c}_j)\colon j < n\}$,
  – *in every condition* p, $i(\mathbf{c}_0^p) < i(\mathbf{c}_1^p) < i(\mathbf{c}_2^p) \ldots$,
  – *for every* $\mathbf{c} \in K$ *and* n *we have* $|\{(f, \mathbf{c}_0^p, \ldots, \mathbf{c}_{n-1}^p)\colon p \in \mathbb{P} \wedge \mathbf{c}_{n-1}^p = \mathbf{c}\}| \leqslant k_{i(\mathbf{c})}$.
  Now for such a sequence $\bar{k}$ *and function* i *we require*:
  (α) $\mathrm{nor}(\mathbf{c}) \in \{\frac{m}{n}\colon n \leqslant k_{i(\mathbf{c})}, m \leqslant k_{i(\mathbf{c})}!!\}$,
  (β) *for every* $p \in \mathbb{P}$, $n \in \omega$, $|\operatorname{pos}(f^p, \mathbf{c}_0^p, \ldots, \mathbf{c}_{n-1}^p)| \ll k_{i(\mathbf{c}_n^p)}$,
  (γ) (*Bigness*) *for every* $p \in \mathbb{P}$, $n \in \omega$, $d\colon \operatorname{pos}(f^p, \mathbf{c}_0^p, \ldots, \mathbf{c}_n^p) \to k_{i(\mathbf{c}_n^p)}$ *there is* $\mathbf{c} \in \Sigma(\mathbf{c}_n^p)$ *such that* $\mathrm{nor}(\mathbf{c}) \geqslant \mathrm{nor}(\mathbf{c}_n^p) - \frac{1}{k_{i(\mathbf{c}_n^p)}}$ *and for every* $g \in \operatorname{pos}(f, \mathbf{c}_0^p, \ldots, \mathbf{c}_{n-1}^p)$, $d \restriction \operatorname{pos}(g, \mathbf{c})$ *is constant,*
  (δ) (*Halving with gluing*) *if* $p \in \mathbb{P}$, $m(*) < \omega$ *then we can find* $q \in \mathbb{P}$ *with the following properties*:
    – $p \leqslant q$,
    – $f^p = f^q$,
    – $\mathbf{c}_m^p = \mathbf{c}_m^q$ *for* $m < m(*)$,
    – *if* $m \geqslant m(*)$ *then* $\mathrm{nor}(\mathbf{c}_m^q) \geqslant \inf\{\mathrm{nor}(\mathbf{c}_\ell^p)\colon \ell \in [m(*), \infty)\} - \frac{1}{k_{i(\mathbf{c}_{m(*)}^p)}}$,
    – *if* $q \leqslant r$, $f^r = f^q$, $\mathbf{c}_m^q = \mathbf{c}_m^r$ *for* $m < m(*)$ *and* $\mathrm{nor}(\mathbf{c}_m^r) \geqslant 1$ *for* $m \geqslant m(*)$ *then there is* q_1 *such that*
      ($*$) $p \leqslant q_1$,
      ($*$) $f^{q_1} = f^p$,
      ($*$) $\mathbf{c}_m^{q_1} = \mathbf{c}_m^p$ *for* $m < m(*)$,
      ($*$) *if* $m \geqslant m(*)$ *then* $\mathrm{nor}(\mathbf{c}_m^q) \geqslant \inf\{\mathrm{nor}(\mathbf{c}_\ell^p)\colon \ell \in [m(*), \infty)\} - \frac{1}{k_{i(\mathbf{c}_{m(*)}^p)}}$,
      ($*$) q_1 *and* r *are equivalent in a strong sense for some* $n(*) \geqslant m(*)$ *we have* $m \geqslant n(*) \to \mathbf{c}_m^{q_1} = \mathbf{c}_m^r$ *and* $\operatorname{pos}(f^{q_1}, \mathbf{c}_0^{q_1}, \ldots, \mathbf{c}_{n(*)-1}^{q_1}) = \operatorname{pos}(f^r, \mathbf{c}_0^r, \ldots, \mathbf{c}_{n(*)-1}^r)$.

Then $\mathbb{P}$ *is* $(T, \mathcal{S}, Y)$*-preserving.*

Proof. Assume that $\chi \geqslant 2^{2^\omega}$ and $N \prec \mathcal{H}(\chi)$ is countable and that $\mathbb{P} \in N$, $p \in N \cap \mathbb{P}$, $T, \mathcal{S}, Y \in N$. Let $\delta = N \cap \omega_1$ and $N \cap \omega_1 \in \mathcal{S}$. Let T be $(Y, \mathcal{S})$-proper. Assume that $p = (\eta^p, \mathbf{c}_0^p, \mathbf{c}_1^p, \ldots)$.

We show that there is $q \geqslant p$ that is $(N, \mathbb{P})$-generic and such that for every $t \in Y(\delta)$,

$$q \Vdash T_{<_T t} \text{ is } \big(N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T\big)\text{-generic}.$$

We enumerate all pairs $(\underset{\sim}{\mathcal{I}}, t)$ of $\mathbb{P}$-names $\underset{\sim}{\mathcal{I}} \in N$ for dense sets in T and $t \in Y(\delta)$ as $\{(\underset{\sim}{\mathcal{I}}_n, t_n)\colon n < \omega\}$, all the $\mathcal{J} \in N$ that are dense in $\mathbb{P}$ as $\{\mathcal{J}_n\colon n < \omega\}$, each object appearing infinitely often in each enumeration.

We choose (p_n, m_n) by induction on $n \in \omega$ such that

(a) $p_n \in \mathbb{P} \cap N$,
(b) $p_n \leqslant p_{n+1}$,
(c) $m_n < m_{n+1} < \omega$, $m_0 = 0$,
(d) $p_n \Vdash (\exists t < t_n)(t \in \bigcap_{k \leqslant n} \underset{\sim}{\mathcal{I}}_k)$,

(e) $p_0 = p$,
(f) $p_n \in \mathcal{J}_n$,
(g) $f^{p_n} = f^p$,
(h) $m_n \geqslant \min\{m > m_{n-1}\colon (\forall r \geqslant m)(\mathrm{nor}(\mathbf{c}_r^{p_{n-1}}) \geqslant n+1)\}$,
(i) if $m < m_n$, then $\mathbf{c}_m^{p_n} = \mathbf{c}_m^{p_{n-1}}$.

If we succeed then we can take the fusion

$$q = \left(f^{p_0}\mathbf{c}_0^{p_0}, \dots, \mathbf{c}_{m_0-1}^{p_0}, \mathbf{c}_{m_0}^{p_1}, \dots, \mathbf{c}_{m_1-1}^{p_1}, \dots\right)$$

and by (h) and (i), q fulfils the norm conditions and hence $q \in \mathbb{P}$, and obviously $q \geqslant p$.

So suppose that p_n and m_n have been defined we are to define p_{n+1}.

Let $i_n = i(\mathbf{c}_{m_n}^{p_n})$ and let $\{g_\ell\colon \ell < \ell_n\}$ list $\mathrm{pos}(f^{p_n}, \mathbf{c}_0^{p_n}, \dots, \mathbf{c}_{m_n-1}^{p_n})$. By the conditions on $\mathbb{P}$, we have $\ell_n \leqslant k_i$.

Now we choose $p_{n,\ell}$ by induction on $\ell < \ell_n$ such that

(a) $p_{n,\ell} \in \mathbb{P} \cap N$,
(b) $p_{n,0} = p_n$,
(c) $f^{p_n} = f^{p_{n,\ell}}$,
(d) if $m < m_n$ then $\mathbf{c}_m^{p_n} = \mathbf{c}_m^{p_{n,\ell}}$,
(e) if $m \geqslant m_n$ then $\mathrm{nor}(\mathbf{c}_m^{p_{n,\ell}}) \geqslant n+1-\frac{1}{k_{i_n}}$,
(f) if there is $q = (g_\ell, \mathbf{c}_{m_n}^q, \mathbf{c}_{m_n+1}^q, \dots) \geqslant p_{n,\ell}$ such that $q \in \mathcal{I}_n$ then $\mathbf{c}_j^q = \mathbf{c}_j^{p_{n,\ell+\frac{1}{2}}}$ for $j \geqslant m_n$; otherwise we apply halving to $(g_\ell, \mathbf{c}_{m_n}^{p_{n,\ell}}, \mathbf{c}_{m_n+1}^{p_{n,\ell}}, \dots)$ and get q as in the halving with gluing, and let again $\mathbf{c}_j^q = \mathbf{c}_j^{p_{n,\ell+\frac{1}{2}}}$ for $j \geqslant m_n$,
(g) if there is $q = (g_\ell, \mathbf{c}_{m_n}^q, \mathbf{c}_{m_n+1}^q, \dots) \geqslant p_{n+\frac{1}{2},\ell}$ such that $q \Vdash (\exists t \in T)(t < t_n \wedge t \in \mathcal{J}_n)$ then $\mathbf{c}_j^q = \mathbf{c}_j^{p_{n,\ell+1}}$ for $j \geqslant m_n$; otherwise we apply halving to $(g_\ell, \mathbf{c}_{m_n}^{p_{n,\ell+\frac{1}{2}}}, \mathbf{c}_{m_n+1}^{p_{n,\ell+\frac{1}{2}}}, \dots)$ and get q as in the halving with gluing, and let again $\mathbf{c}_j^q = \mathbf{c}_j^{p_{n,\ell+1}}$ for $j \geqslant m_n$.

It is easy to carry on the induction. In the end we let $p_{n+1} = p_{n,\ell_n}$. Now we have to show that for each $\mathcal{I}$ and each $(t, \underset{\sim}{\mathcal{J}})$ (that appear under infinitely many indices) after finitely many of these n where e.g. $\mathcal{I}_n = \mathcal{I}$, in items (f) and (g) the first alternative will be applied. This is because of the strong version of bigness. We colour $\mathrm{pos}(\mathbf{c}^p, \mathbf{c}_0^{p_n} \dots \mathbf{c}_{m_n-1}^{p_n}, \mathbf{c}_{m_n}^{p_n})$ by $\{0,1\}$ assigning $c(\hat{g}) = 1$ if there is q with $f^q = \hat{g}$ (no conditions on the rest) and $q \in \mathcal{I}$ (in the case of (g): and $q \Vdash (\exists t')(t' < t \wedge t' \in \underset{\sim}{\mathcal{J}})$). For every $\hat{g} \in \mathrm{pos}(\mathbf{c}^p, \mathbf{c}_0^{p_n} \dots \mathbf{c}_{m_n-1}^{p_n})$ there is a uniform colour. Now we go one level back: For "most" of the $\hat{g} \in \mathrm{pos}(\mathbf{c}^p, \mathbf{c}_0^{p_n} \dots \mathbf{c}_{m_n-1}^{p_n})$, their uniform colour is the same, and for most of the most of the next level and so on. So we get back to the root. Its colour is at some time n the colour 1, since otherwise we succeed in constructing the fusion p_ω that has no extension in $\mathcal{I}$ or no extension in $\{r \in \mathbb{P}\colon r \Vdash (\exists t' < t)(t' \in \underset{\sim}{\mathcal{J}})\}$, so $\mathcal{J}^* = \{s \in T\colon s \not\leq t \vee p_\omega \Vdash s \notin \underset{\sim}{\mathcal{J}}\}$ is dense in T and witnessing that T is not $(\mathcal{S}, Y)$-proper. In any case this is a contradiction. So we get $\mathbf{c}'_{m_n}$ with large norm and colour 1 and are done. There are ℓ_n substeps and in each step we lose maximally $\frac{1}{k_n}$ of $\mathrm{nor}(\mathbf{c}_{m_n}^{p_n})$ so in the end it is still large enough for a fusion. □

3. A sufficient condition for $(T, Y, \mathcal{S})$-preserving for nep forcings

The property of preserving Cohen generic reals over countable elementary submodels proved to be a useful property of forcings. Preserving Cohen reals is slightly stronger than preserving non-meager sets (see [16, Section 3.2]). Preserving Cohen reals is preserved in countable support iterations [22, Chapter XVIII, 3.10]. In this section we show that a relative of this property, namely "$\mathbb{P}$ preserves ω Cohen reals over countable elementary submodels and over certain transitive models called candidates", guarantees that $\mathbb{P}$ preserves Souslin trees. The candidates will replace the elementary $N \prec \mathcal{H}(\chi)$. When a forcing notion $\mathbb{P}$ has also for these non-elementary countable models suitable generic conditions then $\mathbb{P}$ is called "nep" — non-elementary proper. There are many versions of this definition: We can specify which candidates are considered and which conditions are imposed on genericity. A standard reference to non-elementary proper forcing is [23].

Let $N \prec \mathcal{H}(\chi)$. $x \in {}^{\omega}\omega$ is called Cohen over N, if for every comeager G_δ-set $C \subseteq {}^{\omega}\omega$ with code in N, $x \in C$. For Borel codes see [8, Section 25, p. 504]. We recall the original definition for proper forcing with elementary submodels, [16, Definition 3.2.1].

Definition 3.1.

