THE CLUB PRINCIPLE AND THE DISTRIBUTIVITY NUMBER

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Abstract. We give an affirmative answer to Brendle's and Hrušák's question of whether the club principle together with $\mathfrak{h} > \aleph_1$ is consistent. We work with a class of axiom A forcings with countable conditions such that $q \geq_n p$ is determined by finitely many elements in the conditions p and q and that all strengthenings of a condition are subsets, and replace many names by actual sets. There are two types of technique: one for tree-like forcings and one for forcings with creatures that are translated into trees. Both lead to new models of the club principle.

1. Introduction

Ostaszewski [15] introduced the club principle, also written ♣, for a topological construction:

Definition 1.1. *The club principle is the following statement: There is a sequence $\langle A_\alpha : \alpha < \omega_1, \alpha \text{ limit}\rangle$ such that for every α, A_α is cofinal in α and for every uncountable $X \subseteq \omega_1$ the set $\{\alpha \in \omega_1 : A_\alpha \subseteq X\}$ is not empty.*

Replacing "not empty" by "stationary", we get an equivalent principle, see [17, Observation 7.2]. The club principle together with CH is equivalent to the diamond [17, Fact 7.3]. Shelah [17, Theorem 7.4] and Baumgartner [12] gave models for the club principle and non CH. Here we provide more models of this kind. In the technical side of our work, we show that a strengthening of Axiom A that is fulfilled by many tree forcings and many creature forcings [16] leads to models of the club principle in which the continuum and certain cardinal characteristics are $\aleph_2$. We list the used properties axiomatically (see Def. 2.2) in order to show that the technique is quite general and can be applied to Sacks forcing, Miller forcing, Laver forcing, other forcings with normed subtrees of $\omega^{<\omega}$, Blass-Shelah forcing, Mathias forcing, Matet forcing, other forcings with creatures. We will not go into the general theory of the abundance of notions of forcing given in [16].

Hrušák and Brendle [6, Question 9.3] asked whether the club principle together with $\mathfrak{h} = \aleph_2$ is consistent. Using Mathias forcing, we answer this question affirmatively:

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Theorem 1.2. *The club principle together with $\mathfrak{c} = \mathfrak{h} = \aleph_2$ is consistent relative to ZFC.*

(Since $\mathfrak{h} \leq \mathfrak{s}$ in ZFC, also the question on $\mathfrak{s}$ is answered.)

With our techniques, we also find:

Theorem 1.3. *The club principle together with $\mathfrak{u} < \mathfrak{g} = \aleph_2$ is consistent relative to ZFC.*

Actually, there are two models for this theorem: the Matet model (that is, a countable support iteration of length ω_2 of Matet iterands) with $\mathfrak{s} = \aleph_1$ and the Blass-Shelah model with $\mathfrak{s} = \aleph_2$. Using our techniques in an axiomatic manner leads to the following result:

Theorem 1.4. *Any countable support iteration of any length of Axiom A iterands of tree form or of a special (easy) creature form with the finiteness property for $(\leq_n)_{n\in\omega}$ over a ground model of Jensen's diamond yields a model of the club principle. In particular, the club principle holds in the Laver model, the Miller model, the Blass-Shelah model, the Mathias model, the Matet model if the diamond holds in the ground model.*

This was formerly known for the Sacks model (see [14, Cor. 6.12]) and for the side-by-side product with countable support of κ Sacks factors [12].

There is a rich history of models of the club principle: The consistency of club and $\mathrm{cov}(\mathcal{M}) = \kappa = 2^\omega$ for a regular $\kappa \geq \aleph_2$ was shown by Fuchino, Shelah, and Soukup [11], and that of club and $\mathrm{add}(\mathcal{M}) = \aleph_2 = 2^\omega$ by Džamonja and Shelah [9]. Brendle [6] showed the consistency of club and $\mathrm{cov}(\mathcal{N}) = \kappa$ for a regular $\kappa \geq \aleph_2$. By a result of Truss [18], $\stackrel{\bullet}{|}$ (i.e., there is $\langle A_\alpha : \alpha < \omega_1\rangle$, $A_\alpha \in [\omega_1]^\omega$ such that for every uncountable $X \subseteq \omega_1$ there is some α with $A_\alpha \subseteq X$, see [7]) and a fortiori $\clubsuit$ imply that $\mathrm{cov}(\mathcal{M}) = \aleph_1$ or $\mathrm{cov}(\mathcal{N}) = \aleph_1$. So the club principle implies that $\mathrm{add}(\mathcal{N}) = \aleph_1$ and we have a full picture of those cardinals in Cichoń's diagramme (see, e.g., [4]) that can be larger than $\aleph_1$ in the presence of the club principle.

