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L-SPACES AND THE P-IDEAL DICHOTOMY

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Abstract. We extend a theorem of Todorčević: Under the assumption $(\mathcal{K})$ (see Definition 1.11),

$$\boxtimes \qquad \begin{cases} \text{any regular space } Z \text{ with countable tightness such that} \\ Z^n \text{ is Lindelöf for all } n \in \omega \text{ has no } L\text{-subspace.} \end{cases}$$

We assume $\mathfrak{p} > \omega_1$ and a weak form of Abraham and Todorčević's P-ideal dichotomy instead and get the same conclusion. Then we show that $\mathfrak{p} > \omega_1$ and the dichotomy principle for P-ideals that have at most $\aleph_1$ generators together with $\boxtimes$ do not imply that every Aronszajn tree is special, and hence do not imply $(\mathcal{K})$. So we really extended the mentioned theorem.

1. Introduction

A regular topological space X is an *L-space*, if it is hereditarily Lindelöf but not separable. The following theorem is a classical fact about L-spaces.

THEOREM 1.1 (Szentmiklóssy [13]). *Assume* MA_{ω_1}*, and let Z be a compact space with countable tightness. Then Z has no L-subspaces.*

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Using similar techniques as in [11], Todorčević extended Theorem 1.1 to the following:

THEOREM 1.2 (Todorčević [16, Theorem 7.10]). $\boxtimes$ *follows from* $(\mathcal{K})$.

The principle $(\mathcal{K})$ follows from MA_{ω_1}, Martin's Axiom for ω_1 dense sets, and it implies that all Aronszajn trees are special, see [16, Remarks 7.16]. We will review the mentioned principles at the end of this section.

We prove that the conclusion of Theorem 1.2 holds under another assumption (see Statement 1.12):

THEOREM 1.3. $\boxtimes$ *follows from* $\mathfrak{p} > \omega_1$ *and* WPID.

Let us note that neither $\mathfrak{p} > \omega_1$ nor WPID alone are sufficient in Theorem 1.3: as it was noted in [11, p. 266], strong L-spaces remain so in all forcing extensions $V^{\mathbb{P}}$ such that $\mathbb{P}$ has the Knaster property, which means that for any $A \in [\mathbb{P}]^{\omega_1}$ there exists a centred subcollection $B \in [A]^{\omega_1}$. There is a Knaster forcing increasing $\mathfrak{p}$. WPID is consistent with CH, and under CH there are many constructions of regular topological spaces X all of whose finite powers are L-spaces (i.e. *strong L-spaces*).

In the light of Theorems 1.2 and 1.3 it is natural to ask about the relations between MA_{ω_1} and the conjunction of $\mathfrak{p} > \omega_1$ and WPID. Theorem 1.4 below describing such relations is the main result of this paper.

THEOREM 1.4. *The following is consistent:* PID (ω_1*-generated*) *and* $\mathfrak{p} > \omega_1$, $\boxtimes$, *and there is a nonspecial Aronszajn tree. Therefore,* PID (ω_1*-generated*) *and* $\mathfrak{p} > \omega_1$ *does not imply* $(\mathcal{K})$.

If we assume the existence of a supercompact cardinal then the same is true about the (full version of) PID.

Thus Theorem 1.3 adds more cases as compared with Todorčević's Theorem 1.2. In the other direction we have the following consequence of [15, Theorem 7], whose conclusion resembles WPID (ω_1-generated).

THEOREM 1.5. MA_{ω_1} *implies that for every ideal $\mathcal{I}$ on an uncountable set S with ω_1-many generators, either there exists $T \in [S]^{\omega_1}$ locally in $\mathcal{I}$, or there exists $\mathcal{J} \in [\mathcal{I}]^{\omega_1}$ such that $\bigcup \mathcal{J}' \notin \mathcal{I}$ for any infinite $\mathcal{J}' \subset \mathcal{J}$.*

However, the following question remains open.

QUESTION 1.6. *Let* P *be one of the statements* MA_{ω_1} *and* $(\mathcal{K})$, *and* Q *be one of* WPID (ω_1*-generated*), WPID, PID (ω_1*-generated*), PID_{ω_1}. *Does* P *imply* Q?

Since MA_{ω_1} does not put any upper bound on $\mathfrak{b}$, the following theorem shows that MA_{ω_1} does not imply PID.

THEOREM 1.7 (Todorčević [17]). PID *implies* $\mathfrak{b} \leqq \omega_2$.

For the reader's convenience we write a complete proof of Theorem 1.7 at the end of this section, which seems to be not available elsewhere.

Theorems 1.1 and 1.2 arose from the following classical fact:

THEOREM 1.8 (Kunen [11]). *Assume* MA_{ω_1}. *If X is an L-space, then some of its finite powers is not hereditarily Lindelöf.*

Theorems 1.3 and 1.2 imply the following statement, which extends Theorem 1.8:

COROLLARY 1.9. *Assume that* ($\mathfrak{p} > \omega_1$ *and* WPID) *or* MA_{ω_1}. *Then every L-space of size ω_1 has a non-Lindelöf finite power.*

Now we recall the relevant combinatorial principles.

A family $\mathcal{J}$ of infinite subsets of ω is *centred*, if the intersection of any finite subfamily of $\mathcal{J}$ is infinite. We standardly denote by $\mathfrak{p}$ the smallest size of a centred subfamily of $[\omega]^\omega$ without infinite pseudo-intersection.