(1) Let $\mathbb{P}$ be a proper forcing notion. We say $\mathbb{P}$ is *ω-Cohen preserving* iff the following holds: For every $N \prec \mathcal{H}(\chi)$ such that $\mathbb{P} \in N$, for every $p \in \mathbb{P} \cap N$ for every $\{x_n \colon n \in \omega\}$ such that every x_n is a Cohen real over N, there is an $(N, \mathbb{P})$-generic condition $q \geqslant p$ such that

$$q \Vdash (\forall n \in \omega)\bigl(x_n \text{ is Cohen over } N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}]\bigr).$$

(2) $\mathbb{P}$ is *Cohen preserving* iff the above holds for just one Cohen real.

By [22, Chapter XVIII, 3.10] also ω-Cohen preserving is preserved in countable support iterations. Cohen forcing itself is Cohen preserving, whereas random forcing is not, since the ground model reals are a meager set in the extension. For creature forcings [16, Chapter 3] gives some structural properties on the building blocks of the forcing that imply Cohen preserving.

The notion "nep" — non-elementary proper — was introduced and investigated in [23] and it is actually a reach family of notions with many parameters. We give a short introduction to our instance of nep. Our presentation is a compromise between at least covering all the creature forcings from [16] and many technicalities. Explanations and useful work with nep forcings can also be found in [10].

In one respect we introduce more technique than needed for the creature forcings from [16]: We like to allow a parameter $\mathfrak{B}$ with domain $|\mathfrak{B}| \subseteq \mathcal{H}(\omega_1)$ and countable signature.

Why are we so interested in allowing definitions with parameters $\mathfrak{B}$? In the light of the theorem in this section, an interesting question is to consider to which extent forcings specialising Aronszajn trees (by finite approximations, by countable approximation as in [13] or by uncountable conditions as in the NNR forcing from [22, Chapter V, §6]) are nep. Here are some partial answers:

In all ground models in which then NNR does not add reals it is ω-Cohen preserving. However, as our Theorem 3.11 shows, NNR is not ω-Cohen preserving in other models $M[G]$, where G collapses ω_1 of M or it is not nep in the strong sense required in the theorem. The NNR forcings are defined with Aronszajn trees as parameters in the definition. An Aronszajn with its tree order tree can be written as a subset of $H(\omega_1)$ and so still is a parameter allowed in size in the definitions of nep we give.

The forcings from [13] add a real that makes the ground model meager (this is not yet published work by Mildenberger and Shelah), and hence they are not Cohen preserving.

Definition 3.2.

(1) A *fragment* ZFC^* is an $\mathcal{L}(\in)$-theory extending ZC^-, ZFC without replacement and without the power set axiom.
(2) Let K be a class of notions of forcing. We say ZFC^* is *K-good*, if $\mathbb{P}$ is a forcing notion in K and $\beth_\omega(|\mathbb{P}|)$ exists then the forcing $\mathbb{P}$ preserves ZFC^*.

Now let T be a fragment of ZFC and K be a definable class of forcing notions or a set of forcings. Then by using the definability of $\Vdash_{\mathbb{P}}$ for $\mathbb{P} \in K$ [11, Chapter VII, §4] and adding successively the requirements $\Vdash_{\mathbb{P}} \phi$ for ϕ in the previous stage we get a fragment $T_1 \supseteq T$ that $M \models T_1$ ensures $M[G_{\mathbb{P}}] \models T$. Now we iterate and take the union. This need not be a finite fragment anymore. So in practice, in order to get consistency relative to ZFC we take $\mathsf{ZFC}^* = \mathsf{ZC}^-$. Then for every uncountable regular κ, $(H(\kappa), \in) \models \mathsf{ZFC}^*$. Now if no forcing in K collapses κ to ω or to a singular cardinal, ZFC^* is K-good.

We fix $\lambda = (2^{|H(\omega_1)|})^+$, $\chi = |H(\lambda)|^{++}$ the set $K = \{\mathrm{Levy}(\aleph_0, \lambda)\colon\ \lambda \text{ regular uncountable cardinal}, \lambda < \chi\}$ of notions of forcing.

Definition 3.3. A theory $\mathsf{ZFC}^* \subseteq \mathsf{ZFC}$ is called *normal* if the following holds:

$$\text{For every sufficiently large regular } \chi, \mathcal{H}(\chi) \models \mathsf{ZFC}^*.$$

We assume that the forcing $\mathbb{P}$ is defined by formulas $\phi_0(x)$ and $\phi_1(x, y)$ that describe $x \in \mathbb{P}$ and $x \leqslant_{\mathbb{P}} y$. The formulas are in a countable language $\tau \subset H(\omega)$ and use a parameter $\mathfrak{B} \subseteq H(\omega_1)$. We let $\bar{\phi} = (\phi_0, \phi_1)$ for the description of $\mathbb{P}$ and $\leqslant_{\mathbb{P}}$. In the stronger form of nep that is called "explicit nep" we have $\bar{\phi} = (\phi_0, \phi_1, \phi_2)$ with ϕ_0 and ϕ_1 in the same roles, whereas the additional first order formula ϕ_2 describes predense sets.

Definition 3.4. We call $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$ a *definition of a forcing*. We call M a $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-*candidate* if M is a countable transitive ZFC^* model and $\mathfrak{B} \in M$.

This is a simplification, since we say transitive. The evaluation of $\mathbb{P}$ over countable transitive models shall give relevant information about its forcing behaviour in $\mathbf{V}$. Hence it is natural to require $\mathbb{P} = \bigcup_{M \text{ a candidate}} \mathbb{P}^M$, where $\mathbb{P}^M = \phi_0^M$. From the requirement that ϕ_0 is upwards absolute we get $\mathbb{P}^M = \mathbb{P} \cap M$. Then only $\mathbb{P} \subseteq H(\omega_1)$ can fulfil the natural requirement. Fortunately many well-known useful forcings with conditions of size ω have $\mathbb{P} \subseteq H(\omega_1)$. However already iterations of small iterands (i.e., with names in $H(\omega_1)$) of lengths $\geqslant \omega_2$ are not $\subseteq H(\omega_1)$ anymore. As a technical means to handle this situation one can use ord-hc candidates. $M \models \mathsf{ZFC}^*$ instead of transitive models. We refer the reader to [23] and [9], and we will work here only with transitive models.

In the next section, we show that $(T, Y, \mathcal{S})$-preserving is an iterable property. So it is enough to give a sufficient criterion for $(T, Y, \mathcal{S})$-preserving just for one nep iterand. Iterands usually are small and we do not lose any of the creature forcings.

Definition 3.5. If M is a candidate then $G \subseteq \mathbb{P}^M = \{p \in M\colon\ M \models \phi_0(p)\}$ is $(M, \mathbb{P})$-*generic* if for all $A \in M$, if $M \models$ "$A \subseteq \mathbb{P}$ is a maximal antichain", then $|G \cap A| = 1$. (The incompatibility in $\mathbb{P}$ might be not absolute, so $G \cap A \neq \emptyset$ is not enough.) q is called $(M, \mathbb{P})$-*generic* if for all $A \in M$ such that $M \models$ "A is a maximal antichain", $q \Vdash |G \cap A| = 1$.

Definition 3.6. Let K be a class of forcings. $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$ is called a K-$(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-*definition of a nep forcing* if the following hold in $\mathbf{V}$ and in all extensions of $\mathbf{V}$ by members of K:

(a) ϕ_0 defines the set of elements of $\mathbb{P}$ and ϕ_0 is upwards absolute from candidates to $\mathbf{V}$, in $\mathbf{V}$ and in all $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidates,
(b) ϕ_1 defines the quasi ordering $\leqslant_{\mathbb{P}}$ in $\mathbf{V}$ and in every $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidate, ϕ_1 is upwards absolute from candidates to $\mathbf{V}$, in $\mathbf{V}$ and in all $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidates,
(c) if M is a $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidate and $p \in \mathbb{P}^M$ then there is an $(M, \mathbb{P})$-generic $q \geqslant p$.

We isolate a property:

(♡) If M_1 is a $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidate and $M_1 \models$ "M_0 is a $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidate and $p \in \mathbb{P}^{M_0}$" then there is $q \in \mathbb{P}^{M_1}$, $q \geqslant p$ such that $M_1 \models$ "q is $(M_0, \mathbb{P})$-generic" and such that in $\mathbf{V}$, q is $(M_0, \mathbb{P})$-generic.

In the following we show that (♡) follows from quite natural strengthenings of the notion of non-elementary properness. Many well-known forcings are non-elementary proper in one of these strong variants.

Definition 3.7. We add the adverb "explicitly", so say "$(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$ is called a $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-definition of a K-explicitly nep forcing" if $\bar{\phi} = \langle \phi_0, \phi_1, \phi_2 \rangle$ and (ϕ_0, ϕ_1) are as in Definition 3.6 and additionally

(b)$^+$ We assume ϕ_2 is an $(\omega+1)$-place relation that is upward absolute from $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidates. $\phi_2(p_i\colon i \leqslant \omega)$ says $\{p_i\colon i \leqslant \omega\} \subseteq \mathbb{P}$ and $\{p_i\colon i < \omega\}$ is a predense antichain above p_ω not just in $\mathbf{V}$ but in every $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidate which satisfies $\phi_2(p_i\colon i \leqslant \omega)$. In this situation we say $\{p_i\colon i < \omega\}$ is explicitly predense above p_ω.

(c)$^+$ We add to (c) in the definition of nep: There is $q \geqslant p$ with the following property: If $N \models \mathcal{I}$ is a predense antichain above p, so $\mathcal{I} \in N$ then for some list $\langle p_i\colon i < \omega \rangle$ of $\mathcal{I} \cap N$ we have $\phi_2(\langle p_i\colon i < \omega \rangle ^\frown \langle q \rangle)$. We then say "$q$ is explicitly $(N, \mathbb{P})$-generic above p".

In our proof K contains also the Levy collapse, so not only mild forcings. So as soon as the definition of the forcing $\mathbb{P}$ is sensitive to cardinals, K-nep becomes a strong requirement. Think for example again of the forcing specialising a normal Aronszajn tree: After collapsing, the Aronszajn tree is just a perfect tree $\subseteq \omega^{<\omega}$.

So finally to get (♡) we need even more than explicitly nep:

Definition 3.8.

(1) A $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-definition of a forcing notion $\mathbb{P}$ is called *straight* nep if is it K-explicitly nep and in addition For $\ell < 3$ the formula ϕ_ℓ is of the form

$$(\exists t)\bigl[t \in \mathcal{H}(\omega_1) \wedge (\exists s)\bigl((s \in t \vee s = t) \wedge \psi_\ell^{\mathbb{P}}(\bar{x}, s)\bigr)\bigr],$$

where in the formula $\psi_\ell^{\mathbb{P}}$ the quantifiers are of the form $(\exists s' \in s)$ and the atomic formulae are $x \in y$, "x is an ordinal", "$x < y$ are ordinals" and those of $\mathfrak{B}$.

(2) We say *very straight* if it is straight and in addition
   (f) for some Borel functions $\mathbf{B}_1$, $\mathbf{B}_2$, if N is a candidate and $\bar{a}$ lists N and $p \in \mathbb{P}^N$, then $q = \mathbf{B}_1(p, \bar{a}, N)$ is explicitly $(N, \mathbb{P})$-generic and $\mathbf{B}_2(p, \bar{a}, N)$ is a witness, that is it witnesses $p \leqslant q$ and $\phi_2(\langle p_{\mathcal{I},n}\colon n < \omega \rangle, q)$ for some sequence $\langle p_{\mathcal{I},n}\colon n < \omega \rangle$ of members of $\mathcal{I}$ for every predense antichain $\mathcal{I}$ of $\mathbb{P}^N$ in N.

The property from Definition 3.8(1) guarantees: If $p, q \in M_1$ and $p \leqslant q$ in $\mathbf{V}$, then $M_1 \models p \leqslant q$. Upwards absoluteness is included in the more basic canon of nep properties Definition 3.6(a), (b).

The following lemma shows that there are many examples of forcing notions that meet our version non-elementary properness. Its proof is long and will not be repeated here.

Lemma 3.9. *We can use* $\mathsf{ZFC}^* = \mathrm{ZC}^-$ *which is K-good for K from page* 594 *and normal.*

(1) *Suppose that $\mathbb{P}$ is a forcing of one of the following types*:
   (a) $\mathbb{Q}_e^{\mathrm{tree}}(K, \Sigma)$ *for some finitary tree-creating pair (K, Σ) that is t-omittory without a condition on the norm for $e = 0$ and 2-big in the case of $e = 1$ (this covers Sacks forcing).*
   (b) *The Blass–Shelah forcing notion.*

(c) $\mathbb{Q}^*_{\mathrm{w},\infty}(K,\Sigma)$ *for some finitary creating pair which captures singletons, that is* (K,Σ) *is forgetful and for every* $(t_0,\dots,t_n)$ *and for each* $u \in basis(t_0)$ *and* $v \in \mathrm{pos}(u,t_0,\dots,t_n)$ *there is* $(s_0,\dots,s_k)$ *such that* $(t_0,\dots,t_n) \leqslant (s_0,\dots,s_k)$ *and* $m^{t_0}_{\mathrm{dn}} = m^{s_0}_{\mathrm{dn}}$ *and* $m^{t_n}_{\mathrm{up}} = m^{s_k}_{\mathrm{up}}$ *and* $\mathrm{pos}(u,s_0,\dots,s_k) = \{v\}$. (K,Σ) *is forgetful if for every* $t \in K$ *and* $\langle w,u\rangle \in \mathrm{val}[t]$ *and* $w' \in \prod_{n<m^t_{\mathrm{dn}}} H(n)$ *also* $\langle w', w'^\frown u \restriction [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})\rangle \in \mathrm{val}(t)$.

