SPLITTING FAMILIES AND COMPLETE SEPARABILITY

HEIKE MILDENBERGER, DILIP RAGHAVAN, AND JURIS STEPRĀNS

Abstract. We answer a question from Raghavan and Steprāns [5] by showing that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. Then we use this to construct a completely separable maximal almost disjoint family under $\mathfrak{s} \leq \mathfrak{a}$, partially answering a question of Shelah [6].

1. Introduction

The purpose of this short note is to answer a question posed by the second and third authors in [5] and to use this to solve a problem of Shelah [6]. We say that two infinite subsets a and b of ω are *almost disjoint or a.d.* if $a \cap b$ is finite. We say that a family $\mathscr{A}$ of infinite subsets of ω is *almost disjoint or a.d.* if its members are pairwise almost disjoint. A *Maximal Almost Disjoint family, or MAD family* is an infinite a.d. family that is not properly contained in a larger a.d. family.

For an a.d. family $\mathscr{A}$, let $\mathcal{I}(\mathscr{A})$ denote *the ideal on ω generated by $\mathscr{A}$* – that is, $a \in \mathcal{I}(\mathscr{A})$ iff $\exists a_0, \ldots, a_k \in \mathscr{A} \, [a \subset^* a_0 \cup \cdots \cup a_k]$. For any ideal $\mathcal{I}$ on ω, $\mathcal{I}^+$ denotes $\mathcal{P}(\omega) \setminus \mathcal{I}$. An a.d. family $\mathscr{A} \subset [\omega]^\omega$ is said to be *completely separable* if for any $b \in \mathcal{I}^+(\mathscr{A})$, there is an $a \in \mathscr{A}$ with $a \subset b$. Notice that an infinite completely separable a.d. $\mathscr{A}$ must be MAD. Though the following is one of the most well-studied problems in set theory, it continues to remains open.

Question 1 (Erdős and Shelah [3], 1972). *Does there exist a completely separable MAD family $\mathscr{A} \subset [\omega]^\omega$?*

Progress on Question 1 was made by Balcar, Dočkálková, and Simon who showed in a series of papers that completely separable MAD families can be constructed from any of the assumptions $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{s} = \omega_1$, or $\mathfrak{d} \leq \mathfrak{a}$. See [1], [2], and [7] for this work. Then Shelah [6] recently showed that the existence of completely separable MAD families is *almost* a theorem of ZFC. His construction is divided into three cases. The first case is when $\mathfrak{s} < \mathfrak{a}$ and he shows on the basis of ZFC alone that a completely separable MAD family can be constructed in this case. The second and third cases are when $\mathfrak{s} = \mathfrak{a}$ and $\mathfrak{a} < \mathfrak{s}$ respectively and Shelah shows that a completely separable MAD family can be constructed in these cases *provided* that certain PCF type hypotheses are satisfied. More precisely, he shows that there is a completely separable MAD family when $\mathfrak{s} = \mathfrak{a}$ and $U(\mathfrak{s})$ holds, or when $\mathfrak{a} < \mathfrak{s}$ and $P(\mathfrak{s}, \mathfrak{a})$ holds.

Definition 2. For a cardinal $\kappa > \omega$, $U(\kappa)$ is the following principle. There is a sequence $\langle u_\alpha : \omega \le \alpha < \kappa \rangle$ such that

    (1) $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$

    (2) $\forall X \in [\kappa]^\kappa \exists \omega \le \alpha < \kappa \, [|u_\alpha \cap X| = \omega]$.

For cardinals $\kappa > \lambda > \omega$, $P(\kappa, \lambda)$ says that there is a sequence $\langle u_\alpha : \omega \le \alpha < \kappa \rangle$ such that

    (3) $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$

    (4) for each $X \subset \kappa$, if X is bounded in κ and $\text{otp}(X) = \lambda$, then $\exists \omega \le \alpha < \sup(X) \, [|u_\alpha \cap X| = \omega]$.