DEFINITION 1.10. Let S be an uncountable set and let $[S]^{<\omega} = K_0 \cup K_1$ be a partition. Then this is called a *c.c.c. partition* if all singletons are in K_0 and K_0 is closed under subsets and every uncountable subset of K_0 has two elements whose union is in K_0.

DEFINITION 1.11. The principle $(\mathcal{K})$ (see [16, Ch. 7]) says: For any c.c.c. partition (S, K_0, K_1) there is an uncountable $H \subseteqq S$ such that $[H]^2 \subseteqq K_0$. If we replace 2 by the finite number m in the dimension (still partitioning into two parts) we get $(\mathcal{K}_m)$.

It is easy to see that $(\mathcal{K})$ implies also $\mathfrak{p} > \omega_1$: Given a centered subset $\mathcal{F}$ of $[\omega]^\omega$ of size ω_1, take $\mathbb{P}_\mathcal{F}$ from Section 3 and partition $[\mathbb{P}_\mathcal{F}]^{<\omega} = K_0 \cup K_1$, $\{p_0, p_1, \ldots, p_n\} \in K_0$ if there is an upper bound on this set. This is a c.c.c. partition and any uncountable homogeneous set will give a pseudo-intersection to an uncountable part of $\mathcal{F}$, and hence, since $\mathcal{F}$ can assumed to be $\subseteqq^*$-descending, to all of $\mathcal{F}$.

An ideal $\mathcal{I}$ on a set S is a *P-ideal*, if for every countable $\mathcal{J} \subset \mathcal{I}$ there exists $I \in \mathcal{I}$ such that $J \subset^* I$ for all $J \in \mathcal{J}$. We say that $T \subset S$ is *locally in* (resp. *orthogonal to*) the ideal $\mathcal{I}$, if $[T]^\omega \subset \mathcal{I}$ (resp. $\mathscr{P}(T) \cap \mathcal{I} = [T]^{<\omega}$). We consider the ideals containing all singletons. We shall use the following two statements:

WPID: For every P-ideal on an uncountable set S, either S contains an uncountable subset locally in $\mathcal{I}$, or an uncountable subset orthogonal to $\mathcal{I}$.

PID: For every P-ideal on an uncountable set S, either S contains an uncountable subset locally in $\mathcal{I}$, or S can be decomposed into countably many pieces orthogonal to $\mathcal{I}$.

The principle PID (abbreviated from *P-ideal dichotomy*) was introduced in [17]. We write PID_κ for the restriction of PID for sets S of size at most κ and we write PID (ω_1-generated) for the the restriction of PID for P-ideals $\mathcal{I}$ with at most ω_1 generators. Similarly with WPID (here "W" stands for "weak"), but WPID_κ is obviously equivalent to WPID_{ω_1} for every cardinal κ. The WPID (ω_1-generated) was used in [6] in context of S-spaces. The principle PID follows from PFA and is consistent with CH, see [17]. PID_{ω_2} contradicts the existence of Jensen's square sequence at ω_2 (i.e. $\square_{\omega_1}$), and thus any of its consistency proof must involve large cardinals, see [17, p. 258]. Therefore MA does not imply PID_{ω_2}.

Finally, we present the proof of Theorem 1.7. It is a direct consequence of Lemmas 1.12 and 1.13 below. Lemma 1.12 is modelled after the second paragraph of [17], while the proof of Lemma 1.13 can be found in [9, p. 578]. For the reader's convenience we give a complete proof of Lemma 1.12, which seems not to be available elsewhere. Given a relation R on ω and $x, y \in \omega^\omega$, we denote the set $\big\{n \in \omega : x(n)\, R\, y(n)\big\}$ by $[xRy]$.

Let κ, λ be regular cardinals. A (κ, λ)-*pregap* $\big\langle \{f_\alpha\}_{\alpha<\kappa}, \{g_\beta\}_{\beta<\lambda} \big\rangle$ is a pair of transfinite sequences $\langle f_\alpha : \alpha < \kappa\rangle$ and $\langle g_\beta : \beta < \lambda\rangle$ of nondecreasing sequences f_α, g_β of natural numbers such that $f_{\alpha_1} \leqq^* f_{\alpha_2} \leqq^* g_{\beta_2} \leqq^* g_{\beta_1}$ for all $\alpha_1 \leqq \alpha_2 < \kappa$ and $\beta_1 \leqq \beta_2 < \lambda$. As usual, $f \leqq^* g$ means that the set $[f > g]$ is finite. A (κ, λ)-pregap is called a (κ, λ)-*gap*, if there is no $h \in \omega^\omega$ such that $f_\alpha \leqq^* h \leqq^* g_\beta$ for all α, β.

LEMMA 1.12. *Suppose that there exists a (κ, λ)-gap such that κ is regular, $\mathfrak{b} > \kappa > \omega_1$, and $\lambda \geqq \omega_1$. Then PID_λ fails.*

PROOF. Assume that such a gap $\big\langle \{f_\alpha\}_{\alpha<\kappa}, \{g_\beta\}_{\beta<\lambda} \big\rangle$ exists and, contrary to our claim, PID_λ holds. Set

$$\mathcal{I} = \Big\{A \in [\lambda]^\omega : \exists \alpha \in \kappa\, \forall \gamma \geqq \alpha\, \forall n \in \omega \Big(\big| \big\{\beta \in A : [f_\gamma > g_\beta] \subset n\big\} \big| < \omega \Big) \Big\}.$$

We claim that $\mathcal{I}$ is a P-ideal. Indeed, let $\{A_i : i \in \omega\}$ be a sequence of mutually disjoint elements of $\mathcal{I}$, α_i be a witness for $A_i \in \mathcal{I}$, and $\alpha = \sup\{\alpha_i : i \in \omega\}$. Fix $\gamma \geqq \alpha$. Let $B_i(\gamma) = \big\{\beta \in A_i : [f_\gamma > g_\beta] \subset i\big\}$. Then $B_i(\gamma)$ is a finite subset of A_i by the definition of $\mathcal{I}$. Since $\kappa < \mathfrak{b}$, there exists a sequence $\langle B_i : i \in \omega\rangle$ such that each B_i is a finite subset of A_i and for every $\gamma \geqq \alpha$, $B_i(\gamma) \subset B_i$ for all but finitely many i.