Then $\mathbb{P}$ *is an explicitly nep very straight forcing notion with a Souslin definition* (*see* [23, *Definition* 1.9]).

(2) *Suppose that* $\mathbb{P}$ *is a forcing of one of the following types*:

(a) $\mathbb{Q}^{\mathrm{tree}}_e(K,\Sigma)$ *for some countable tree-creating pair* (K,Σ) *that is t-omittory without a condition on the norm* (*see Definition* 2.15) *for* $e = 0$ *and* 2*-big in the case of* $e = 1$ (*this Miller forcing and Laver forcing*).

(b) $\mathbb{Q}^*_\infty(K,\Sigma)$ *for some finitary growing pair* (K,Σ). *This covers the Mathias forcing notion.* (K,Σ) *is called growing if for any sequence* $(t_0,\dots,t_{n-1})$ *with* $m^{t_i}_{\mathrm{dn}} = m^{t_{i-1}}_{\mathrm{up}}$ *for* $i < n$ *there is* $t \in \Sigma(t_0,\dots,t_{n-1})$ *such that* $\mathrm{nor}(t) \geqslant \max_{i<n} \mathrm{nor}(t_i)$.

Then $\mathbb{P}$ *is an explicitly nep very straight forcing notion.*

For a proof see [16, Proposition 3.2].

Definition 3.10. Suppose $\mathfrak{B}$ is a model with domain $\subseteq \mathcal{H}(\omega_1)$, $\mathbb{P}$ is a $(\bar\varphi,\mathfrak{B},\mathsf{ZFC}^*)$-definition of a nep forcing. $\mathbb{P}$ is called *ω-Cohen preserving for $(\bar\varphi,\mathfrak{B},\mathsf{ZFC}^*)$-candidates*, iff the following holds: If N is a $(\bar\varphi,\mathfrak{B},\mathsf{ZFC}^*)$-candidate and for each n, $x_n \in {}^\omega\omega$ is a Cohen real over N and $p \in \mathbb{P}^N$ then there is $q \in \mathbb{P}$, $q \geqslant p$ that is $(N,\mathbb{P})$-generic and $q \Vdash (\forall n)$ (x_n is a Cohen real over $N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}]$).

For proper forcings that are given by a definition $(\bar\varphi,\mathfrak{B},\mathsf{ZFC}^*)$ this is a strengthening of Definition 3.1, since for countable $M \prec H(\chi)$, the transitive collapse is a candidate.

So finally we can state the main theorem in this section:

Theorem 3.11. *Suppose $\mathfrak{B}$ is a model with domain $|\mathfrak{B}| \subseteq H(\omega_1)$ and countable signature, the definition $(\bar\varphi,\mathfrak{B},\mathsf{ZFC}^*)$ of $\mathbb{P}$ is explicitly very straightly nep, ZFC^* is normal and K-good. Suppose that $\mathbb{P}$ is ω-Cohen preserving for $(\bar\varphi,\mathfrak{B},\mathsf{ZFC}^*)$-candidates. Then $\mathbb{P}$ is $(T,Y,\mathcal{S})$-preserving for all triples $(T,Y,\mathcal{S})$.*

Proof. Let $\lambda \geqslant 2^{|H(\omega_1)|}$ be large enough such that $(\forall\lambda' \geqslant \lambda)(\mathcal{H}(\lambda') \models \mathsf{ZFC}^*)$. Let $\lambda_1 = |\mathcal{H}(\lambda)|$, and let $\chi > \lambda_1 \geqslant \lambda$. Let $N' \prec \mathcal{H}(\chi)$ be countable such that $\{\lambda_1, p, \mathcal{S}, T, \bar\varphi, \mathfrak{B}\} \subset N'$. Our aim is to show that N' is as in Definition 1.16.

Let N be the Mostowski collapse of N' and say $\pi^{N'} : N' \to N$ is the collapsing function. Let $\delta = N' \cap \omega_1$. So $N \models \delta = \aleph_1$. Assume that $N' \cap \omega_1 \in \mathcal{S}$.

We let $\mathbb{R} = \mathrm{Levy}(\aleph_0, \pi(\lambda_1))^N$.

Claim 1. *In* $\mathbf{V}$, *there is* $\mathbf{g}$ *such that*

(a) $\mathbf{g}$ *is* $\mathbb{R}$*-generic over* N,
(b) *if* $t \in \pi^{N'}(Y(\delta))$ *then* $\pi^{N'}(T_{<t})$ *is* $\pi(T)$*-generic over* $N[\mathbf{g}]$.

Proof. $\mathbb{R}$ is a forcing notion in N and hence in $\mathbf{V}$. Let $\mathbf{g}_1$ be generic over $\mathbf{V}$ not just over N. We first show that $\mathbf{g}_1$ would be as desired, were it in $\mathbf{V}$. In $\mathbf{V}[\mathbf{g}_1]$, let $t \in \pi^{N'}(Y(\delta))$. We show that $\pi^{N'}(T_{<t})$ is generic over $N[\mathbf{g}_1]$. Let $\mathcal{I} \in N[\mathbf{g}_1]$ be a subset of $\pi(T_{<\delta})$ such that $N[\mathbf{g}_1] \models \mathcal{I}$ is dense and open in the forcing $\pi((T,<_T))$. By the forcing theorem there is $p \in \mathbf{g}_1 \cap \mathbb{R}$ such that $N \models p \Vdash_{\mathbb{R}} \underset{\sim}{\mathcal{I}}$ is dense and open. Assume towards a contradiction that $\mathbf{V} \models [p \Vdash_{\mathbb{R}} \pi(T_{<t}) \cap \underset{\sim}{\mathcal{I}} = \emptyset]$.

The set

$$\mathcal{I}^* = \bigl\{q \in \text{Cohen: } \bigl(\exists \nu \in \pi^{N'}(T_{<t})\bigr)(q \nVdash_{\mathbb{R}} \nu \notin \mathcal{I})\bigr\}$$

is dense and open in the Cohen poset, since $\mathbb{R}$ is just Cohen forcing, and the iteration of two Cohen forcings is equivalent to the iteration in the reversed order. So there is $\nu \in \pi^{N'}(T_{<_T t})$ and there is $q \geqslant p$ such that $q \Vdash_{\mathbb{R}} \nu \in \mathcal{I}$ which is a contradiction.

Now $\mathbf{g}_1$ is not in $\mathbf{V}$. The requirements on $\mathbf{g}_1$ have only quantifiers bounded by sets and hence are absolute for transitive models

$$\begin{aligned}&(\forall D \in N)\bigl(D \text{ dense in } \operatorname{Levy}\bigl(\aleph_0, \pi(\lambda_1)\bigr)^N \to D \cap \mathbf{g}_1 \neq \emptyset\bigr) \wedge \\ &\quad \bigl(\forall t \in \pi\bigl(Y(\delta)\bigr)\bigr)\bigl(\forall D \in N[\mathbf{g}_1] \text{ that are dense in } \pi(T)\bigr)\bigl(\exists s \in \pi(T_{<t}) \cap D\bigr).\end{aligned}$$

So "there is such a $\mathbf{g}$ with these properties" is a Σ_1 sentence with parameters in $\mathbf{V}$ that is true in $\mathbf{V}[\mathbf{g}_1]$. By absoluteness, it is true also in $\mathbf{V}$.

Let $t \in Y \cap \pi^{N'}(T)$ be given. Note that $\pi^{N'}(T) = T_{<\delta}$. By the assumption that T is $(Y, \mathcal{S})$-proper, and hence after the Levy collapse, $\pi(T_{<_T t})$ is $(N, \pi^{N'}(T))$-generic. Let $M = N[\mathbf{g}]$ with a $\mathbf{g}$ as in the claim. Then $\pi^{N'}(T_{<t})$ is also $(M, \pi^{N'}(T))$-generic by the choice of $\mathbf{g}$. M is a candidate since $\mathbb{P}$ is nep as in the condition of the theorem and as $\mathbb{R} \in K$.

Now we use that $\mathbb{P}$ is ω-Cohen preserving for the $(\bar{\phi}, \mathfrak{B}, \mathsf{ZFC}^*)$-candidate M. We choose a dense embedding $h\colon (\omega^{<\omega}, \triangleleft) \to \pi^{N'}(T, <_T)$, $h \in N[\mathbf{g}]$. Note the $N[\mathbf{g}]$ thinks that $\pi^{N'}(T)$ is countable, since $\mathbb{R}$ collapses $\pi^{N'}(\omega_1) < \pi^{N'}(\lambda_1)$ from N to ω. So in $N[\mathbf{g}]$ the Cohen forcing $(\omega^{<\omega}, \triangleleft)$ and $\pi^{N'}(T, <_T)$ are equivalent. We let $\eta_t \in {}^\omega\omega$ be such that $n < \omega \to h(\eta_t \restriction n) < t$. So η_t is Cohen over M iff $\pi^{N'}(T_{<_T t})$ is $(M, \pi^{N'}(T))$ generic, and this holds also for extensions of M since there is the isomorphism h in them. Since $\pi^{N'}(T_{<_T t})$ is $(M, \pi^{N'}(T))$-generic, η_t is Cohen generic over M. Now we use that $\mathbb{P}$ is Cohen preserving for the candidate M. So there is $q \geqslant \pi(p)$ that is $(M, \mathbb{P})$-generic and $q \Vdash (\forall t \in \pi^{N'}(Y(\delta)))$ (η_t is Cohen over $M[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}]$). So

$$q \Vdash \bigl(\forall t \in \pi\bigl(Y(\delta)\bigr)\bigr)\bigl(\pi(T_{<_T t}) \text{ is } \bigl(M[\mathbf{G}_{\mathbb{P}}], \pi(T)\bigr)\text{-generic}\bigr) \text{ and } q \text{ is } (M, \mathbb{P})\text{-generic.} \tag{3.1}$$

Now we get from the latter

$$\bigl(\exists q_3 \geqslant \pi(p)\bigr)\bigl(q_3 \Vdash \text{“}\bigl(\forall t \in \pi\bigl(Y(\delta)\bigr)\bigr)\ \pi(T_{<_T t}) \text{ is } \bigl(N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], \pi(T)\bigr)\text{-generic” and } q_3 \text{ is } (N, \mathbb{P})\text{-generic}\bigr). \tag{3.2}$$

Why? We use nep again. We take χ_1 such that $N \models (\chi_1$ is sufficiently large such that $\mathcal{P}(\mathbb{P}) \in H(\chi_1)$ and χ_1 is sufficiently small so that 2^{χ_1} exists). Let $N_1 = N \restriction H(\chi_1)^N$. In N, N_1 is a candidate.

By ($\heartsuit$) there is $q_1 \geqslant \pi(p)$, $q_1 \in N \subseteq M$, $N \models$ "$\pi(p) \leqslant q_1$ and q_1 is $(N_1, \mathbb{P})$-generic" and $q_1 \geqslant \pi(p)$ also in $\mathbf{V}$ and q_1 is $(N_1, \mathbb{P})$-generic also in $\mathbf{V}$.

We claim: q_1 is as required in the first half of (3.2), that is: For any $\mathcal{I} \in M[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}]$ that has a $\mathbb{P}$-name in N and is (forced by the weakest condition to be) a dense set in $\pi(T, <_T)$ in the sense of $M[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}]$, $q_1 \Vdash (\forall t \in \pi(Y(\delta)))(\underset{\sim}{\mathcal{I}} \cap \pi(T_{<_T t}) \neq \emptyset)$. All $\underset{\sim}{\mathcal{I}} \in N$ that are $\mathbb{P}$-names for dense sets in $\pi(T, <_T)$ have names $\underset{\sim}{\mathcal{I}} \in N_1$. Now we argue in N: For any $q_2 \geqslant q_1$, $q_2 \in M$, there is $q \geqslant q_2$, q is $(M, \mathbb{P})$-generic and $q \Vdash (\forall t \in \pi(Y(\delta)))(\underset{\sim}{\mathcal{I}} \cap \pi(T_{<_T t}) \neq \emptyset)$, as we have shown above, in Eq. (3.1), proved for q_1 instead of $\pi(p)$ and proved in $N \models \mathsf{ZFC}^*$ instead of in $\mathcal{H}(\chi)$ and $\mathbf{V}$. Then $q \Vdash \pi^{N'}(T_{<_T t}) \cap \underset{\sim}{\mathcal{I}} \cap N_1 \neq \emptyset$ since $(\pi^{N'}(T_{<_T t}))^{N[\mathbf{G}_{\mathbb{P}}]} = (\pi(T_{<_T t}))^{M[\mathbf{G}_{\mathbb{P}}]} \subseteq N_1$. So $q \Vdash \pi^{N'}(T_{<_T t}) \cap \underset{\sim}{\mathcal{I}} \cap N \neq \emptyset$ and hence $\pi^{N'}(T_{<_T t})$ is $(N[\mathbf{G}_{\mathbb{P}}], \mathbb{P})$-generic. Since N is elementary equivalent to $\mathcal{H}(\chi)$, also in $\mathbf{V}$ we have: For every $t \in \pi^{N'}(Y(\delta))$, for every $\mathcal{I} \in M[G]$ that has a $\mathbb{P}$-name in N and is a dense set in $\pi^{N'}(T, <_T)$, $q_1 \Vdash (\forall t \in \pi(Y(\delta)))(\underset{\sim}{\mathcal{I}} \cap \pi(T_{<_T t}) \neq \emptyset)$. So q_1 has the property required of q_3 in (3.2).