It is easy to see that both $U(\mathfrak{s})$ and $P(\mathfrak{s}, \mathfrak{a})$ are satisfied when $\mathfrak{s} < \aleph_\omega$, so in particular, the existence of a completely separable MAD family is a theorem of ZFC when $\mathfrak{c} < \aleph_\omega$. Shelah [6] asked whether all uses of PCF type hypotheses can be eliminated from the second and third cases.

The second and third authors modified the techniques of Shelah [6] in order to treat MAD families with few partitioners in [5] (see the introduction there). In that paper they introduced a cardinal invariant $\mathfrak{s}_{\omega,\omega}$, which is a variation of the splitting number $\mathfrak{s}$. They showed that if $\mathfrak{s}_{\omega,\omega} \le \mathfrak{b}$, then there is a weakly tight family. Recall that an a.d. family $\mathscr{A} \subset [\omega]^\omega$ is called *weakly tight* if for every countable collection $\{b_n : n \in \omega\} \subset \mathcal{I}^+(\mathscr{A})$, there is $a \in \mathscr{A}$ such that $\exists^\infty n \in \omega \, [|b_n \cap a| = \omega]$. The question of whether $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ was raised in [5], and the authors pointed out that an affirmative answer to this question could help eliminate the use of PCF type hypotheses from the second case of Shelah's construction.

In this paper we answer this question from [5] by proving that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. We then use this information to partially answer the question from Shelah [6]. We show that the second case can be done without any additional hypothesis. So it is a theorem of ZFC alone that a completely separable MAD family exists when $\mathfrak{s} \le \mathfrak{a}$. We give a single construction from this assumption, so Shelah's first and second cases are unified into a single case.

The question of whether the hypothesis $P(\mathfrak{s}, \mathfrak{a})$ can be eliminated from the case when $\mathfrak{a} < \mathfrak{s}$ remains open.

2. $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$

In this section we answer Question 21 from [5] by showing that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. For a set $x \subset \omega$, x^0 is used to denote x and x^1 is used to denote $\omega \setminus x$. This notation will be used in the next section also. Recall the following definitions.

Definition 3. For $x, a \in \mathcal{P}(\omega)$, x *splits* a if $\left|x^0 \cap a\right| = \left|x^1 \cap a\right| = \omega$. $\mathcal{F} \subset \mathcal{P}(\omega)$ is called a *splitting family* if $\forall a \in [\omega]^\omega \exists x \in \mathcal{F} \, [x \text{ splits } a]$. $\mathcal{F} \subset \mathcal{P}(\omega)$ is said to be (ω, ω)-*splitting* if for each countable collection $\{a_n : n \in \omega\} \subset [\omega]^\omega$, there exists $x \in \mathcal{F}$ such that $\exists^\infty n \in \omega \, \left[\left|x^0 \cap a_n\right| = \omega\right]$ and $\exists^\infty n \in \omega \, \left[\left|x^1 \cap a_n\right| = \omega\right]$. Define

$$\mathfrak{s} = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{P}(\omega) \wedge \mathcal{F} \text{ is a splitting family}\}$$

$$\mathfrak{s}_{\omega,\omega} = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{P}(\omega) \wedge \mathcal{F} \text{ is } (\omega, \omega) - \text{splitting}\}.$$

Obviously every (ω, ω)-splitting family is a splitting family. So $\mathfrak{s} \le \mathfrak{s}_{\omega,\omega}$. It was shown in Theorem 13 of [5] that if $\mathfrak{s} < \mathfrak{b}$, then $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. We reproduce that result here for the reader's convenience.