Set $A = \bigcup_{i\in\omega} A_i \setminus B_i$ and fix $n \in \omega$ and $\gamma \geqq \alpha$. If $\beta \in A_i$ is such that $[f_\gamma > g_\beta] \subset n$ and $i \geqq n$, then $\beta \in B_i(\gamma)$. Let $j_\gamma \geqq n$ be such that $B_i \supset B_i(\gamma)$ for all $i \geqq j_\gamma$. Then for every $i \geqq j_\gamma$ the set $\big\{\beta \in A_i : [f_\gamma > g_\beta] \subset n\big\}$ is a subset of B_i, and hence

$$\big\{\beta \in A : [f_\gamma > g_\beta] \subset n\big\} \subset \Big\{\beta \in \bigcup_{i<j_\gamma} A_i : [f_\gamma > g_\beta] \subset n\Big\}.$$

Since $\gamma \geqq \alpha = \sup_{i\in\omega} \alpha_i$, the latter set is finite.

Applying PID_λ to $\mathcal{I}$ we conclude that one of the following alternatives is true:

1. There exists $S \in [\lambda]^{\omega_1}$ such that $[S]^\omega \subset \mathcal{I}$. Passing to an uncountable subset of S if necessary, we can assume that $S = \{\beta_\xi : \xi < \omega_1\}$ and $\beta_\xi < \beta_\eta$ for any $\xi < \eta < \omega_1$. For every ξ we denote by S_ξ the set $\{\beta_\zeta : \zeta < \xi\}$.

By the definition of $\mathcal{I}$ for every ξ there exists $\alpha_\xi \in \kappa$ witnessing for $S_\xi \in \mathcal{I}$. Let $\alpha = \sup\{\alpha_\xi : \xi < \omega_1\}$. There exists $n \in \omega$ such that the set $C = \{\xi < \omega_1 : [f_\alpha > g_{\beta_\xi}] \subset n\}$ is uncountable. Let ξ_0 be the ω-th element of C. Then $\alpha \geqq \alpha_{\xi_0}$ and for all $\xi \in C \cap \xi_0$ we have $[f_\alpha > g_{\beta_\xi}] \subset n$. On the other hand, $S_{\xi_0} \in \mathcal{I}$, and hence there should be only finitely many such $\xi \in \xi_0$, a contradiction.

2. $\lambda = \bigcup_{m\in\omega} S_m$ such that S_m is orthogonal to $\mathcal{I}$ for all $m \in \omega$. This means that for every $m \in \omega$ and $\alpha \in \kappa$ there exists $\gamma_\alpha > \alpha$ and $n_{m,\alpha} \in \omega$ such that $[f_{\gamma_\alpha} > g_\beta] \subset n_{m,\alpha}$ for all $\beta \in S_m$. There is no (κ, ω)-gap for $\kappa < \mathfrak{b}$ (see the proof of Theorem 29.8 on page 578 in [9]), and hence we assume that $\mathrm{cf}\,(\lambda) > \omega$. Therefore there exists $m \in \omega$ such that S_m is cofinal in λ. Let $n \in \omega$ be such that the set $J = \{\alpha \in \kappa : n_{m,\alpha} = n\}$ is cofinal in κ. For every k let $h(k) = \max\{f_{\gamma_\alpha}(k) : \alpha \in J\}$. From the above it follows that $[g_\beta < h] \subset n$ for all $\beta \in S_m$, and hence h contradicts the fact that $\langle \{f_\alpha\}_{\alpha<\kappa}, \{g_\beta\}_{\beta<\lambda}\rangle$ is a gap. $\square$

LEMMA 1.13. *If $\mathfrak{b} > \omega_2$ then there is an (ω_2, λ)-gap for some uncountable λ.*

2. Proof of Theorem 1.3

We recall that a family $\mathcal{V}$ of subsets of a set X is *point-finite*, if for every $x \in X$ the set $\{V \in \mathcal{V} : x \in V\}$ is finite. The following statement could be classified as folklore.