Now we use nep again and find $q_3 \geqslant q_1$ that is $(N, \mathbb{P})$-generic. So q_3 witnesses that (3.2) is proved. Now after taking the reverse image of the Mostowski collapse (the nep forcing $\mathbb{P}$ is moved from model to model by just taking its interpretation) we have $q_4 \geqslant p$ such that

$$q_4 \Vdash \text{“}\big(\forall t \in Y(\delta)\big) T_{<_T t} \text{ is } \big(N'[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T\big)\text{-generic” and } q_4 \text{ is } \big(N', \mathbb{P}\big)\text{-generic.} \qquad \square$$

Remark 3.12. In the special case that the $Y \cap T_\delta$ is a singleton (or empty) for all $\delta \in \mathcal{S}$, we need only a weaker form of Cohen preserving, with one Cohen generic η. In this special case “T is $(Y, \mathcal{S})$-proper” implies $T \restriction \{\sup(a)\colon a \in \mathcal{S} \wedge Y(\sup(a)) \neq \emptyset\}$ has no specialisation, see [22, Chapter IX].

We may also consider the well-known stronger variant of $(Y, \mathcal{S})$-properness for forcing with finite products of T: If $N \prec \mathcal{H}(\chi)$ and $\delta = N \cap \omega_1$ and $N \cap \omega_1 \in \mathcal{S}$ and $t_0, \dots, t_{n-1}$ are pairwise distinct then $\{\bar{s} \in {}^n(T_{<\delta})\colon \bigwedge_{\ell<n} s_\ell <_T t_\ell\}$ is $({}^n(T_{<\delta}), N)$-generic. Also preserving this kind of $(Y, \mathcal{S})$-properness in a consequence of ω-Cohen-preserving and nep, by the same proof as above. By an analogue of the results of the next section, this preservation property is iterable.

The following theorem is similar to Theorem 1.17, and in the case of nep forcing it strengthens Theorem 1.17 by adding the intermediate step in the implication $\mathrm{Pr}^2(\mathbb{P}) \to \mathbb{P}$ is ω-Cohen preserving $\to \mathbb{P}$ is $(T, Y, \mathcal{S})$-preserving.

Theorem 3.13.

(1) *If* $\mathrm{Pr}^2(\mathbb{P})$*, then* $\mathbb{P}$ *is* ω*-Cohen preserving.*
(2) *Assume* $\alpha(*) = \omega_1$ *and* $\mathcal{S} \subseteq [\omega_1]^\omega$ *is stationary. If* $\mathrm{Pr}^2_{\mathcal{S}}(\mathbb{P})$*, then* $\mathbb{P}$ *is* ω*-Cohen preserving for Cohen reals over* N *with* $N \cap \omega_1 \in \mathcal{S}$*.*

Proof. (1): Assume $N \prec H(\chi)$, $N \cap \omega_1 \in \mathcal{S}$, $N \cap \omega_1 = \delta$, and $\mathbb{P} \in N$, $p \in N \cap \mathbb{P}$, and assume for every $i < \omega$, x_i be Cohen generic over N. Let $x = \mathbf{st}$ for a winning strategy $\mathbf{st}$ for player COM in $\Game^2(N, \mathbb{P}, p)$. We show that there is a q as required.

Let $\{\underset{\sim}{\mathcal{I}}_k\colon k \in \omega\}$ list all $\mathbb{P}$-names in N of comeager sets and let $\{\mathcal{J}_n\colon n \in \omega\}$ list all the dense sets in $\mathbb{P}$ in N. Now take a play $\langle(\bar{p_n}, \bar{q_n})\colon n \in \omega\rangle$ in which COM plays according to $\mathbf{st}$. By Lemma 1.4 COM can strengthen his moves and still wins. COM plays in every round n in every part $p_{n,\ell}$, $\ell < \ell_n$, so strong that $p_{n,\ell} \in \bigcap_{r<n} \mathcal{J}_r$ such that for every $i < n$

$$p_{n,\ell} \Vdash_{\mathbb{P}} x_i \in \bigcap_{k<n} \underset{\sim}{\mathcal{I}}_k.$$

Such $p_{n,\ell}$ exist for the following reason: Since $\bigcap_{k<n} \underset{\sim}{\mathcal{I}}_k$ is (forced by the weakest condition to be) comeager, for every $n \in \omega$, the set $\mathcal{J}_n = \{s \in \mathbb{C}\colon \{q \in \mathbb{P}\colon q \nVdash_{\mathbb{P}} [s] \cap \bigcap_{k<n} \underset{\sim}{\mathcal{I}}_k = \emptyset\}$ is dense and open in $\mathbb{P}\}$ is open and dense in the Cohen forcing $\mathbb{C}$ in the ground model. The Cohen real x_i fulfils $x_i \in \mathcal{J}_n$. So for every i, $\{q \in \mathbb{P}\colon q \nVdash_{\mathbb{P}} x_i \notin \bigcap_{k<n} \underset{\sim}{\mathcal{I}}_k\}$ is dense in $\mathbb{P}$.

COM wins the play because he played according to the strategy. So for every u, in particular for $u = \omega$, there is $q_u \geqslant p$ such that

$$q_u \Vdash \big(\exists^\infty n \in u\big)(\exists \ell < \ell_n)(p_{n,\ell} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}). \tag{3.3}$$

Let $k \in \omega$ and $q' \geqslant q_u$ be given. Then there is $q'' \geqslant q'$ and $n \geqslant k$ such that $q'' \Vdash n \in u$. So there is $i < \ell_n$, $q'' \Vdash q_{n,i} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ and hence

$$q'' \Vdash_{\mathbb{P}} x_k \in \bigcap_{k'<n} \underset{\sim}{\mathcal{I}}_{k'}. \tag{3.4}$$

Now we unfreeze k and combine Eqs. (3.3) and (3.4) and thus get

$$q_u \Vdash (\forall k < \omega)\big(x_k \text{ is } \big(N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T\big)\text{-generic}\big).$$

From $q_{n,i} \in \bigcap_{r<n} \mathcal{J}_r$ we also get that q_u is $(N, \mathbb{P})$-generic. □

Remark 3.14. ω-Cohen preserving is not a necessary condition for preserving Souslin trees: By [23, Lemma 3.1], Blass–Shelah forcing is nep in the strong form that is used in Theorem 3.11. Blass–Shelah forcing is not Cohen preserving. This follows from the fact that the generic real is not split by any real in the ground model (see [3]). Hence the ground model is meager after Blass–Shelah forcing. So Blass–Shelah forcing is not positivity preserving for the meager ideal in the sense of [10, Definition 3.1]. So by [10, Lemma 5.6] is it not true positivity preserving [10, Definition 5.5]. Now by [10, Lemma 5.8] Blass–Shelah does not preserve the Cohen genericity. So it is a nep forcing not covered by Theorem 3.11. Nevertheless Blass–Shelah preserves Souslin trees by Theorem 2.32.

4. Preserving the Souslinity of an ω_1-tree

The topic of the section is the preservation of properties of notions of forcing in countable support iterations. We return to the ω_1-trees and the properties of $(T, Y, \mathcal{S})$ from the first section. In this section we give a self-contained proof of the following theorem:

Theorem 4.1. *Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta\colon \alpha \leqslant \gamma, \beta < \gamma\rangle$ be a countable support iteration of proper forcings. Suppose that T is a Souslin tree in $\mathbf{V}$ and that for every $\alpha < \gamma$, in $\mathbf{V}^{\mathbb{P}_\alpha}$, $\Vdash_{\mathbb{Q}_\alpha}$ "T is Souslin". Then T is Souslin in $\mathbf{V}^{\mathbb{P}_\gamma}$.*

By Lemma 1.15 T is Souslin in $\mathbf{V}^{\mathbb{P}_\gamma}$ iff T is $(Y, \mathcal{S})$-proper for a stationary $\mathcal{S} \subseteq [\omega_1]^\omega$ and $Y = \bigcup\{T_{\sup(a)}\colon a \in \mathcal{S}\}$. So Theorem 4.1 is a special case of the following theorem ([22, Chapter XVIII, Conclusion 3.9 F]):

Theorem 4.2. *Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta\colon \alpha \leqslant \gamma, \beta < \gamma\rangle$ be an countable support iteration of proper forcings. Suppose that T is an ω_1-tree that is $(Y, \mathcal{S})$-proper and that for every $\alpha < \gamma$, in $\mathbf{V}^{\mathbb{P}_\alpha}$, $\underset{\sim}{\mathbb{Q}}_\alpha$ is $(T, Y, \mathcal{S})$-preserving. Then $\mathbb{P}_\gamma$ is $(T, Y, \mathcal{S})$-preserving and T is $(Y, \mathcal{S})$-proper in $\mathbf{V}^{\mathbb{P}_\gamma}$.*

The proof of Theorem 4.2 is not more complex than the proof of Theorem 4.1. It involves preserving unbounded families in certain relations. A simple similar example is to preserve a $\leqslant^*$-unbounded family. The relations are binary relations on spaces ${}^a a$ for countable sets a. For the proof of Theorem 4.2 we need that the union of all considered a covers ω_1. For each fixed a, ${}^a a$ is just a copy of the Baire space ω^ω, which means a has the discrete topology and ${}^a a$ carries the product topology. We consider $\aleph_1$ different sets a, and on each fixed a we work with countably many relations $R_{\alpha,a}$, $\alpha \in a$.

Let $\mathcal{S} \subseteq [\omega_1]^\omega$. Let for $a \in \mathcal{S}$, $\mathbf{g}_a \in {}^a a$ and for $\alpha \in a$, $R_{\alpha,a} \subseteq {}^a a \times {}^a a$ be a relation.

We assume that a fixed $\mathbf{g}_a$, the sets

$$\{f \in {}^a a\colon\ f R_{\alpha,a} \mathbf{g}_a\} \tag{4.1}$$

are closed in ${}^a a$. We will see that open relations do not harm since they are the union of countably many closed relations. Let $N \prec (H(\chi), \in, <)$. Now suppose that we have a collection $\mathscr{G}$ of $\mathbf{g}_a$'s such that

$$(\forall f \in N \cap {}^a a) \bigvee_{\alpha \in a} \bigvee_{\mathbf{g}_a \in \mathscr{G}} f R_{\alpha,a} \mathbf{g}_a. \tag{4.2}$$

Is there an $(N, \mathbb{P}, p)$-generic condition such that q forces that q forces (4.2) holds for $f \in N[G]$? Suppose the answer is positive for each iterand, what can be said about the countable support limit?

The proof of the iteration theorem in its basic form uses only that

$$\{f \in {}^a a\colon\ f R_{\alpha,a} \mathbf{g}_a\}$$

is closed for each $R_{\alpha,a}$, $\mathbf{g}_a$. There is an example of relations $R_{\alpha,a}$, $\mathbf{g}_a$, such that $(\bar{R}, \mathcal{S}, \bar{\mathbf{g}})$ preserving coincides with $(T, \mathcal{S}, Y)$-preserving.

The reader can jump ahead to Definition 4.9 to see what particular a, g_a and $R_{\alpha,a}$ we use for the proof of Theorem 4.2. We let

$$\alpha(*) = \bigcup \mathcal{S} = \omega_1.$$

We point to the sources in [22]: Our presentation belongs to Case (b) from [22, Chapter XVIII, Context 3.1]. Within this Case (b) we focus onto the Possibilities (also sometimes called "Cases" there) A and C in [22, Chapter XVIII, Definitions 3.3, 3.4]. From Definition 4.9 one reads off that in the triple $(\bar{R}, \mathcal{S}, \bar{\mathbf{g}})$ describing $(T, Y, \mathcal{S})$-preservation has closed and open relations $\bar{R}$. So for the relevant relations $\bar{R}$, preservation for Possibility A and for Possibility C are equivalent, see Lemma 4.11.

In our presentation in contrast to [6,2], $\bar{R}$ is of size ω_1 and $\alpha(*) = \omega_1$. The widely known presentations of iteration theorems for the relations on the real numbers [6,2] have usually countably many relations and the equivalent to $\alpha(*)$ is ω. The relations we preserve are still on the Baire space and its topology matters for all the considered possibilities. However, there is for each $a \in \mathcal{S}$ an incarnation ${}^a a$ of the Baire space. Moreover, since $a = N \notin N$ names $f \in N$ for functions $f : \mathrm{dom}(f) \to \omega_1$ such that $a \subseteq \mathrm{dom}(f)$, $f(x) \in a$ for $x \in a$ now necessarily are names for functions with larger domains. This does not cause problems, since the evaluation will be always on a.