Lemma 4 (Theorem 13 of [5]). *If $\mathfrak{s} < \mathfrak{b}$, then $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.*

Proof. Let $\langle e_\alpha : \alpha < \kappa \rangle$ witness that $\kappa = \mathfrak{s}$. Suppose $\{b_n : n \in \omega\} \subset [\omega]^\omega$ is a countable collection such that $\forall \alpha < \kappa \exists i \in 2 \forall^\infty n \in \omega \left[b_n \subset^* e_\alpha^i \right]$. By shrinking them if necessary we may assume that $b_n \cap b_m = 0$ whenever $n \neq m$. Now, for each $\alpha < \kappa$ define $f_\alpha \in \omega^\omega$ as follows. We know that there is a unique $i_\alpha \in 2$ such that there is a $k_\alpha \in \omega$ such that $\forall n \geq k_\alpha \left[\left| b_n \cap e_\alpha^{i_\alpha} \right| < \omega \right]$. We define $f_\alpha(n) = \max \left(b_n \cap e_\alpha^{i_\alpha} \right)$ if $n \geq k_\alpha$, and $f_\alpha(n) = 0$ if $n < k_\alpha$. As $\kappa < \mathfrak{b}$, there is a $f \in \omega^\omega$ with $f \,^*\!> f_\alpha$ for each $\alpha < \kappa$. Now, for each $n \in \omega$, choose $l_n \in b_n$ with $l_n \geq f(n)$. Since the b_n are pairwise disjoint, $c = \{l_n : n \in \omega\} \in [\omega]^\omega$. So by definition of $\mathfrak{s}$, there is $\alpha < \kappa$ such that $\left| c \cap e_\alpha^0 \right| = \left| c \cap e_\alpha^1 \right| = \omega$. In particular, $c \cap e_\alpha^{i_\alpha}$ is infinite. However we know that there is an $m_\alpha \in \omega$ such that $\forall n \geq m_\alpha \left[f_\alpha(n) < f(n) \right]$. So there exists $n \geq \max\{m_\alpha, k_\alpha\}$ with $l_n \in b_n \cap e_\alpha^{i_\alpha}$. But this is a contradiction because $l_n \leq f_\alpha(n) < f(n)$. $\dashv$

In the case when $\mathfrak{b} \leq \mathfrak{s}$ it turns out that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ can still be proved by considering the following notion appearing in [4].

Definition 5. $\mathcal{F}$ is called *block-splitting* if given any partition $\langle a_n : n \in \omega \rangle$ of ω into finite sets there is a set $x \in \mathcal{F}$ such that there are infinitely many n with $a_n \subset x$ and there are infinitely many n with $a_n \cap x = 0$.

It was proved by Kamburelis and Węglorz [4] that the least size of a block splitting family is $\max\{\mathfrak{b}, \mathfrak{s}\}$. Therefore, when $\mathfrak{b} \leq \mathfrak{s}$, there is a block splitting family of size $\mathfrak{s}$.

Theorem 6. $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.

Proof. In view of Lemma 4, we may assume that $\mathfrak{b} \leq \mathfrak{s}$. By results of Kamburelis and Węglorz [4] fix $\langle x_\alpha : \alpha < \mathfrak{s} \rangle \subset \mathcal{P}(\omega)$, a block splitting family. We show that $\langle x_\alpha : \alpha < \mathfrak{s} \rangle$ is an (ω, ω)-splitting family. Let $\{a_n : n \in \omega\} \subset [\omega]^\omega$ be given. For $n \in \omega$, define $s_n \in [\omega]^{<\omega}$ as follows. Suppose $\langle s_i : i < n \rangle$ have been defined. Put $s = \bigcup_{i<n} s_i$. Put $s_n = \{\min(\omega \setminus s)\} \cup \{\min(a_i \setminus s) : i \leq n\}$. Note that $\langle s_n : n \in \omega \rangle$ is a partition of ω into finite sets and that $\forall i \in \omega \forall^\infty n \in \omega \left[s_n \cap a_i \neq 0 \right]$. Now choose $\alpha < \mathfrak{s}$ such that $\exists^\infty n \in \omega \left[s_n \subset x_\alpha^0 \right]$ and $\exists^\infty n \in \omega \left[s_n \subset x_\alpha^1 \right]$. So for each $i \in \omega$, $\exists^\infty n \in \omega \left[s_n \cap a_i \cap x_\alpha^0 \neq 0 \right]$ and $\exists^\infty n \in \omega \left[s_n \cap a_i \cap x_\alpha^1 \neq 0 \right]$. Since the s_n are pairwise disjoint, it follows that $\left| a_i \cap x_\alpha^0 \right| = \left| a_i \cap x_\alpha^1 \right| = \omega$, for each $i \in \omega$. $\dashv$