LEMMA 2.1. *If a topological space T has an uncountable point-finite family of open subsets, then it is not hereditarily Lindelöf.*

PROOF. Assume that T is hereditarily Lindelöf and there exists an uncountable point-finite family $\mathcal{V} = \{V_\alpha : \alpha < \omega_1\}$ of open nonempty subsets of T. Then we can construct a transfinite sequence $(\alpha_\beta)_{\beta<\omega_1}$ of countable ordinals such that for all $\beta < \omega_1$,

$$\bigcup_{\alpha \geqq \sup\{\alpha_\xi : \xi<\beta\}} V_\alpha = \bigcup_{\alpha_\beta > \alpha \geqq \sup\{\alpha_\xi : \xi<\beta\}} V_\alpha.$$

Let $\zeta = \sup\{\alpha_n : n \in \omega\}$. From the above it follows that $V_\zeta \subset \bigcup_{\alpha_{n+1}>\alpha\geqq\alpha_n} V_\alpha$ for all $n \in \omega$, and hence any point $t \in V_\zeta$ is a witness for the family $\{V_\alpha : \alpha < \omega_1\}$ being not point-finite, a contradiction. $\square$

Let us recall some definitions. A family $\mathcal{U}$ of subsets of a set X is an ω-*cover* (resp. γ-*cover*) of X [7], if $X \notin \mathcal{U}$ and for every finite $F \subset X$ the set $\{U \in \mathcal{U} : F \subset U\}$ is nonempty (resp. for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$

is finite.) It is clear that $\mathcal{U}$ is a γ-cover of X if and only if $\mathcal{V} = \{X \setminus U : U \in \mathcal{U}\}$ is point-finite. A topological space X is called an *ε-space* (resp. a *γ-space*) [7], if any open ω-cover of X contains a countable ω-subcover (resp. γ-subcover). It is easy to check [7] that X is an ε-space iff all finite powers of X are Lindelöf. We recall from [5] that a *Σ-product* of a family $\{X_\alpha : \alpha \in A\}$ of topological spaces with base-point $(p_\alpha)_{\alpha\in A}$ is the dense subspace Σ of $\prod_{\alpha\in A} X_\alpha$ consisting of those points x for which $x_\alpha = p_\alpha$ for all but countably many α's. In what follows we shall be interested in the Σ-product of ω_1-many unit intervals with the base-point (0), and we denote this space by Σ_{ω_1}. The space Σ_{ω_1} has countable tightness [5].

Let $\mathcal{B}$ be the standard base of the topology of $[0,1]^{\omega_1}$ consisting of products $\prod_{\alpha\in\omega_1} U_\alpha$, where each U_α is an open interval with rational end-points, and $U_\alpha = [0,1]$ for all but finitely many α.

Given any topological space X with a base $\mathcal{C}$, let us denote by $\mathcal{I}_{X,\mathcal{C}}$ the set of all at most countable point-finite subfamilies of $\mathcal{C}$.

Lemma 2.2. *Let V_0, V_1, V_2 be three models of* ZFC *such that $\omega_1^{V_0} = \omega_1^{V_1} = \omega_1^{V_2}$, $V_0 \subset V_1 \subset V_2$, and $\omega^\omega \cap V_0 = \omega^\omega \cap V_1$. Then there is no ε-subspace $X \in V_2$ of Σ_{ω_1} containing a dense subspace $Y \in V_0$ with the following properties:*

(i) *Y is an L-subspace in V_2 (and hence in V_0 and V_1);*

(ii) $V_0 \models$ *"$\mathcal{I}_{Y,\mathcal{B}|Y}$ is a P-ideal";*

(iii) $V_1 \models$ "WPID *is true for $\mathcal{I}^{V_0}_{Y,\mathcal{B}|Y}$";*

(iv) $V_0 \models$ *"Every countable ω-cover of Y by elements of $\mathcal{B}|Y$ contains a γ-subcover"; and*

(v) *for every α there exists $x \in Y$ such that $x_\alpha = 1$.*

Proof. Assume that such X and Y exist. Let us note that $\mathcal{B}^{V_0} = \mathcal{B}^{V_1}$ and $(\mathcal{B}|Y)^{V_0} = (\mathcal{B}|Y)^{V_1} = (\mathcal{B}|Y)^{V_2}$, where $\mathcal{B}|Y = \{B \cap Y : B \in \mathcal{B}\}$. Indeed, the first equality follows from the fact that there are no new reals in V_1, while the second is a consequence of $Y \in V_0$. Therefore we can simply write $\mathcal{B}|Y$ in what follows. For every $\alpha \in \omega_1$ let $U_\alpha = \mathrm{pr}_\alpha^{-1}(1/2,1] \cap Y$ and $O_\alpha = \mathrm{pr}_\alpha^{-1}[0,1/2) \cap X$. Applying (in V_1) WPID to the P-ideal $\left(\mathcal{I}_{Y,\mathcal{B}|Y}\right)^{V_0}$ restricted to the family $\{U_\alpha : \alpha \in \omega_1\}$, we conclude that one of the subsequent alternatives holds.

1. There is an uncountable family $\mathcal{W} \in V_1$ of $\{U_\alpha : \alpha \in \omega_1\}$ such that each countable subset $\mathcal{U} \in V_0$ of $\mathcal{W}$ belongs to $\left(\mathcal{I}_{Y,\mathcal{B}|Y}\right)^{V_0}$. Since V_1 and V_0 have the same reals, this means that in V_1, $\mathcal{W}$ is a point-finite family of open subsets of Y of size ω_1. Applying Lemma 2.1, we conclude that Y is not an L- space in V_1, a contradiction.