Note that we change one Definition, namely [22, Chapter XVIII, Definition 3.4]. So our version of "$(\bar{R}, \mathcal{S}, \mathbf{g})$-preserving for Possibility A" has not been named in the definitions in the book nor anywhere else. Namely items (iv) and (vi) of Definition 4.5 are new.

However, Definition 4.5 is the one used in the proof of the preservation theorem for the limit case in [22, Chapter XVIII, Theorem 3.6]. Our Possibility A here and the proofs here (which are the ones from the book with some additional explanations) do not need the distinction whether reals are added or not. The original definition of Possibility A in the book works as well, however, the proofs are longer. There are two proofs based on the old definition: For forcings that add reals the technique is much shorter [6] than for the general case that was proved later by Goldstern and Kellner [7]. Our proof given here is short and works in the general case.

The letter $\mathbb{Q}$ now stands for an iterand. We let $(2^{|\mathbb{Q}|})^+ < \chi$, $\mathcal{S} \subseteq [\omega_1]^{<\omega_1}$ be stationary, usually the elements of $\mathcal{S}$ are of the form $N \cap \omega_1$ for a countable $N \prec \mathcal{H}(\chi)$. In the language of [22], we have for $a \in \mathcal{S}$, $d[a], c[a] = a \notin a$ and we are in Case (b) of [22, Chapter XVIII, Context 3.1], $d[a] \notin a$, and $d'[a] = c'[a] = \omega_1$. We will not mention the functions c, d, c', d' henceforth since they are fixed. We stay with our special case of $\mathcal{S}$ and $\alpha(*) = \omega_1$. So we cut down a lot in comparison to the rich Section 3 of Chapter XVIII of [22]. On the other hand, we add numerous proofs to claims that are written there without a proof.

Definition 4.3.

(0) N is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good means: $a := N \cap \bigcup \mathcal{S} \in \mathcal{S}$, and for every $f \in N$, $f : \bigcup \mathcal{S} \supset\!\!\to \bigcup \mathcal{S}$ with $a \subseteq \mathrm{dom}(f)$ for some $\beta \in a \cap \alpha(*)$ we have $f \restriction a R_{\beta,a} \mathbf{g}_a$.

(1) We say $(\bar{R}, \mathcal{S}, \mathbf{g})$ *covers in* $\mathbf{V}$ iff for sufficiently large χ for every $x \in H(\chi)^{\mathbf{V}}$ there is a countable $N \prec \mathcal{H}(\chi)$ to which $(\bar{R}, \mathcal{S}, \mathbf{g})$ and x belong such that N is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good.
(2) Let $\mathcal{S}$ be stationary. We say $(\bar{R}, \mathcal{S}, \mathbf{g})$ *fully covers in* $\mathbf{V}$ iff for some $x \in H(\chi)$, for every countable $N \prec \mathcal{H}(\chi)$ to which $(\bar{R}, \mathcal{S}, \mathbf{g})$ and x belong and which fulfils $N \cap \bigcup \mathcal{S} \in \mathcal{S}$ we have that N is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good.

Definition 4.4.

(1) We say $(\bar{R}, \mathcal{S}, \mathbf{g})$ *strongly covers in Case A* iff it covers in $\mathbf{V}$ and each $R_{\alpha,a}$ is a closed or an open binary relation on ${}^a a$. We assume from now on that for $\alpha \in a$, $a \in \mathcal{S}$,

$$\{f : a \to a\colon\ f R_{\alpha,a} \mathbf{g}_a\}$$

is closed. This is sufficient.
(2) We say $(\bar{R}, \mathcal{S}, \mathbf{g})$ *strongly covers in Case C* iff it covers in $\mathbf{V}$ and in addition for each $a \in \mathcal{S}$ in any forcing extension (or at least for any forcing extension in by a forcing in a family of forcings we are interested in) of $\mathbf{V}$ player II has a winning strategy in the following game: In the n-th move player I chooses N_n, H_n such that
   (a) $(N_n, \in\restriction N_n)$ is a countable not necessarily transitive model of ZFC$^-$, $N_n \cap \bigcup \mathcal{S} = a \in \mathcal{S}$, a, $\mathcal{S}$, $\mathbf{g}$, $\bar{R} \in N_n$, $\ell < n \to N_\ell \subset N_n$ and $N_n \models (\bar{R}, \mathcal{S}, \mathbf{g})$ covers, and $f \in N_n \to (f \restriction a) R_a \mathbf{g}_a$,
   (b) $H_n \subseteq \{\langle f_0, \ldots, f_{n-1}\rangle\colon$ for some finite $d \subseteq \omega_1$, $(\forall \ell < n)(f_\ell \in {}^d\omega_1)\}$ and $H_n \in N_n$ is not empty,
   (c) if $\langle f_0, \ldots, f_{n-1}\rangle \in H_n$ and $d \subseteq \operatorname{dom}(f_0)$ is finite then $\langle f_0 \restriction d, \ldots, f_{n-1} \restriction d\rangle \in H_n$,
   (d) if $\langle f_0, \ldots, f_{n-1}\rangle \in H_n$ and $\operatorname{dom}(f_0) \subseteq d$, d finite, $d \subseteq \omega_1$ then for some $\langle f'_0, \ldots, f'_{n-1}\rangle \in H_n$ we have $\operatorname{dom}(f'_\ell) = d$ and $f_\ell \subseteq f'_\ell$,
   (e) $m < n$ and $\langle f_0, \ldots, f_{n-1}\rangle \in H_n \to \langle f_0, \ldots, f_{m-1}\rangle \in H^*_m$ (see below).
   Player II chooses $\langle f^n_0, \ldots, f^n_{n-1}\rangle \in H_n \cap N_n$, $f^n_\ell \supseteq f^m_\ell$ for $\ell \leqslant m < n$. Finally, the definition $H^*_n = \{\langle f_0, \ldots, f_{n-1}\rangle$ for each ℓ the functions f_ℓ, f^n_ℓ are compatible$\}$ completes the induction step.
   In the end player II wins if for every $m < \omega$, $\bigcup_{n \geqslant m} f^n_m$ is a function with domain a and $\bigvee_{\alpha \in a} \bigcup_{m \geqslant n} f^n_m R_{\alpha,a} \mathbf{g}_a$

If $R_{\alpha,a}$ is open then we can write $R_{\alpha,a} = \bigcup_{n \in \omega} R_{\alpha,n,a}$ where each $R_{\alpha,n,a}$ is closed and use $\omega_1 = \alpha(*) = \omega\alpha(*)$, $R'_{\omega\alpha+n,a} = R_{\alpha,n,a}$ and work with the closed relations $R'_{\beta,n}$, $\beta < \alpha(*)$.

As we already mentioned, we changed the following definition in Possibility A in comparison to the definition [22, Chapter XVIII, Definition 3.4] in the book, so that it is fitting to the proof in the book. The items (iv) and (vi) are changed.

Definition 4.5. We say $\mathbb{Q}$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-*preserving for Possibility A* iff the following holds for any χ, χ_1, N, $p \in \mathbb{Q} \cap N$, $k < \omega$: Assume

$(*)$
   (i) χ_1 is large enough and $\chi > 2^{\chi_1}$,
   (ii) $N \prec \mathcal{H}(\chi)$ is countable, $N \cap \bigcup \mathcal{S} = a \in \mathcal{S}$, and $\mathbb{Q}, \mathcal{S}, \mathbf{g}, \chi_1 \in N$,
   (iii) N is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good and $p \in \mathbb{Q} \cap N$,
   (iv) $k \in \omega$ and for $\ell < k$ we have a $\mathbb{Q}$-name for a function $\underset{\sim}{f_\ell} \in N$, and $\Vdash_{\mathbb{Q}} \operatorname{dom}(\underset{\sim}{f_\ell}) \supseteq a$,
   (v) for $\ell < k$, $m < \omega$, $f^*_{m,\ell}$ is a function from a to a in N,
   (vi) for $n < \omega$, $p \leqslant p_n \leqslant p_{n+1}$,
   (vii) for $x \in \operatorname{dom}(f^*_{m,\ell})$, $\ell < k$, for every m there is n_0 such that for $n \geqslant n_0$, $p_n \Vdash \underset{\sim}{f_\ell}(x) = f^*_{m,\ell}(x)$,
   (viii) for $\ell < k$, $m < \omega$, $f^*_{m,\ell} R_{\beta^m_\ell, a} \mathbf{g}_a$ for some $\beta^m_\ell \in a$, $\beta^{m+1}_\ell \leqslant \beta^m_\ell$, and $\beta^*_\ell = \lim_{m \to \omega} \beta^m_\ell$,
   (ix) if $\mathcal{I}$ is a dense open set of $\mathbb{Q}$ and $\mathcal{I} \in N$, then for some n, $p_n \in \mathcal{I}$.

Then there is $q \geqslant p$, $q \in \mathbb{Q}$ that is $(N, \mathbb{Q})$-generic and

(a) for $\ell < k$, $q \Vdash_{\mathbb{Q}} (\exists \gamma_\ell \in a, \gamma_\ell \leqslant \beta^*_\ell)(\underset{\sim}{f}_\ell \upharpoonright a R_{\gamma_\ell, a} \mathbf{g}_a)$,
(b) $q \Vdash N[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}]$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good.

Note that conclusion (a) expresses a sort of directedness: $\mathbf{g}_a$ is the same for any $\underset{\sim}{f}_\ell$, $\ell < k$. We will use the possibility to work with unboundedly many k in the proof of the preservation of "$(\bar{R}, \mathcal{S}, \mathbf{g})$-preserving for Possibility A" for iterations when the cofinality of the iteration length is countable.

Definition 4.6. We say $\mathbb{Q}$ *is* $(\bar{R}, \mathcal{S}, \mathbf{g})$*-preserving for Possibility C* iff the following holds: Assume

(i) χ_1 is large enough and $\chi > 2^{\chi_1}$,
(ii) $N \prec \mathcal{H}(\chi)$ is countable, $N \cap \bigcup \mathcal{S} = a \in \mathcal{S}$, and $\mathbb{Q}, \mathcal{S}, \mathbf{g}, \chi_1 \in N$,
(iii) N is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good and $p \in \mathbb{Q} \cap N$.

Then there is $q \geqslant p$, $q \in \mathbb{Q}$ that is $(N, \mathbb{Q})$-generic and $q \Vdash N[\mathbf{G}_{\mathbb{Q}}]$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good.

Lemma 4.7.

(1) *If* $(\bar{R}, \mathcal{S}, \mathbf{g})$ *covers in* $\mathbf{V}$ *and* $\mathbb{Q}$ *is an* $(\bar{R}, \mathcal{S}, \mathbf{g})$*-preserving forcing notion* (*for any Possibility*) *then in* $\mathbf{V}^{\mathbb{Q}}$, $(\bar{R}, \mathcal{S}, \mathbf{g})$ *still covers.*
(2) *The property "*$(\bar{R}, \mathcal{S}, \mathbf{g})$*-preserving for Possibility A* (*respectively C*)*" is preserved by composition of forcing notions.*

Proof. (1) Let $\mathbf{G}$ be $\mathbb{P}$-generic over $\mathbf{V}$. $N[\mathbf{G}] \prec \mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]}$ for N being $(\bar{R}, \mathcal{S}, \mathbf{g})$-good in $\mathbf{V}$ is a witness for covering in $\mathbf{V}[\mathbf{G}]$. For Possibility A, we can take $k = 0$, so $(*)$ is vacuously true. Conclusion (b) suffices.

(2) The proof for Possibility A fits the old ([22, Chapter XVIII, Definition 3.4]) and the new definition of Possibility A (Definition 4.5). We fix $\mathbb{Q} = \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$, χ, χ_1, N, a, k, $\underset{\sim}{f}_\ell$, β_ℓ, $f^*_{m,\ell}$ for $\ell < k$, $m < \omega$, $p = (q^0_0, \underset{\sim}{q}^0_1)$, $p^n = p_n = (q^n_0, \underset{\sim}{q}^n_1)$ as in $(*)$ of Definition 4.5 of Possibility A. We take $p^0 = p$. By condition (vi) of $(*)$ for each $n < m < \omega$, $q^m_0 \Vdash_{\mathbb{Q}_0} \underset{\sim}{q}^0_1 \leqslant_{\mathbb{Q}_1} q^n_1 \leqslant_{\mathbb{Q}_1} q^m_1$ hence without loss of generality by clause (ix) of $(*)$ by taking different names $\underset{\sim}{q}^n_1$ that are above q^n_0 the same,

$(*)_1$ $\Vdash_{\mathbb{Q}_0} \underset{\sim}{q}^0_1 \leqslant_{\mathbb{Q}_1} \underset{\sim}{q}^n_1 \leqslant_{\mathbb{Q}_1} \underset{\sim}{q}^m_1$, and
$(*)_2$ for every $x \in a$ or every sufficiently large $n < \omega$, $(\emptyset, q^n_1)$ forces $\underset{\sim}{f}_\ell(x)$ to be equal to some specific $\mathbb{Q}_0$-name $\underset{\sim}{f}'_{n,\ell}(x) \in N$ for each $\ell < k$.