3. Constructing a completely separable MAD family from $\mathfrak{s} \leq \mathfrak{a}$

As $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ and as every (ω, ω)-splitting family is also a splitting family, fix once and for all a sequence $\langle x_\alpha : \alpha < \kappa \rangle$ witnessing that $\kappa = \mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. We will construct a completely separable MAD family assuming that $\kappa \leq \mathfrak{a}$. The construction closely follows the proof of Lemma 8 in [5], which in turn is based on Shelah [6]. An important point of the construction is that if $\mathscr{A}$ is an arbitrary a.d. family and $b \in \mathcal{I}^+(\mathscr{A})$, then every (ω, ω)-splitting family contains an element which splits b into two *positive* pieces.

Lemma 7. *Let $\mathscr{A} \subset [\omega]^\omega$ be any a.d. family. Suppose $b \in \mathcal{I}^+(\mathscr{A})$. Then there is $\alpha < \kappa$ such that $b \cap x_\alpha^0 \in \mathcal{I}^+(\mathscr{A})$ and $b \cap x_\alpha^1 \in \mathcal{I}^+(\mathscr{A})$.*

Proof. See proof of Lemma 7 of [5]. $\dashv$

At a stage $\delta < \mathfrak{c}$, an a.d. family $\mathscr{A}_\delta = \langle a_\alpha : \alpha < \delta \rangle \subset [\omega]^\omega$ is given. Moreover we assume that there is also a family $\langle \sigma_\alpha : \alpha < \delta \rangle \subset 2^{<\kappa}$ such that for each $\alpha < \delta$,

$\forall \xi < \mathrm{dom}(\sigma_\alpha) \left[a_\alpha \subset^* x_\xi^{\sigma_\alpha(\xi)} \right]$. We say that σ_α is *the node associated with* a_α. The next lemma says that under the assumption $\kappa \leq \mathfrak{a}$, such an a.d. family must be "nowhere maximal", which is of course a property that we need to maintain in order to end up with a completely separable MAD family.

Definition 8. Let $\eta \in 2^{<\kappa}$. Define $\mathcal{I}_\eta = \left\{ a \in \mathcal{P}(\omega) : \forall \xi < \mathrm{dom}(\eta) \left[a \subset^* x_\xi^{\eta(\xi)} \right] \right\}$.

Lemma 9 (Main Lemma). *Assume $\kappa \leq \mathfrak{a}$. Let $\delta < \mathfrak{c}$. Suppose that $\mathscr{A}_\delta = \langle a_\alpha : \alpha < \delta \rangle$ and $\langle \sigma_\alpha : \alpha < \delta \rangle$ are as above. Assume also that $\forall \alpha, \beta < \delta \left[\alpha \neq \beta \implies \sigma_\alpha \neq \sigma_\beta \right]$. Let $b \in \mathcal{I}^+(\mathscr{A}_\delta)$. Then there exist $a \in [b]^\omega$ and $\sigma \in 2^{<\kappa}$ such that*

    (1) $\forall \alpha < \delta \left[|a \cap a_\alpha| < \omega \right]$.
    (2) for each $\alpha < \delta$, $\sigma \not\subset \sigma_\alpha$ and $a \in I_\sigma$.

Proof. Applying Lemma 7, let $\alpha_0 < \kappa$ be least such that $b \cap x_{\alpha_0}^0 \in \mathcal{I}^+(\mathscr{A}_\delta)$ and $b \cap x_{\alpha_0}^1 \in \mathcal{I}^+(\mathscr{A}_\delta)$. Define $\tau_0 \in 2^{\alpha_0}$ by stipulating that

$$\forall \xi < \alpha_0 \forall i \in 2 \left[\tau_0(\xi) = i \leftrightarrow b \cap x_\xi^i \in \mathcal{I}^+(\mathscr{A}_\delta) \right].$$