2. There exists $A \in [\omega_1]^{\omega_1} \cap V_1$ such that no infinite subset of $\{U_\alpha : \alpha \in A\}$ belongs to $\left(\mathcal{I}_{Y,\mathcal{B}|Y}\right)^{V_0}$. $\{O_\alpha : \alpha \in A\}$ is an ω-cover of X, and hence there exists a countable $I \subset A$, $I \in V_2$, such that $\{O_\alpha : \alpha \in I\}$ is an ω-cover of X. Since $A \in V_1$ and $\omega_1^{V_1} = \omega_1^{V_2}$, there exists a countable subset $I' \in V_1$ of A containing I, and therefore $I' \in V_0$. Consequently, $\{O_\alpha \cap Y : \alpha \in I'\}$ is a countable ω-cover of Y in V_0, and hence it contains a countable γ-subcover. This means that there exists a countable subset $I'' \in V_0$ of I' such that $\{O_\alpha \cap Y : \alpha \in I''\}$ is a γ-cover of Y, and hence $\{U_\alpha : \alpha \in I''\} \in \left(\mathcal{I}_{Y,\mathcal{B}|Y}\right)^{V_0}$, which contradicts the choice of A. $\square$

COROLLARY 2.3. *Assume that* WPID *holds. Then there is no ε-space X with countable tightness containing an L-subspace Y with the following properties:*

(i) *$\mathcal{I}_{Y,\tau}$ is a P-ideal, where τ is the topology of Y; and*

(ii) *Y is a γ-space.*

PROOF. Assume that that there are such X and Y. Without loss of generality [10], $Y = \{y_\alpha : \alpha \in \omega_1\}$, $Y_\beta := \{y_\alpha : \alpha < \beta\}$ is closed in Y for all $\beta < \omega_1$, and Y is dense in X. Set $F_\beta = \mathrm{cl}_X(Y_\beta)$. From the above it follows that $F_\beta \cap Y = Y_\beta$. Since X has countable tightness, $X = \bigcup_{\beta<\omega_1} F_\beta$.

For every $\alpha < \omega_1$ there exists an open neighbourhood U_α of y_α such that $\mathrm{cl}_X(U_\alpha) \cap F_\alpha = \emptyset$. Let $f_\alpha : X \to [0,1]$ be a continuous function such that $f|F_\alpha = 0$ and $f(y_\alpha) = 1$, and $f : X \to [0,1]^{\omega_1}$ be the diagonal product of f_α's, i.e. $f(x)_\alpha = f_\alpha(x)$. Then $f(X)$ and $f(Y)$ fulfil all the conditions of Lemma 2.2 with $V = V_0 = V_1 = V_2$, a contradiction. $\square$

We recall that $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of ω^ω with respect to $\leqq^*$. It is well-known (and easy) that $\mathfrak{p} \leqq \mathfrak{b}$.

PROOF OF THEOREM 1.3. Follows from Corollary 2.3 and the following well-known observations.

CLAIM 2.4. *Let $|T| < \mathfrak{b}$, $\mathcal{F}$ a family of subsets of T, and V the family of all at most countable point-finite subfamilies of $\mathcal{F}$. Then V is a P-ideal on $\mathcal{F}$.*

PROOF. Let us fix any $\{\mathcal{V}_n : n \in \omega\} \subset \mathsf{V}$. There is no loss of generality to assume that each $\mathcal{V}_n$ is infinite. Let $\mathcal{V}_n = \{V_{n,m} : m \in \omega\}$ be a bijective enumeration of $\mathcal{V}_n$. Using the point-finiteness of $\mathcal{V}_n$'s, for every $t \in T$ find a number sequence $\left(m_t(n)\right)_{n\in\omega}$ such that $t \notin V_{n,m}$ for all $n \in \omega$ and $m \geqq m_t(n)$. Since $|T| < \mathfrak{b}$, there exists a number sequence $\left(m(n)\right)_{n\in\omega}$ such that for every $t \in T$ the inequality $m_t(n) \leqq m(n)$ holds for all but finitely many $n \in \omega$. A direct verification shows that $\mathcal{V} = \left\{V_{n,m} : m \geqq m(n)\right\} \in \mathsf{V}$ and $\mathcal{V}_n \subset^* \mathcal{V}$ for all $n \in \omega$, which finishes our proof. $\square$

CLAIM 2.5. *Every countable ω-cover $\mathcal{V}$ of a set T of size $|T| < \mathfrak{p}$ contains a γ-subcover.*

PROOF. Since $\mathcal{V}$ is an ω-cover, the family $\{\{V \in \mathcal{V} : z \in V\} : z \in T\}$ is a centred family of infinite subsets of $\mathcal{V}$ of size $< \mathfrak{p}$, and hence it has a pseudo-intersection $\mathcal{V}'$. Then $\mathcal{V}'$ is a γ-cover of T. $\square$

PROOF OF COROLLARY 1.9. Follows from Theorem 1.3 and the following fact: For every L-space T of size ω_1 there exists a continuous surjection $f : T \to Z$ onto some L-space Z of countable tightness. The latter can be proved using the same arguments as in the second paragraph of Corollary 2.3. $\square$

3. Proof of Theorem 1.4

Todorčević's result, that $(\mathcal{K})$ implies that all Aronszajn trees are special, gives the second part of the theorem. For the proof of the first part, we build on a deep work by Hirschorn [8] and on a preservation theorem by Abraham [1] and just add an epsilon to it.

The following notion was introduced by Shelah for his proof [12, Ch. IX] that the non-existence of Souslin trees does not imply that all Aronszajn trees are special.

DEFINITION 3.1 [12, Definition IX.4.5]. We call a forcing notion $\mathbb{P}$ (T,S)-*preserving* if the following holds: T is an Aronszajn tree, $S \subseteqq \omega_1$, and for every $\lambda > \left(2^{|\mathbb{P}|+\aleph_1}\right)^+$ and countable $N \prec \left(H(\lambda), \in\right)$ such that $\mathbb{P}, T, S \in N$ and $\delta = N \cap \omega_1 \notin S$, and every $p \in N \cap \mathbb{P}$ there is some $q \geqq p$ (bigger conditions are stronger) such that

(1) q is $(N, \mathbb{P})$ generic; and

(2) for every $x \in T_\delta$, if $\left(x \in A \to (\exists y <_T x) y \in A\right)$ for all $A \in \mathscr{P}(T) \cap N$, then $q \Vdash_{\mathbb{P}} \left(x \in \dot{A} \to (\exists y < x) y \in \dot{A}\right)$ for every $\mathbb{P}$-name $\dot{A} \in N$ such that $\Vdash_{\mathbb{P}} \dot{A} \subset T$.