Since $\mathbb{Q}_0$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$ preserving there is $q_0 \in \mathbb{Q}_0$ which is $(N, \mathbb{Q}_0)$-generic and is above q^0_0 in $\mathbb{Q}_0$ and forces $N[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_0}]$ to be $(\bar{R}, \mathcal{S}, \mathbf{g})$-good and for some $\gamma'^*_\ell \leqslant \beta^*_\ell$, we have $q_0 \Vdash_{\mathbb{Q}_0} \bigwedge_{\ell<k} \underset{\sim}{f}_\ell R_{\gamma'_\ell, a} \mathbf{g}_a$.

Let $\mathbf{G}_0 \subseteq \mathbb{Q}_0$ be generic over $\mathbf{V}$ and $q_0 \in \mathbf{G}_0$. We want to apply Definition 4.5 with $N[\mathbf{G}_0]$, $\underset{\sim}{q}^0_1[\mathbf{G}_0]$, $\langle \underset{\sim}{q}^n_1[\mathbf{G}_0]\colon n < \omega\rangle$, $\langle \underset{\sim}{f}_\ell[\mathbf{G}_0]\colon \ell < k\rangle$, $\langle \underset{\sim}{f}'_{n,\ell}[\mathbf{G}_0]\colon \ell < k, n < \omega\rangle$, $\langle \gamma'^*_\ell\colon \ell < k\rangle$, $\underset{\sim}{\mathbb{Q}}_1[G_0]$ there in $(*)$ and check that all the items are fulfilled.

Clause (i) follows from clause (i) for $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$,

clause (ii): as q_0 is $(N, \mathbb{Q}_0)$-generic we have $N[\mathbf{G}_0] \cap \bigcup \mathcal{S} = N \cap \bigcup \mathcal{S} \in \mathcal{S}$, clause (iii) holds by the choice of q_0 and by conclusion (b) in Definition 4.5 for $\mathbb{Q}_0$,

clause (iv) follows from clause (iv) for $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$,

clause (v): if $x \in a$ then there are ℓ and a $\mathbb{Q}_0$-name $\underset{\sim}{\tau} \in N$ such that $\Vdash_{\mathbb{Q}_0} [q^1_\ell \Vdash_{\underset{\sim}{\mathbb{Q}}_1} \underset{\sim}{f}'_{m,\ell}(x) = \underset{\sim}{\tau} \in a]$, as the set of $(r_0, \underset{\sim}{r}_1) \in \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$ such that $r_0 \Vdash_{\mathbb{Q}_0} \underset{\sim}{r}_1 \Vdash_{\underset{\sim}{\mathbb{Q}}_1} \underset{\sim}{f}'_{m,\ell}(x) = \underset{\sim}{\tau}$ for some $\mathbb{Q}_0$-name $\underset{\sim}{\tau}$ is a dense open subset of $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$ some (q^0_ℓ, q^1_ℓ) is in it and there is such a $\underset{\sim}{\tau}$, by properness w.l.o.g. $\underset{\sim}{\tau} \in N$. So $\underset{\sim}{f}'_{m,\ell}[\mathbf{G}_0] = \underset{\sim}{\tau}[\mathbf{G}_0] \in a$.

clause (vi) was ensured by our choice $(*)_1$,

clause (vii): by the choice of $\underset{\sim}{f}'_{m,\ell}$ and $\langle q^n_1\colon\ n < \omega\rangle$,

clause (viii): by the choice of q_0 and γ'_ℓ,

clause (ix) follows from clause (ix) for $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$ and a density argument as in (v). In details: If $N[\mathbf{G}_0] \models \mathcal{I} \subseteq \underset{\sim}{\mathbb{Q}}_1$ is dense and open, then since $\mathcal{I} \in N[\mathbf{G}_0]$ for some $\underset{\sim}{\mathcal{I}}' \in N$ we have $\Vdash_{\mathbb{Q}_0} \underset{\sim}{\mathcal{I}}'$ is a dense open subset of $\underset{\sim}{\mathbb{Q}}_1$ and $\underset{\sim}{\mathcal{I}}'[\mathbf{G}_0] = \mathcal{I}$. Let $\mathcal{J} = \{(r_0, \underset{\sim}{r}_1) \in \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1\colon \Vdash \underset{\sim}{r}_1 \in \underset{\sim}{\mathcal{I}}'\}$. $\mathcal{J} \in N$ is a dense open subset of $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$. Hence for every sufficiently large ℓ, $(q^0_\ell, \underset{\sim}{q}^1_\ell) \in \mathcal{J}$ and so $\underset{\sim}{q}^1_\ell[\mathbf{G}_0] \in \underset{\sim}{\mathcal{I}}'[\mathbf{G}_0] = \mathcal{I}$ and we finish.

The proof for Possibility C is particularly easy: We read the definition of $(\bar{R}, \mathcal{S}, \mathbf{g})$-preserving in this case and see that given $N \prec \mathcal{H}(\chi)$, $N \cap \omega_1 \in \mathcal{S}$, $p \in \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1 \cap N$, $p = (q^0_0, q^0_1)$ there is $q \geqslant p$, $q = (q^1_0, q^1_1)$, that is $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$-generic and

$$(q^1_0, \underset{\sim}{q}^1_1) \Vdash_{\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1} N[\mathbf{G}_{\mathbb{Q}_0} * \mathbf{G}_{\underset{\sim}{\mathbb{Q}}_1[\mathbf{G}_{\mathbb{Q}_0}]}] \text{ is } (\bar{R}, \mathcal{S}, \mathbf{g})\text{-good.}$$

First we take by the hypothesis on $\mathbb{Q}_0$ an $(N, \mathbb{Q}_0)$-generic condition $q^1_0 \geqslant q^0_0$ such that

$$q^1_0 \Vdash_{\mathbb{Q}_0} N[\mathbf{G}_{\mathbb{Q}_0}] \text{ is } (\bar{R}, \mathcal{S}, \mathbf{g})\text{-good.}$$

Then we take $\mathbf{G}_{\mathbb{Q}_0}$, $\mathbb{Q}_0$-generic over $\mathbf{V}$ such that $q^1_0 \in \mathbf{G}_{\mathbb{Q}_0}$. Now in $\mathbf{V}[\mathbf{G}_{\mathbb{Q}_0}]$, $N[\mathbf{G}_{\mathbb{Q}_0}] \cap \omega_1 \in \mathcal{S}$ and hence there is $q^1_1 \geqslant \underset{\sim}{q}^1_0[\mathbf{G}_{\mathbb{Q}_0}]$ such that,

$$q^1_1 \Vdash_{\underset{\sim}{\mathbb{Q}}_1[\mathbf{G}_0]} N[\mathbf{G}_{\mathbb{Q}_0} * \mathbf{G}_{\underset{\sim}{\mathbb{Q}}_1[\mathbf{G}_{\mathbb{Q}_0}]}] \text{ is } (\bar{R}, \mathcal{S}, \mathbf{g})\text{-good.} \qquad \square$$

The following theorem is central.

Theorem 4.8. *Suppose that $(\bar{R}, \mathcal{S}, \mathbf{g})$ strongly covers in $\mathbf{V}$ for Possibility A (resp. C), and that $\mathbb{P} = \langle \mathbb{P}_i, \mathbb{Q}_j\colon\ i \leqslant \alpha, j < \alpha\rangle$ is a countable support iteration of proper $(\bar{R}, \mathcal{S}, \mathbf{g})$-preserving forcing notions for Possibility A (resp. C). Then $\mathbb{P}_\alpha$ is an $(\bar{R}, \mathcal{S}, \mathbf{g})$-preserving forcing notion for Possibility A (resp. C) and $(\bar{R}, \mathcal{S}, \mathbf{g})$ strongly covers in $\mathbf{V}^{\mathbb{P}}$ for the respective Possibility.*

Proof. We prove by induction of $\zeta \leqslant \alpha$ that for every $\xi \leqslant \zeta$, $\mathbb{P}_\zeta/\mathbb{P}_\xi$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-preserving for Possibility A (resp. C) in $\mathbf{V}^{\mathbb{P}_\xi}$, moreover in Definition 4.5 we can get $\mathrm{dom}(q) \setminus \xi = \zeta \cap N$. For $\zeta = 0$ there is nothing to prove, for ζ successor we use the previous lemma. So let ζ be a limit. We first consider $\mathrm{cf}(\zeta) = \omega$. We fix a strictly increasing sequence $\langle \zeta_\ell\colon\ \ell < \omega\rangle$ with $\zeta_0 = \xi$ and $\sup \zeta_\ell = \zeta$.

First we consider Possibility A. We let $\{\tau_j\colon\ j \in \omega\}$ list the $\mathbb{P}_\zeta$-names of ordinals which belong to N. Let N be $(\bar{R}, \mathcal{S}, \mathbf{g})$-good. In the following we use the convention that the first index indicates that we deal with a $\mathbb{P}_{\zeta_\ell}$-name τ or $\underset{\sim}{f}$ (for a $\mathbb{P}_\zeta/\mathbb{P}_{\zeta_\ell}$-name) and the second index is for the enumeration of the particular subset of N.

We choose by induction on j, $k_j < \omega$ such that

(A) $k_j < k_{j+1}$,
(B) there is a sequence $\langle \underset{\sim}{\tau}_{\ell,j}\colon \ell < j\rangle$ such that $\underset{\sim}{\tau}_{\ell,j}$ is a $\mathbb{P}_{\zeta_\ell}$-name and
 (α) $p_{k_j} \restriction [\zeta_j,\zeta) \Vdash_{\mathbb{P}_\zeta} \underset{\sim}{\tau}_j = \underset{\sim}{\tau}_{j,j}$,
 (β) for $\ell < j$ we have $p_{k_j} \restriction [\zeta_\ell,\zeta_{\ell+1}) \Vdash_{\mathbb{P}_{\zeta_{\ell+1}}} \underset{\sim}{\tau}_{\ell+1,j} = \underset{\sim}{\tau}_{\ell,j}$,
(C) if $j = i+1$, $\ell < i$ then $\Vdash_{\mathbb{P}_{\zeta_{\ell+1}}} p_{k_i} \restriction [\zeta_\ell,\zeta_{\ell+1}) \leqslant p_{k_j} \restriction [\zeta_\ell,\zeta_{\ell+1})$,
(D) if $j = i+1$ then $\Vdash_{\mathbb{P}_\zeta} p_{k_i} \restriction [\zeta_i,\zeta) \leqslant p_{k_j} \restriction [\zeta_i,\zeta)$.

Given k_i, $\langle \tau_{\ell,i}\colon \ell < i\rangle$ we by induction hypothesis the p that fulfil the requirement for $p_{k_{i+1}}$ are dense in $\mathbb{Q} \cap N$, hence by (ix) there is a k_{i+1} such that $p_{k_{i+1}}$ is in that dense set.

Now let $\underset{\sim}{f}_\ell$, $\ell < k$, be given as in ($*$) of Definition 4.5. Let $\{\underset{\sim}{f}_j\colon \ell < j < \omega\}$ list the $\mathbb{P}_\zeta$-names of members on N that are extensions of functions from a to a. For $\ell < k$ let them be the $f^*_{m,\ell}$ as given in ($*$) of Definition 4.5. Since N is $(\bar{R},\mathcal{S},\mathbf{g})$-good, p_n from above can serve as p_n in ($*$). We will now show how to choose $\underset{\sim}{f}^*_{m,j} \in N$, $m < \omega$, $j < \omega$.

Let $h(j,x) < \omega$ be such that $\underset{\sim}{\tau}_{h(j,x)} = \underset{\sim}{f}_j(x)$. We can now define for $n < \omega$, $j < \omega$, $\underset{\sim}{f}^*_{n,j}$ a $\mathbb{P}_{\zeta_n}$-name of a function from a to a. Let $\underset{\sim}{f}^*_{n,j}(x) = \tau_{n,h(j,x)}$ if $h(j,x) \geqslant n$ and $\tau_{h(j,x),h(j,x)}$ if $h(j,x) < n$. So $\underset{\sim}{f}^*_{0,j}(x) = f_j(x)$ for $j < k$. So also for the names $\underset{\sim}{f}^*_{m,j}(x)$ we have (viii) of the hypothesis ($*$), since (viii) holds objects $f_{m,j}$ from there and the sets

$$\{f \in {}^a a\colon\ f R_{\beta,a} \mathbf{g}_a\}$$

are closed for $\beta \in a$. We choose by induction on n, q_n, $\underset{\sim}{\alpha}^n_\ell$ for $\ell < k+n$ such that

(a) $q_n \in \mathbb{P}_{\zeta_n}$, $\operatorname{dom}(q_n) \setminus \xi = N \cap \zeta_n$, $q_{n+1} \restriction \zeta_n = q_n$,
(b) q_n is $(N,\mathbb{P}_{\zeta_n})$-generic,
(c) $q_n \Vdash_{\mathbb{P}_{\zeta_n}} N[\mathbf{G}_{\mathbb{P}_{\zeta_n}}]$ is $(\bar{R},\mathcal{S},\mathbf{g})$-good,
(d) $p_0 \restriction \zeta_0 \leqslant q_0$ in $\mathbb{P}_{\zeta_0}$,
(e) $q_{n+1} \restriction \zeta_n \Vdash_{\mathbb{P}_{\zeta_n}} p_n \restriction [\zeta_n,\zeta_{n+1}) \leqslant p_{n+1} \restriction [\zeta_n,\zeta_{n+1})$ (in $\mathbb{P}_{\zeta_{n+1}}/\mathbb{P}_{\zeta_n}$),
(f) for $\ell < k+n$, $\underset{\sim}{\alpha}^n_\ell$ is a $\mathbb{P}_{\zeta_n}$-name of an ordinal in a, $q_{n+1} \Vdash \underset{\sim}{\alpha}^{n+1}_\ell \leqslant \underset{\sim}{\alpha}^n_\ell$, $\alpha^0_\ell \leqslant \beta^*_\ell$, for $\ell < k$,
(g) for $\ell < k+n$, $q_n \Vdash_{\mathbb{P}_{\zeta_n}} \underset{\sim}{f}^*_{n,\ell}\, R_{\underset{\sim}{\alpha}^n_\ell,a} \mathbf{g}_a$.