By choice of α_0 and by the hypothesis that $b \in \mathcal{I}^+(\mathscr{A}_\delta)$, τ_0 is well defined. Now, construct two sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset \kappa$ and $\langle \tau_s : s \in 2^{<\omega} \rangle \subset 2^{<\kappa}$ such that

    (3) $\forall s \in 2^{<\omega} \forall i \in 2 \left[\alpha_s = \mathrm{dom}(\tau_s) \wedge \alpha_{s^\frown\langle i \rangle} > \alpha_s \wedge \tau_{s^\frown\langle i \rangle} \supset \tau_s{}^\frown\langle i \rangle \right]$.
    (4) for each $s \in 2^{<\omega}$ and for each $\xi < \alpha_s$, $x_\xi^{1-\tau_s(\xi)} \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \in \mathcal{I}(\mathscr{A}_\delta)$.
        Here when $s = 0$, $\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)}$ is taken to be ω.
    (5) for each $s \in 2^{<\omega}$, both $x_{\alpha_s}^0 \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \in \mathcal{I}^+(\mathscr{A}_\delta)$ and $x_{\alpha_s}^1 \cap b \cap$
        $\left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \in \mathcal{I}^+(\mathscr{A}_\delta)$.

α_0 and τ_0 are already defined. Suppose that α_s and τ_s are given. By (5) for each $i \in 2$, $x_{\alpha_s}^i \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \in \mathcal{I}^+(\mathscr{A}_\delta)$. Apply Lemma 7 to let $\alpha_{s^\frown\langle i \rangle}$ be the least $\alpha < \kappa$ such that both $x_{\alpha_s}^i \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \cap x_\alpha^0$ and $x_{\alpha_s}^i \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \cap x_\alpha^1$ are in $\mathcal{I}^+(\mathscr{A}_\delta)$. Again define $\tau_{s^\frown\langle i \rangle} \in 2^{\alpha_{s^\frown\langle i \rangle}}$ by stipulating that

$$\forall \xi < \alpha_{s^\frown\langle i \rangle} \forall j \in 2 \left[\tau_{s^\frown\langle i \rangle}(\xi) = j \leftrightarrow x_{\alpha_s}^i \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \cap x_\xi^j \in \mathcal{I}^+(\mathscr{A}_\delta) \right]$$

$\tau_{s^\frown\langle i \rangle}$ is well defined because $x_{\alpha_s}^i \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \in \mathcal{I}^+(\mathscr{A}_\delta)$ and because of the choice of $\alpha_{s^\frown\langle i \rangle}$. Now, for each $\xi < \alpha_s$, $x_{\alpha_s}^i \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \subset b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right)$ and, by (4), $b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \cap x_\xi^{1-\tau_s(\xi)} \in \mathcal{I}(\mathscr{A}_\delta)$. It follows that $\alpha_{s^\frown\langle i \rangle} \geq \alpha_s$ and that for each $\xi < \alpha_s$, $\tau_s(\xi) = \tau_{s^\frown\langle i \rangle}(\xi)$. Next, since $x_{\alpha_s}^i \cap b \cap \left(\bigcap_{t \subsetneq s} x_{\alpha_t}^{\tau_s(\alpha_t)} \right) \cap x_{\alpha_s}^{1-i} = 0$, $\alpha_{s^\frown\langle i \rangle} > \alpha_s$, and $\tau_{s^\frown\langle i \rangle} \supset \tau_s{}^\frown\langle i \rangle$. Now, it is clear that (4) and (5) hold for $s^\frown\langle i \rangle$. This completes the construction of $\langle \alpha_s : s \in 2^{<\omega} \rangle$ and $\langle \tau_s : s \in 2^{<\omega} \rangle$.