Let $\mathcal{F}$ be a centred family of infinite subsets of ω of size $|\mathcal{F}| = \omega_1$. For such an $\mathcal{F}$ we let $\mathbb{P}_{\mathcal{F}} = (P, \leqq_P)$ be defined by $P = \left\{(s, F) : F \in [\mathcal{F}]^{<\omega}, s \in [\omega]^{<\omega}\right\}$. $(t, G) \geqq_P (s, F)$ if $t \supseteqq s$, $t \smallsetminus s \subseteqq \bigcap F$ and $G \supseteqq F$.

LEMMA 3.2. *Let T be an Aronszajn tree, $S = \emptyset$, and $\mathcal{F}$ be a centred subfamily of $[\omega]^\omega$ of size ω_1. Then the forcing $\mathbb{P}_{\mathcal{F}}$ is (T,S)-preserving.*

PROOF. Let $\mathcal{F} = \{a_\alpha : \alpha < \omega_1\}$ and $\bar{g} = \langle g_\alpha : \alpha < \omega_1\rangle$ be a sequence of infinite subsets of ω such that g_α is a pseudo-intersection of a_β, $\beta < \alpha$. Fix an $N \prec (H(\lambda), \in)$ with T, S, P, $\bar{g}$, $p = (s, F) \in N$. Let $\delta = N \cap \omega_1$. We take $\delta^* = \sup\left\{\varphi(\delta) + 1 : \varphi \in N,\ \varphi(\delta) \text{ is an ordinal } < \omega_1\right\}$.

We show that $q = p$ is as desired. Let $q_1 \geqq q$, $q_1 \in \mathbb{P}_{\mathcal{F}}$. We show that there is some $q_2 \geqq q_1$ such that $q_2 \Vdash x \notin \dot{A}$ or $q_2 \Vdash (\exists y <_T x)(y \in \dot{A})$. If $q_1 \Vdash x \notin \dot{A}$, then we can choose $q_2 = q_1$. Otherwise there is $q_3 \geqq q_1$,

$q_3 \Vdash x \in \dot{A}$. We take any $q_4 = \big(s(q_4), F(q_4)\big) \geqq q_3$ and show that q_4 can be extended to a condition that forces $y \in \dot{A}$ for some $y <_T x$. Then by density it follows that q_3 forces $(x \in \dot{A} \to (\exists y <_T x) y \in A)$.

The proof is an adaption of the steps given on pp. 456–457 in [12]. We define a function $f\colon T \to \omega_1 \cup \{\omega_1\}$ as follows:

$$f(y) = \sup \Big\{ \alpha \in \omega_1 : \text{there are pairwise disjoint nonempty finite } t_i,\ i \in \omega, \\ t_i \subset g_\alpha \text{ such that } \forall i\, \exists F_{1,i}\ \big(t_i \cup s(q_4), F_{1,i}\big) \Vdash y \in \dot{A} \big) \Big\}.$$

$f \in N$, as it is defined by a first order formula in $(H(\lambda), \in)$ with parameters in N. Let

$$A^* = \big\{ y \in T : f(y) = \omega_1 \big\}.$$

$A^* \in N$ is a set, not just a name for a set.

Let $f^*\colon \omega_1 \to \omega_1$ be defined by

$$f^*(\alpha) = \sup \big\{ f(y) + 1 : y \in T_\alpha, f(y) < \omega_1 \big\}.$$

By the definition of δ^*, $f^*(\delta) < \delta^*$. The condition q_4 exemplifies that $f(x) \geqq \delta^*$, since we can take witnesses t_i that are just pairwise disjoint nonempty finite subsets of $\bigcap F(q_4) \cap g_\xi$, where $\xi > \delta^*$ is any countable ordinal above all η such that $a_\eta \in F(q_4)$. So $f(x) = \omega_1$ and $x \in A^*$. Now we use the hypothesis and get some $y <_T x$, $y \in A^*$. Say the highest index in one of the members of $F(q_4)$ is α_0. Then since $f(y) = \omega_1$ there are $\alpha_1 > \alpha_0$ and pairwise disjoint nonempty finite t_i, $i \in \omega$, all subsets of g_{α_1}, such that for every i for some $F_{1,i}$

$$\big(t_i \cup s(q_4), F_{1,i}\big) \Vdash y \in \dot{A}.$$

But then some of the t_i is not only a subset of the pseudo-intersection $g_{\alpha_1} \subseteqq^* \bigcap F(q_4)$ but really a subset of $\bigcap F(q_4)$ and hence for this t_i we have $\big(t_i \cup s(q_4), F_{1,i} \cup F(q_4)\big) \Vdash y \in \dot{A}$ and $\big(t_i \cup s(q_4), F_{1,i} \cup F(q_4)\big) \geqq q_4$. $\square$