The induction step is by the induction hypothesis and by Definition 4.5 of Possibility A with $k+n$ in the role of k. In the end we let $q = \bigcup_{n<\omega} q_n$.

We show that q is $(N,\mathbb{P}_\zeta)$-generic and that is satisfies conclusions (a) and (b) of Definition 4.5. Let $q \in \mathbf{G}_{\mathbb{P}_\zeta} \subseteq \mathbb{P}_\zeta$, $\mathbf{G}_{\mathbb{P}_\zeta}$ be $\mathbb{P}_\zeta$-generic over $\mathbf{V}$. $\mathbf{G}_{\mathbb{P}_\xi} = \mathbf{G}_{\mathbb{P}_\zeta} \cap \mathbb{P}_\xi$ for $\xi < \zeta$ and $\mathbf{G}_{\mathbb{P}_{\zeta_n}} = \mathbf{G}_{\mathbb{P}_\zeta} \cap \mathbb{P}_{\zeta_n}$. Now for each $\mathbb{P}_\zeta$-name $\underset{\sim}{\tau}$ for an ordinal there is some j such that $\underset{\sim}{\tau} = \underset{\sim}{\tau}_j$. q_j forces $\underset{\sim}{\tau}_{j,j} \in N$ and $p_j \restriction [\zeta_j,\zeta)$ forces $\underset{\sim}{\tau}_j = \underset{\sim}{\tau}_{j,j}$. $p_j \restriction [\zeta_j,\zeta) \leqslant q$ by (d) and (e). So q forces $\underset{\sim}{\tau}_j = \underset{\sim}{\tau}_{j,j}$ and $q \Vdash \underset{\sim}{\tau}_j \in N \cap \mathrm{On}$, so q is $(N,\mathbb{P}_\zeta)$-generic.

For each ℓ, $\langle \alpha^n_\ell\colon \ell \leqslant n < \omega\rangle$ is not increasing by (f) and hence eventually constant, say with value α^*_ℓ. If $x \in a$, $j < \omega$, then for $n > h(j,x)$, $p_n \Vdash \underset{\sim}{f}_j(x) = \underset{\sim}{f}^*_{n,j}(x)$. So for every finite $b \subseteq a$, $\langle (\underset{\sim}{f}^*_{n,j} \restriction b)[\mathbf{G}_{\mathbb{P}_{\zeta_n}}]\colon n < \omega\rangle$ is eventually constant, equal to $(\underset{\sim}{f}_j \restriction b)[\mathbf{G}_{\mathbb{P}_\zeta}]$. By (g), for sufficiently large n,

(1) $q \Vdash_{\mathbb{P}_\zeta} (\underset{\sim}{f}_j \restriction b)[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_\zeta}] = (\underset{\sim}{f}^*_{n,j} \restriction b)[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\zeta_n}}]$ and
(2) $q_n \Vdash \underset{\sim}{f}^*_{n,j}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\zeta_n}}] R_{\alpha^n_j,a} \mathbf{g}_a$ and
(3) $\alpha^n_j = \alpha^*_j$.

Since $R_{\alpha^*_j,a}$ is closed, and $\langle \underset{\sim}{f}^*_j \restriction b[\mathbf{G}_{\mathbb{P}_\zeta}]\colon\ b \subseteq a, b \text{ finite}\rangle$ converges to $\underset{\sim}{f}_j$, we get $q \Vdash_{\mathbb{P}_\zeta} \underset{\sim}{f}_j\, R_{\alpha^*_j,a}\mathbf{g}_a$. This finishes the proof of (b), that $q \Vdash_{\mathbb{P}_\zeta} N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_\zeta}]$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good. Now for (a) note that for there is n such that for $\ell < k$, $p_n \Vdash \alpha^*_\ell \leqslant \alpha^n_\ell[\mathbf{G}_{\mathbb{P}_\zeta}] \leqslant \beta^*_\ell$. Thus we finished the proof for the limit of countable cofinality for Possibility A.

Again the proof for Possibility C is short. Let $\langle \underset{\sim}{f}_\ell\colon\ \ell < \omega\rangle$ enumerate the $\mathbb{P}_\zeta$-names $\underset{\sim}{f}: \omega_1 \to \omega_1$ with $\underset{\sim}{f} \in N$. Let $\langle \underset{\sim}{\tau}_n\colon\ n < \omega\rangle$ list the $\mathbb{P}_\zeta$-names of ordinals which belong to N. We choose by induction on n, $\underset{\sim}{p}_n$, q_n, $\underset{\sim}{H}_n$, $\langle \underset{\sim}{f}^n_\ell\colon\ \ell \leqslant n\rangle$ such that

(a) $q_n \in \mathbb{P}_{\zeta_n}$, $\operatorname{dom}(q_n) \setminus \xi \subseteq N \cap \zeta_n$, $q_{n+1} \restriction \zeta_n = q_n$,
(b) q_n is $(N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\zeta_n}}], \mathbb{P}_{\zeta_n})$-generic,
(c) $q_n \Vdash N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\zeta_n}}]$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good,
(d) $\underset{\sim}{p}_n$ is a $\mathbb{P}_{\zeta_n}$-name of a member of $\mathbb{P}_\zeta \cap N$, $q_n \Vdash_{\mathbb{P}_{\zeta_n}} \underset{\sim}{p}_n \restriction \zeta_n \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\zeta_n}}$,
(e) $\underset{\sim}{H}_n$ is a $\mathbb{P}_{\zeta_n}$-name, $\underset{\sim}{H}_n = \{\langle g_0, \dots, g_{n-1}\rangle\colon\ d \subseteq a$ is finite and $\underset{\sim}{p}_n \nVdash_{\mathbb{P}_\zeta/\underset{\sim}{\mathbf{G}}_{\zeta_n}} \langle \underset{\sim}{f}_0 \restriction d, \dots, \underset{\sim}{f}_{n-1} \restriction d\rangle \neq \langle g_0, \dots, g_{n-1}\rangle\}$,
(f) $\underset{\sim}{f}^n_\ell$ is a $\mathbb{P}_{\zeta_n}$-name such that

$$q_n \Vdash_{\mathbb{P}_{\zeta_n}} \langle \underset{\sim}{f}^n_\ell\colon\ \ell < n\rangle \in \underset{\sim}{H}_n \quad \text{and for every } m \leqslant n \text{ we have} \quad \underset{\sim}{p}_{n+1} \nVdash_{\mathbb{P}_\zeta/\mathbb{P}_{\zeta_n}} \neg \bigwedge_{\ell<m} \underset{\sim}{f}_\ell \supseteq \underset{\sim}{f}^m_\ell,$$

(g) $q_n \Vdash_{\mathbb{P}_{\zeta_n}} \underset{\sim}{p}_{n+1}$ forces a value to $\underset{\sim}{\tau}_n$.

There is no problem to carry out the definition and we still have the freedom to choose $\langle \underset{\sim}{f}^n_\ell\colon\ \ell < \omega\rangle$. For this we use the winning strategy from Possibility C of Definition 4.4 choosing there the n-th move of player I as $N_n = N[\mathbf{G}_{\mathbb{P}_{\zeta_n}}]$ and

$$\underset{\sim}{H}_n[\mathbf{G}_{\mathbb{P}_{\zeta_n}}] = \bigl\{\langle g_0, \dots, g_{n-1}\rangle\colon \text{ for some finite } d \subseteq a \text{ we have } g_\ell \in {}^d\omega_1 \text{ for } \ell < n \text{ and}$$
$$\underset{\sim}{p}_n[\mathbf{G}_{\mathbb{P}_{\zeta_n}}] \nVdash_{\mathbb{P}_\zeta/\mathbf{G}_{\mathbb{P}_{\zeta_n}}} \langle \underset{\sim}{f}_0 \restriction d, \dots, \underset{\sim}{f}_{n-1} \restriction d\rangle \neq \langle g_0, \dots, g_{n-1}\rangle\bigr\}$$

so the n-th move is defined in $\mathbf{V}^{\mathbb{P}_{\zeta_n}}$ according to the winning strategy for $\mathbb{P}_{\zeta_{n+1}}$. We can work in $\mathbf{V}^{\mathrm{Levy}(\aleph_0,(2^{|\mathbb{P}_\alpha|})^+)}$. Now of course while playing the universe changes but as the winning strategy is absolute there is no problem.

We let $q = \bigcup q_n$. In the end player II wins, that means for every $m < \omega \bigcup_{n \geqslant n} f^n_m$ is a function which has domain a and $\bigcup_{n \geqslant m} f^n_m R_a \mathbf{g}_a$. By the choice of players I's moves for every $m < \omega$, $q \Vdash_{\mathbb{P}_\zeta} \underset{\sim}{f}_m = \bigcup_{n \geqslant m} f^n_m$. So N is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good. Moreover, also for $\mathbb{P}_\alpha$ player II has a winning strategy in the game: It is just the winning strategy sketched above. If I plays a proper, non-empty subset of the $\underset{\sim}{H}_n$ from Definition 4.4 then II takes an n-tuple from the subset according to his winning strategy in $\mathbf{V}^{\mathbb{P}_{\zeta_n}}$. Thus we finish the limit step of countable cofinality.

Now we continue to look at Possibility C:

If $\operatorname{cf}(\zeta) > \aleph_0$, suppose an initial part of the game $\langle \underset{\sim}{N}_n, \underset{\sim}{H}_n, \langle \underset{\sim}{f}^n_\ell\colon\ \ell \leqslant n\rangle\colon\ n \leqslant m\rangle$ for $\mathbb{P}_\zeta$ is played. We choose $\xi < \zeta$ such these finitely many names for countable objects are $\mathbb{P}_\xi$-names. One can collapse all objects and thus get them into $H(\omega_1)$ and hence they are hereditarily countable and then [3, Lemma 5.13] is applicable and such a ξ exists. Now player II can play according to the strategy for $\mathbb{P}_\xi$. Then player I moves and we choose a new ξ for catching the longer initial segment of the game. In the end II wins, since $\mathbb{P}_\xi \lessdot \mathbb{P}_\zeta$ for all the ξ's on the way and each $f \in N[\mathbf{G}_{\mathbb{P}_\zeta}]$, $: a \to a$ also appears at some stage $\xi < \zeta$, we know that $N[\mathbf{G}_{\mathbb{P}_\zeta}]$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good.

In the case of Possibility A, all the $\mathbb{P}_\zeta$-names $\underset{\sim}{f} \in N$ for functions in ${}^a a$ and the $\underset{\sim}{f}_\ell$, $\ell < k$, are $\mathbb{P}_\xi$-names for a $\xi < \zeta$. $\square$

Definition 4.9. Now fix $(T, Y, \mathcal{S})$ as in Section 1. Assume that $\mathcal{S} \subseteq [\omega_1]^\omega$ stationary and $\delta = \sup(a)$ for $a \in \mathcal{S}$. $Y \subseteq T$ is given, and we fix an enumeration of it as follows: $Y(\delta) = \{t^\delta_n\colon n < \gamma_\delta\} \subseteq T_\delta$ and γ_δ is finite or ω. Now we choose $R_{\alpha,a}$ and $\mathbf{g}_a$ by defining $\{f : a \to a\colon f R_{\alpha,a} \mathbf{g}_a\}$ for $\sup(a) = \delta$ as

(α) $\alpha = 0$ and $f(0) \in \gamma_\delta$ and $f^{-1}[\{1\}] \cap \{s \in T_{<\delta} \cap a\colon 0 <_T s <_T t^\delta_{f(0)}\} \neq \emptyset$, or
(β) $0 < \alpha < \delta$ and $f(0) \in \gamma_\delta$ and $f^{-1}[\{1\}] \cap \{s \in T_{<\delta} \cap a\colon t^\delta_{f(0)} \restriction \alpha \leqslant s\} = \emptyset$,
(γ) $f(0) \notin \gamma_\delta$.