    For each $f \in 2^\omega$, put $\alpha_f = \sup \{ \alpha_{f\restriction n} : n \in \omega \}$ and $\tau_f = \bigcup_{n \in \omega} \tau_{f\restriction n}$. As $\kappa = \mathfrak{s}$, $\mathrm{cf}(\kappa) > \omega$. Therefore, $\alpha_f < \kappa$. Note that $\tau_f \in 2^{\alpha_f}$. Also, if $f, g \in 2^\omega$, $f \neq g$, and $n \in \omega$ is least such that $f(n) \neq g(n)$, then $\tau_f \supset \tau_s{}^\frown\langle i \rangle$ and $\tau_g \supset \tau_s{}^\frown\langle 1 - i \rangle$, where $s = f \restriction n = g \restriction n$ and $i \in 2$. So there cannot be $\alpha < \delta$ such that both $\tau_f \subset \sigma_\alpha$ and $\tau_g \subset \sigma_\alpha$ hold. Therefore, it is possible to find $f \in 2^\omega$ such that $\tau_f \notin \{ \sigma \in 2^{<\kappa} : \exists \alpha < \delta \left[\sigma \subset \sigma_\alpha \right] \}$. Fix such f and for each $n \in \omega$, define e_n to

be $b \cap \left(\bigcap_{m<n} x_{\alpha_f \restriction m}^{\tau_f(\alpha_f \restriction m)} \right)$. By (5) each $e_n \in \mathcal{I}^+(\mathscr{A}_\delta)$. Moreover, $e_{n+1} \subset e_n \subset b$. Therefore, by a standard argument, there is $e \in [b]^\omega \cap \mathcal{I}^+(\mathscr{A}_\delta)$ such that $\forall n \in \omega \, [e \subset^* e_n]$.

Now, suppose $\xi < \alpha_f$. Since for all $n \in \omega$, $\alpha_{f \restriction n+1} > \alpha_{f \restriction n}$, it follows that $\xi < \alpha_{f \restriction n}$ for some n. By (4) applied to $s = f \restriction n$, $x_\xi^{1-\tau_f(\xi)} \cap e_n \in \mathcal{I}(\mathscr{A}_\delta)$. Since $e \subset^* e_n$, $x_\xi^{1-\tau_f(\xi)} \cap e \in \mathcal{I}(\mathscr{A}_\delta)$. Thus we conclude that $\forall \xi < \alpha_f \left[x_\xi^{1-\tau_f(\xi)} \cap e \in \mathcal{I}(\mathscr{A}_\delta) \right]$. So for each $\xi < \alpha_f$, fix $F_\xi \in [\delta]^{<\omega}$ such that

$$\left(x_\xi^{1-\tau_f(\xi)} \cap e \right) \subset^* \left(\bigcup_{\alpha \in F_\xi} a_\alpha \right)$$

Now, put $\mathcal{F} = \bigcup_{\xi < \alpha_f} F_\xi$ and $\mathcal{G} = \{\alpha < \delta : \sigma_\alpha \subset \tau_f\}$. Note that $|\mathcal{F} \cup \mathcal{G}| < \kappa \le \mathfrak{a}$ because of the assumption that $\forall \alpha, \beta < \delta \, [\alpha \ne \beta \implies \sigma_\alpha \ne \sigma_\beta]$. Since $e \in \mathcal{I}^+(\mathscr{A}_\delta)$, there is $a \in [e]^\omega$ such that $\forall \alpha \in \mathcal{F} \cup \mathcal{G} \, [|a \cap a_\alpha| < \omega]$. Note that for each $\xi < \alpha_f$, $x_\xi^{1-\tau_f(\xi)} \cap a$ is finite. Thus putting $\sigma = \tau_f$, we have that $\forall \alpha < \delta \, [\sigma \not\subset \sigma_\alpha]$ and $a \in I_\sigma$. In order to finish the proof, it is enough to check that $\forall \alpha < \delta \, [|a_\alpha \cap a| < \omega]$.