Now we prove Theorem 1.4. Assume that GCH holds in the ground model. We will use an iterated forcing construction $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha : \beta \leqq \omega_2, \alpha < \omega_2 \rangle$ which takes care about all posets of the form $\mathbb{P}_{\mathcal{F}}$ defined before Lemma 3.2, along the odd α's. This way we shall guarantee $\mathfrak{p} > \omega_1$ in $V^{\mathbb{P}_{\omega_2}}$. Concerning even α's (those of the form $2 \cdot \beta$), $\dot{\mathbb{Q}}_\alpha$ is the $\mathbb{P}_\alpha$ name of the Abraham–Todorčević forcing $\mathbb{Q}_{I_\alpha}$ (again [2, p. 172]) corresponding to an ω_1-generated P-ideal I_α in $V^{\mathbb{P}_\alpha}$. This is a proper ω_2-c.c. forcing of size ω_2 (provided GCH holds in the ground model) which does not add reals and such that the P-ideal dichotomy for I_α holds in the extension [2]. This forcing satisfies the

ω_2-properness isomorphism condition by [2, Lemma 3.6], and hence the resulting poset $\mathbb{P}_{\omega_2}$ has the ω_2-c.c., see [12, Ch. V]. Combining Lemma 3.2 with the two deep results below, we conclude that every non-special Aronszajn tree in V remains non-special in $V^{\mathbb{P}_{\omega_2}}$.

THEOREM 3.3 (Hirschorn [8]). *Let T be an Aronszajn tree and let $S = \emptyset$. The Abraham–Todorčević forcing is (T, S)-preserving.*

THEOREM 3.4 (Abraham [1]). *If every iterand $\dot{\mathbb{Q}}_\alpha$ is forced to be (T, S)-preserving, then the countable support iteration is (T, S)-preserving.*

A standard book-keeping of all possible ω_1 generated P-ideals allows us to get PID (ω_1-generated) in $V^{\mathbb{P}_{\omega_2}}$. In order to get $\boxtimes$ in addition, we need to be a little bit more careful when we set up our book-keeping. We use an auxiliary Lemma 3.5 for this.

A subset C of an ordinal is called ω_1*-closed* if it is closed under increasing limits of cofinality ω_1. An ω_1*-club (in ω_2)* is an unbounded subset (of ω_2) that is ω_1-closed. A set is called ω_1*-stationary* if it intersects every ω_1-club.

LEMMA 3.5. *Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha : \beta \leqq \omega_2, \alpha < \omega_2 \rangle$ be a countable support iteration of proper iterands such that* CH *holds in $V^{\mathbb{P}_\beta}$ for all $\beta < \omega_2$, $\mathfrak{p} > \omega_1$ holds in $V^{\mathbb{P}_{\omega_2}}$, $\mathbb{P}_{\omega_2}$ has the ω_2-c.c., and the iterands at even stages force the P-ideal dichotomy for ideals with ω_1 generators.*

Let $\beta < \omega_2$ and let $X \in V^{\mathbb{P}_\beta}$ be a subspace of Σ_{ω_1} of size ω_1. Let $\psi(X)$ = "$\mathcal{I}_{X,\mathcal{B}|X}$ is a P-ideal, and each countable ω-cover of X contains a γ-subcover". Then the set of all those $\alpha < \omega_2$ such that $V^{\mathbb{P}_\alpha} \models \psi(X)$ is ω_1-closed and unbounded in ω_2.

PROOF. The closure under increasing ω_1-sequences is proved as in model theory. Just note that there are no new names for countable ω-covers in the limit model. Now for the unboundedness: Let α_0 be given. We choose a continuously increasing sequence α_i, $i < \omega_1$, such that in $V^{\mathbb{P}_{\alpha_{i+1}}}$, $P_{X,\mathcal{B}|X}$ is a P-ideal and each countable ω-cover of X in $V^{\mathbb{P}_{\alpha_i}}$ (by the CH there are at most ω_1 tasks here) contains a γ-subcover. Then $\alpha_{\omega_1} := \sup_{\alpha<\omega_1} \alpha_i$ is in the set. $\square$

By [4, Lemma 5.2] for every subspace Y of Σ_{ω_1} of size ω_1 in $V^{\mathbb{P}_{\omega_2}}$ there exists $\beta < \omega_2$ and a $\mathbb{P}_\beta$-name $\dot{Y}$ of Y. Let Y_α, $\alpha < \omega_2$, enumerate all such subspaces in $V^{\mathbb{P}_{\omega_2}}$ and assume that Y_α has a $\mathbb{P}_{\beta_\alpha}$-name $\dot{Y}_\alpha$ (which is also a $\mathbb{P}_\varepsilon$-name for all $\varepsilon \in [\beta_\alpha, \omega_2]$). Let C_α be an ω_1-club of $\xi \geqq \beta_\alpha$ such that $\psi(Y_\alpha)$ holds in $V^{\mathbb{P}_\xi}$ (as in the lemma before). We divide $\{\alpha < \omega_2 : \operatorname{cof}(\alpha) = \omega_1\}$ into ω_2 pairwise disjoint ω_1-stationary sets (as in [9, p. 94]) $\langle S_\alpha : \alpha < \omega_2 \rangle$. Now, if our book-keeping was arranged in such a way that the $\mathbb{P}_\xi$ name