$R_{\alpha,a}$ is a countable union of closed relations, so Possibility A applies.

Lemma 4.10.

(1) *Iff T is $(Y, \mathcal{S})$-proper, then $(\bar{R}, \mathcal{S}, \mathbf{g})$ fully covers.*
(2) *If $(\bar{R}, \mathcal{S}, \mathbf{g})$ covers then $(\bar{R}, \mathcal{S}, \mathbf{g})$ strongly covers for Possibility A.*

Proof. (1) We read the meaning of $\{f \in {}^a a\colon f R_{\beta,a} \mathbf{g}_a\}$ from Definition 4.9. $f(0) \in a \cap \omega_1$. The part (a) of the disjunction means $f^{-1}[\{1\}]$ is a subset of the forcing $T_{<\delta}$, and $\{s\colon s <_t t^\delta_{f(0)}\}$ meets $f^{-1}[\{1\}]$. This is an open relation. Note that it is not written that $f^{-1}[\{1\}]$ be dense. The disjunction (b) means $f^{-1}[\{1\}]$ is not dense in $(T \cap N, <_T)$ since above $t^\delta_{f(0)} \restriction \alpha$ there is no element. This is a closed relation. The disjunction (c) means $t^\delta_{f(0)}$ need not be considered as a generic filter in $T_{<\delta}$ as it is not in $Y(\delta)$. This is a clopen relation, since it speaks only about $f(0)$. Now $(\bar{R}, \mathcal{S}, \mathbf{g})$ covers N iff T is $(Y, \mathcal{S})$-proper.

(2) The $R_{\beta,a}$ are open or closed. $\square$

Lemma 4.11. *$(\bar{R}, \mathcal{S}, \mathbf{g})$ be as in Definition 4.9. A forcing notion $\mathbb{Q}$ is $(\bar{R}, \mathcal{S}, \mathbf{g})$-preserving for Possibility C iff it is so for Possibility A.*

Proof. "$\leftarrow$": A winning strategy for player II is: In the n-th round he uses that I played $N_n \supseteq N_\ell$ and H_n for $n \geqslant \ell$ such that N_n is as in (a) of Definition 4.4(2). In particular $N_n \models (\bar{R}, \mathcal{S}, \mathbf{g})$ covers. So there is $N' \in N_n$ that is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good and $N' \models (\bar{R}, \mathcal{S}, \mathbf{g})$ covers and is $(\bar{R}, \mathcal{S}, \mathbf{g})$-good.

Player II plays $\langle f^n_0, \dots, f^n_{n-1}\rangle \in H_n \cap N_n$ and $d_n \supseteq d_\ell$ for $\ell < n$ and $\beta^n_k \leqslant \beta^\ell_k$ for $k \leqslant n$ and $\ell < n$ such that $\mathrm{dom}(f^n_\ell) = d_n$, for $\ell < n$ and such that $(\forall \ell < n)(\exists m)(f^*_{m,\ell} \supset f^n_\ell \restriction d_n \wedge f^*_{m,\ell} : a \to a \wedge f^*_{m,\ell} R_{\beta^m_\ell, a} \mathbf{g}_a)$, and such that $\bigcup d_n = a$. Since for $\ell < \omega$, the β^m_ℓ, $m < \omega$, become eventually constant to β^*_ℓ, and since $R_{\beta^*_\ell, a}$ is closed, he thus ensures $(\forall \ell \in \omega)(\bigcup_{n \geqslant \ell} f^n_\ell R_{\beta^*_\ell, a} \mathbf{g}_a)$.

"$\rightarrow$": We take the conditions from Possibility A: Let $N \prec \mathcal{H}(\chi)$ be countable, $(\bar{R}, \mathcal{S}, \mathbf{g}) \in N$ and p, $\langle p_n\colon n < \omega\rangle$, $\langle \underset{\sim}{f}_\ell\colon \ell < k\rangle$, $\langle f^*_{m,\ell}\colon \ell < k, m < \omega\rangle$, $\langle \beta^m_\ell, \beta^*_\ell\colon \ell < k, m < \omega\rangle$ be as in $(*)$ of Definition 4.5.

Let $\delta = N \cap \omega_1$. Let $w_m = \{\ell < k\colon f^*_{m,\ell}(0) < \gamma_\delta \wedge \beta^m_\ell \neq 0\}$. For some m_0 all of the finitely many possible w_m appeared. Fix such an m_0 and fix the finitely many witnesses $x_{m,\ell}$, $m < m_0$, $\ell < k$. For $\ell \in k \setminus w_m$ choose $t^\delta_{f^*_{m,\ell}(0)} \restriction \beta^m_\ell \leqslant_T x_{m,\ell} \in T \cap N$ such that $x_{m,\ell} <_T t^\delta_{f^*_{m,\ell}(0)}$ and $\bigvee p_n \Vdash$ "$\underset{\sim}{f}_\ell(x_{m,\ell}) = 1 \vee \underset{\sim}{f}_\ell(0) \geqslant \gamma_\ell$". So for some $n(*)$,

$$p_{n(*)} \Vdash \bigwedge_{\ell \in k \setminus w} \underset{\sim}{f}_\ell(x_{m,\ell}) = 1 \vee \underset{\sim}{f}_\ell(0) \geqslant \gamma_\ell.$$

Let

$$\mathcal{I} = \big\{q \in \mathbb{Q}\colon \text{ for each } \ell \in w, q \text{ forces a value to } \underset{\sim}{f}_\ell, \text{ say } m_\ell \text{ and}$$
$$\text{it forces a truth value to } (\exists x)\big(t^\delta_{m_\ell} \restriction \beta_\ell <_T x \wedge \underset{\sim}{f}_\ell(x) = 1\big)\big\}.$$

So for some $n > n(*)$ we have $p_n \in \mathcal{I}$, and hence all truth values it forces are false, since for $f^*_{m,\ell}$ they are false, since $f^*_\ell R_{\beta^*_\ell,a}\mathbf{g}_a$ and since for any x there is $p_{n'} \geqslant p_n$ and m forcing $\underset{\sim}{f}_\ell(x) = f^*_{m,\ell}(x)$ and we let $\beta^{**}_\ell = \max\{\beta^m_\ell\colon m < m_0\}$. So any $(N,\mathbb{Q})$-generic $q \geqslant p_{n'}$ we have

$$q \Vdash \bigwedge_{m\in\gamma_\delta} \{s \in \delta\colon s <_t t^\delta_m\} \text{ is } \big(N[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}], T\big) \text{ generic},$$

or, in other words, $q \Vdash N[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}]$ is $(\bar{R},\mathcal{S},\mathbf{g})$-good. The existence of such a q follows from Possibility C. For $\underset{\sim}{f}_\ell$, $\ell < k$ moreover $q \Vdash \underset{\sim}{f}_\ell R_{\beta^{*}_\ell{}^*,a}\mathbf{g}_a$ and $\beta^*_\ell{}^* \leqslant \gamma_\ell$. So the conclusions (a) and (b) of Definition 4.5 of preserving in Case A are shown. $\square$

Lemma 4.12. $(\bar{R},\mathcal{S},\mathbf{g})$ *be as in Definition 4.9. A forcing notion $\mathbb{Q}$ is $(\bar{R},\mathcal{S},\mathbf{g})$-preserving for Possibility A iff it is $(T,\mathcal{S},Y)$-preserving.*

Proof. For the forward direction we read the meaning of $(\bar{R},\mathcal{S},\mathbf{g})$. It is easy to fulfil $(*)$ of Definition 4.5 for $k = 0$. Conclusion (b) in Definition 4.5 for Possibility A ensures that $\mathbb{Q}$ is $(T,\mathcal{S},Y)$-preserving.

For the backward direction: We look at Possibility A. All the $\{f\colon fR_{\alpha,a}\mathbf{g}_a\}$ are open or closed, and by reorganising, we can assume that all of them are closed. Fix $k \in \omega$ and p, $\underset{\sim}{f}_\ell$, $\ell < k$, $f^*_{m,\ell}$, $\ell, m < \omega$, β^m_ℓ, $m < \omega$, $\langle p_n\colon n < \omega\rangle$ as in $(*)$ of Definition 4.5. We have to find a particular generic $q \geqslant p$ that also satisfies conclusions (a) and (b) of Definition 4.5.

Now

$$\mathcal{I} = \big\{q \in \mathbb{Q} \cap N\colon\ q \nVdash (\exists \ell < k)(\neg \underset{\sim}{f}_\ell R_{\beta^*_\ell,a}\mathbf{g}_a)\big\}$$

is dense above some p_n in $\mathbb{P}$ since $\bigcup_{\ell<k}\{f \in {}^a a\colon \neg f R_{\beta^*_\ell,a}\mathbf{g}_a\}$ is open in the Baire space ${}^a a$. So first we take $p_1 \geqslant p$, $p_1 \in \mathcal{I} \cap N$, and then we take, according to preserving in Possibility C, $q \geqslant p_1$ such that q is $(N,\mathbb{P})$-generic and $q \Vdash N[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}]$ is $(\bar{R},\mathcal{S},\mathbf{g})$-good. $\square$

So now for the proof of Theorem 4.2 we can use Lemma 4.6 and Theorem 4.8 for Possibility A or for Possibility C, and we are done.

References

[1] Uri Abraham, Proper forcing, in: Matthew Foreman, Akihiro Kanamori (Eds.), Handbook of Set Theory, Springer, 2010.
[2] Tomek Bartoszyński, Haim Judah, Set Theory, on the Structure of the Real Line, A.K. Peters, 1995.
[3] Andreas Blass, Saharon Shelah, There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin–Keisler ordering may be downward directed, Ann. Pure Appl. Logic 33 (1987) 213–243.
[4] Andreas Blass, Saharon Shelah, Near coherence of filters III: A simplified consistency proof, Notre Dame J. Form. Log. 30 (1989) 530–538.
[5] Jörg Brendle, Cardinal invariants of the continuum and combinatorics on uncountable cardinals, Ann. Pure Appl. Logic 144 (2006) 43–72.
[6] Martin Goldstern, Tools for your forcing construction, in: Judah Haim (Ed.), Set Theory of the Reals, in: Israel Math. Conf. Proc., vol. 6, 1993, pp. 305–360.
[7] Martin Goldstern, Jakob Kellner, New reals: Can live with them, can live without them, Math. Log. Q. 52 (2) (2006) 115–124.

[8] Thomas Jech, Set Theory. The Third Millennium Edition, revised and expanded, Springer, 2003.
[9] Jakob Kellner, Non-elementary proper forcing, Rend. Semin. Mat. Univ. Politec. Torino (2011), in press, arXiv:0910.2132v2 [math.LO].
[10] Jakob Kellner, Saharon Shelah, Preserving preservation, J. Symbolic Logic 70 (3) (2005) 914–945.
[11] Kenneth Kunen, Set Theory, an Introduction to Independence Proofs, North-Holland, 1980.
[12] Heike Mildenberger, The club principle and the distributivity number, J. Symbolic Logic 76 (1) (2011) 34–46.
[13] Heike Mildenberger, Saharon Shelah, Specializing Aronszajn trees by countable approximations, Arch. Math. Logic 42 (2003) 627–648.
[14] Arnold Miller, Arnie Miller's problem list, in: Judah Haim (Ed.), Set Theory of the Reals, Ramat Gan, 1991, in: Israel Math. Conf. Proc., vol. 6, Bar-Ilan Univ., Ramat Gan, 1993, pp. 645–654.
[15] Adam J. Ostaszewski, A perfectly normal countably compact scattered space which is not strongly zero-dimensional, J. Lond. Math. Soc. (2) 14 (1) (1976) 167–177.
[16] Andrzej Rosłanowski, Saharon Shelah, Norms on Possibilities I: Forcing with Trees and Creatures, Mem. Amer. Math. Soc., vol. 671, Amer. Math. Soc., 1999.
[17] Andrzej Rosłanowski, Saharon Shelah, The last forcing standing, preprint [RoSh:942], 2009.
[18] Andrzej Rosłanowski, Saharon Shelah, Lords of the iteration, in: Set Theory and its Applications, in: Contemp. Math., vol. 533, Amer. Math. Soc., Providence, RI, 2011, pp. 287–330.
[19] Andrzej Rosłanowski, Saharon Shelah, Reasonable ultrafilters, again, Notre Dame J. Form. Log. 52 (2) (2011) 113–147.
[20] Fritz Rothberger, Sur un ensemble toujours de première catégorie qui est dépourvu de la propriété λ, Fund. Math. 32 (1939) 294–300.
[21] Saharon Shelah, On cardinal invariants of the continuum, in: J. Baumgartner, D. Martin, S. Shelah (Eds.), Axiomatic Set Theory. Proceedings of the Conference in Set Theory, Boulder, June 1983, in: Contemp. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1984, pp. 183–207.
[22] Saharon Shelah, Proper and Improper Forcing, second ed., Springer, 1998.
[23] Saharon Shelah, Properness without elementaricity, J. Appl. Anal. 10 (2004) 168–289, arXiv:math.LO/9712283.
[24] Peter Vojtáš, Set theoretic characteristics of summability and convergence of series, Comment. Math. Univ. Carolin. 28 (1987) 173–184.