Fix $\alpha < \delta$. If $\alpha \in \mathcal{G}$, then $|a \cap a_\alpha| < \omega$ simply by choice of a. Suppose $\alpha \notin \mathcal{G}$. Then there must be $\xi \in \text{dom}(\sigma_\alpha) \cap \alpha_f$ such that $\sigma_\alpha(\xi) = 1 - \tau_f(\xi)$. However since $a_\alpha \subset^* x_\xi^{\sigma_\alpha(\xi)}$ and $a \cap x_\xi^{1-\tau_f(\xi)}$ is finite, it follows that $|a \cap a_\alpha| < \omega$. $\dashv$

Theorem 10. *If $\mathfrak{s} \le \mathfrak{a}$, then there is a completely separable MAD family.*

Proof. Fix an enumeration $\langle b_\alpha : \alpha < \mathfrak{c} \rangle$ of $[\omega]^\omega$. Let $\langle x_\alpha : \alpha < \kappa \rangle$ witness $\kappa = \mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. Build two sequences $\langle a_\delta : \delta < \mathfrak{c} \rangle$ and $\langle \sigma_\delta : \delta < \mathfrak{c} \rangle$ such that the following hold.

(1) for each $\delta < \mathfrak{c}$, $a_\delta \in [\omega]^\omega$, $\sigma_\delta \in 2^{<\kappa}$, and $a_\delta \in I_{\sigma_\delta}$
(2) $\forall \gamma, \delta < \mathfrak{c} \, [\gamma \ne \delta \implies (|a_\gamma \cap a_\delta| < \omega \wedge \sigma_\gamma \ne \sigma_\delta)]$
(3) for each $\delta < \mathfrak{c}$, if $b_\delta \in \mathcal{I}^+(\mathscr{A}_\delta)$, then $a_\delta \subset b_\delta$, where $\mathscr{A}_\delta = \{a_\alpha : \alpha < \delta\}$

Note that if we succeed in this, then $\mathscr{A}_\mathfrak{c} = \{a_\delta : \delta < \mathfrak{c}\}$ will be completely separable for given any $b \in \mathcal{I}^+(\mathscr{A}_\mathfrak{c})$, b is in $\mathcal{I}^+(\mathscr{A}_\delta)$ for every $\delta < \mathfrak{c}$ and so there is a $\delta < \mathfrak{c}$ where $b_\delta = b$ and $b_\delta \in \mathcal{I}^+(\mathscr{A}_\delta)$, whence by (3), $a_\delta \subset b$.

At a stage $\delta < \mathfrak{c}$ suppose $\langle a_\alpha : \alpha < \delta \rangle$ and $\langle \sigma_\alpha : \alpha < \delta \rangle$ are given. If $b_\delta \in \mathcal{I}^+(\mathscr{A}_\delta)$, then let $b = b_\delta$, else let $b = \omega$. In either case, the hypotheses of Lemma 9 are satisfied. So find $a_\delta \in [b]^\omega$ and $\sigma_\delta \in 2^{<\kappa}$ such that

(4) $\forall \alpha < \delta \, [|a_\delta \cap a_\alpha| < \omega]$
(5) for each $\alpha < \delta$, $\sigma_\delta \not\subset \sigma_\alpha$ and $a_\delta \in I_{\sigma_\delta}$.

It is clear that a_δ and σ_δ are as needed. $\dashv$

References

1. B. Balcar, J. Dočkálková, and P. Simon, *Almost disjoint families of countable sets*, Finite and infinite sets, Vol. I, II (Eger, 1981), Colloq. Math. Soc. János Bolyai, vol. 37, North-Holland, Amsterdam, 1984, pp. 59–88.
2. B. Balcar and P. Simon, *Disjoint refinement*, Handbook of Boolean algebras, Vol. 2, North-Holland, Amsterdam, 1989, pp. 333–388.
3. P. Erdős and S. Shelah, *Separability properties of almost-disjoint families of sets*, Israel J. Math. **12** (1972), 207–214.
4. A. Kamburelis and B. Węglorz, *Splittings*, Arch. Math. Logic **35** (1996), no. 4, 263–277.
5. D. Raghavan and J. Steprāns, *On weakly tight families*, to appear.
6. S. Shelah, *Mad families and sane player*, preprint, 0904.0816.
7. P. Simon, *A note on almost disjoint refinement*, Acta Univ. Carolin. Math. Phys. **37** (1996), no. 2, 89–99, 24th Winter School on Abstract Analysis (Benešova Hora, 1996).