Appendix to "The Coherence of Semifilters: a Survey"

The Semifilter Generated by $\{A + A : a \in [\mathbb{N}]^\omega\}$ Is Comeagre and Pinning Down Some Cardinals

Heike Mildenberger

We answer the first question in Problem 28.1 in the article [1] and also several questions on cardinal characteristics from the same article.

Problem 28.1

A subset $\mathcal{S} \subseteq [\mathbb{N}]^\omega$ such that $(\forall X \in \mathcal{S})(\forall Y)(X \setminus Y$ finite $\to Y \in \mathcal{S})$ is called a semifilter. Every $\mathcal{B} \subseteq [\mathbb{N}]^\omega$ generates a semifilter $\mathcal{S} = \{a + b : a \in A, b \in B\}$.

Theorem A.1. *The semifilter $\mathcal{S}$ generated by $\{A+A : A \in [\mathbb{N}]^\omega\}$ is comeagre.*

Proof – Let $N \subseteq [\mathbb{N}]^\omega$ be not meagre. We show that (the semifilter generated by $\{A + A : A \in [\mathbb{N}]^\omega\}) \cap N \neq \emptyset$. Then the semifilter generated by $\{A + A : A \in [\mathbb{N}]^\omega\}$ has no non-meagre complement, and hence is comeagre. We assume that (the semifilter generated by $\{A + A : A \in [\mathbb{N}]^\omega\}) \cap N = \emptyset$. Since semifilters are closed under supersets, we can assume that N is closed under subsets.

For $n \in \omega$ we choose a part of a real x_n such that $x_n : [2^{2n}, 2^{2n+2}) \to 2$, $x_n(2^{2n}) = 1$, $x_n(2^{2n} + 2^{2i}) = 1$ for $i = 0, \cdots n$; on the other points in $[2^{2n}, 2^{2n+2})$, x_n can be arbitrary.

We use the following characterizations of meagreness from [2; 2.2.4]: A set M is meagre if there is an increasing function $f : \omega \to \omega$ ans a sequence $\langle x_n : n < \omega \rangle$, $x_n : [f(n), f(n+1)) \to 2$, such that

$$M = \big\{x \in 2^\omega : (\forall^\infty n)(x \restriction [f(n), f(n+1)) \neq x_n)\big\}.$$

Now we apply this with $f(n) = 2^{2n}$ and x_n above, to the non-meagre set N and get

$$(\exists x \in N)(\exists^\infty n)(x \restriction [f(n), f(n+1)) = x_n).$$

We fix such an $x \in N$ and enumerate the infinitely many n such that $x \restriction [f(n), f(n+1)) = x_n$ in increasing order, say by $\langle n_i : i \in \omega \rangle$, and let $A = \{2^{n_i} : i < \omega\}$. Then $A + A \subseteq \{m : (\exists n)(x(m) = x_n(m) = 1)\}$, and hence, since N is the complement of a semifilter, and by assumption closed under subsets, we get $A+A = \{2^{n_i}+2^{n_j} : i \leq j < \omega\} \in N$ and hence $\{A+A : A \in [\mathbb{N}]^\omega\}) \cap N \neq \emptyset$. $\square$ **Please, check if the close bracket is correct.**

Problem 28.2

Now we change the topic. We go over to the table in the last section of [1].

Theorem A.2. *In the Cohen model* $\mathfrak{g}_f = \aleph_1$.

Proof – This follows from Blass' work [3]. The proof of [3; Theorem 2] actually wotks with groupwise dense ideals. So Corollary there gives that in the Cohen model and in the random model $\mathfrak{g}_f = \aleph_1$. $\square$

Theorem A.3. *In the Cohen model* $\mathfrak{g}_u = \aleph_1$.

Proof – From [6; Prop. 2.1.c] it follows $\mathfrak{g}_u \leq \mathfrak{mcf}$. And from Canjar's thesis [5] it follows that in the Cohen model $\mathfrak{mcf} = \aleph_1$. $\square$

Since $\mathfrak{g}_f \leq \mathfrak{g}_u$, we thus get anotehr proof of the previous theorem.

Theorem A.4. *In the model called Blass, we have* $\mathfrak{s} = \aleph_1$.

Proof – Both Cohen forcing and Miller forcing preserve $\sqsubseteq^{Cohen}$ from [2; 6.3.15], see [2; 6.3.18 and 7.3.46]. Hence by [2; 6.3.20] in the final model V_0 is not meagre and $\mathfrak{s} \leq \mathrm{unif}(\mathcal{M}) = \aleph_1$. $\square$

For Problem 28.3 we remark that in the model from [4] we have 2^{2^ω} near coherence classes of ultrafilters, so also so many coherence classes of semifilters.

References

[1] Banakh, T. and Zdomskyy, L.: The coherence of semifilters: a survey, *Quad. Mat.* **18**, Dept. Math., Seconda Univ. Napoli, Caserta (2006), 53–103.

[2] Bartoszyński, T. and Judah, H.: *Set Theory: On the Structure of the Real Line*, A.K. Peters, Wellesly, Massachusets (1995).

[3] Blass, A.: Applications of superperfect forcing and its relatives, in *Set Theory and its Applications* (J. Steprans and S. Watson, eds.), Lecture Notes in Mathematics **1401** (1989), 18–40.

[4] Blass, A. and Shelah, S.: Ultrafilters with small generating sets, *Israel Journal of Mathematics* **65** (1989), 259–270.

[5] Canjar, R.M.: Countable ultraproducts without CH, *Annals of Pure and Applied Logic* **37** (1988), 1–79.

[6] Mildenberger, H.: Groupwise dense families, *Archive for Mathematical Logic* **40** (2) (2001), 93–112.