$\dot{\mathbb{Q}}_{\mathcal{I}_{Y_\alpha,\mathcal{B}|Y_\alpha}}$ [1] is used for some $\xi = \xi_\alpha \in S_\alpha \cap [\beta_\alpha, \omega_2) \cap C_\alpha$, the extension $V^{\mathbb{P}_\alpha}$ has the required properties. Indeed, assume that in $V^{\mathbb{P}_{\omega_2}}$ there exists an ε-space X of countable tightness and an L-subspace Y of X. By the same argument as in the proof of Corollary 2.3 we can assume that $X \subset \Sigma_{\omega_1}$, $|Y| = \omega_1$, and for every $\beta \in \omega_1$ there exists $y \in Y$ such that $y_\beta = 1$. Let $\alpha \in \omega_2$ be such that $Y = Y_\alpha$. Now $V_0 = V^{\mathbb{P}_{\xi_\alpha}}$, $V_1 = V^{\mathbb{P}_{\xi_\alpha+1}}$, $V_2 = V^{\mathbb{P}_{\omega_2}}$, X, and Y fulfil the premises of Lemma 2.2, a contradiction. This finishes the proof of Theorem 1.4. $\square$

4. Applications in C_p-theory

The properties of a topological space X appearing in Section 2 have counterparts among the properties of $C_p(X)$, the space of all continuous functions $f\colon X \to \mathbb{R}$ endowed with the topology of pointwise convergence. Namely:

• A regular space X is a γ-space if and only if $C_p(X)$ has the Fréchet–Urysohn property, see [7]. We recall that a topological space Y has the Fréchet–Urysohn property if for every $y \in Y$ and $A \subset Y$ such that $y \in \overline{A}$, there exists a sequence of elements of A convergent to y.

• The existence of an uncountable point-finite family of open subsets of a topological space X is equivalent [14] to the existence of a copy in $C_p(X)$ of the one-point compactification $\alpha\,\omega_1$ of ω_1 with the discrete topology.

• The methods of [3] imply that for a perfectly normal (= every open subset is F_σ) space X, the ideal $P_{X,\tau}$ is a P-ideal iff $C_p(X)$ has the property α_1. (Here τ denotes the topology of X.) Recall that a topological space Y has the property α_1 at a point $y \in Y$, if for every countable family $\mathcal{A}$ of convergent to y sequences and for each $A \in \mathcal{A}$ there exists $B_A \in [A]^{<\omega}$ such that $\bigcup_{A\in\mathcal{A}}(A \setminus B_A)$ converges to y. Y has the property α_1, if it has this property at every $y \in Y$.

Therefore it could be possible to apply some ideas from previous sections to the space of continuous functions. We illustrate this by giving a simple[2] (modulo the equivalences above) proof of Corollary 1.9 based on the following straightforward observation: A topological space Y has the property α_1 at a point $y \in Y$ if and only if the family of all countable subsets of Y converging to y constitutes P-ideal.

LEMMA 4.1. (PID) *Let Y be a topological space. If Y has the property α_1 at a point $y \in Y$, then either Y contains a copy Z of $\alpha\,\omega_1$ with y being the unique non-isolated point of Z, or $Y \setminus \{y\}$ can be written as a countable*

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[1] I.e. for every $\mathbb{P}_\xi$- generic filter G_ξ the interpretation $\dot{\mathbb{Q}}^{G_\xi}_{\mathcal{I}_{Y_\alpha,\mathcal{B}|Y_\alpha}}$ coincides with the Abraham–Todorčević forcing corresponding to the P-ideal $\mathcal{I}_{Y_\alpha,\mathcal{B}|Y_\alpha}$ computed in $V[G_\xi]$.

[2] The price for the simplicity of that proof is that we use PID there instead of WPID.

union $\bigcup_{n\in\omega} Y_n$ *such that for every* n *there is no convergent to* y *sequence of elements of* Y_n*.*

If, in addition, Y *has the Fréchet–Urysohn property at* y*, then none of* Y_n*'s contains* y *in its closure, i.e.* Y *has countable pseudo-character at* y*.*
□

Applying Lemma 4.1 to $C_p(X)$, we obtain the following

PROPOSITION 4.2 (PID). *Let* X *be a Tychonov space. If* $C_p(X)$ *has the property* α_1 *and the Fréchet–Urysohn property, then either* $C_p(X)$ *contains a copy of* $\alpha\,\omega_1$*, or* X *is separable.*

PROOF. If $C_p(X)$ does not contain a copy of $\alpha\,\omega_1$, then $C_p(X)$ must have countable pseudo-character by Lemma 4.1. The above means that we can find a sequence $(F_n,\varepsilon_n)_{n\in\omega}$, where $F_n \in [X]^{<\omega}$ and $\varepsilon_n > 0$, such that a continuous function $f\colon X \to \mathbb{R}$ coincides with 0 provided $\left|f(x)\right| < \varepsilon_n$ for all $x \in F_n$ and $n \in \omega$. Together with regularity of X this clearly implies that $\bigcup_{n\in\omega} F_n$ is dense in X. □

ALTERNATIVE PROOF OF COROLLARY 1.9. Assume, to the contrary, that X is an L-space of size ω_1 and, in addition, X is an ε-space. Since $\mathfrak{p} > \omega_1$, $C_p(X)$ has the Fréchet–Urysohn property (every open ω-cover contains a countable ω-subcover by the definition of an ε-space, and every open ω-cover of X contains a γ-subcover by Claim 2.5, and therefore X is a γ-space). In addition, $C_p(X)$ has the property α_1 by Claim 2.4 and [3, Theorem 11]. From the above it follows that X fulfills the premises of Proposition 4.2. Since X is an L-space, it is not separable, and hence $C_p(X)$ contains a copy of $\alpha\,\omega_1$. Therefore there exists [14] an uncountable point-finite family of open subsets of X, which contradicts Lemma 2.1. □

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