Devlin and Shelah's weak diamond [8], i.e., $2^{\aleph_0} < 2^{\aleph_1}$, can be arranged to hold or not to hold:

Remark 1.5. *Since starting with* CH*, all our forcings have the $\aleph_2$-c.c., $2^{\aleph_1}$ can be anything while $2^{\aleph_0} = \aleph_2$ is all the mentioned models. So Devlin and Shelah's weak diamond may hold or may not hold in our models.*

Our notation is fairly standard. For general set-theoretic notation, we refer the reader to [13] and for cardinal characteristics, [1] and [4] are good sources. We follow the Israeli convention that the stronger condition is the larger one.

2. Axiom A forcings with a finiteness condition on $\leq_n$

First, we isolate a class of axiom A forcings with an additional property that allows to determine under $q \geq p$ whether the stronger $q \geq_n p$ holds just from looking at a suitable finite part of p and of q.

Our work builds on Baumgartner and Laver's [2] and on Hrušák's [12].

Definition 2.1. *A notion of forcing* $(\mathbb{P}, \leq_{\mathbb{P}})$ *has* Axiom A *if there are relations* $\leq_n$*,* $n \in \omega$*, with the following properties:*

(1) $\leq_0 = \leq_{\mathbb{P}}$*,*

(2) $p \leq_{n+1} q$ *implies* $p \leq_n q$*,*

(3) if $p_n \leq_n p_{n+1}$ *for* $n \in \omega$*, then there is some* $q \in \mathbb{P}$ *such that for all* n*,* $p_n \leq_n q$*,*

(4) for every $p \in \mathbb{P}$ *and every* n *and every antichain* A *in* $\mathbb{P}$ *there is some* $q \geq_n p$ *such that* $\{r \in A : r \not\perp q\}$ *is countable.*

We now add a property that will be useful for handling conditions in countable support iterations. We want that parts of the conditions are determined and not just names. We give an axiomatisation for tree forcings:

Definition 2.2. *A notion of forcing* $(\mathbb{P}, \leq_{\mathbb{P}})$ *whose elements* p *are subsets of* $2^{<\omega}$ *or of* $\omega^{<\omega}$ *has the* finiteness property for $(\leq_n)_{n\in\omega}$ *iff*

(1) $q \geq p$ *implies* $q \subseteq p$*, and*

(2) there is a function $f \colon \mathbb{P} \times \omega \to \omega$ *such that for every* n*,* p*,* q*:*

$$p \leq_n q \text{ iff } (p \leq_{\mathbb{P}} q \text{ and } q \cap f(p,n)^{f(p,n)} = p \cap f(p,n)^{f(p,n)}). \tag{2.1}$$

In the case of $2^{<\omega}$ *we can write* $2^{f(p,n)}$ *instead of* $f(p,n)^{f(p,n)}$*.*

An important consequence of the finiteness property together with the countability of the domain of a condition and the fact that $q \geq p$ implies $q \subseteq p$ is: If $p \leq_n q$, then there are a finite subset $a \subseteq q$ and a strengthening $p\langle a \rangle$ (see Definition 3.2) of p such that $p \leq_n p\langle a \rangle \leq_{n+1} q$ (see Lemma 3.4). Since p is countable, there are countable many possibilities for these sets a, for all possible q together. In our stepping up Lemma 3.5, we will perform a construction along these a's. Besides this, the proof of the club principle uses the original diamond in the ground model and the properness of the iterands and some hereditary countability of names and of forcing conditions (after some collapse).

Examples of tree forcings: *Sacks forcing* $\mathbb{S}$. Conditions are perfect subtrees of $2^{<\omega}$. For $s, t \in \omega^{<\omega}$ we write $s \trianglelefteq t$ iff s in an initial segment of t. For a Sacks condition p, $r \in p$ is called the trunk of p, if for all $s \triangleleft r$, s has only one immediate successor in p (this is called "s does not split") and $r ^\frown 0 \in p$ and $r ^\frown 1 \in p$. The length of the trunk $+1$ is called the first splitting level. A level $k \geq n$ of p is called the n-th splitting level if there are 2^n points $s_i \in p \cap 2^k$, $i < 2^n$, such that $\{r \in p : \exists i \leq 2^n, r \trianglelefteq s_i\}$ can be mapped in a $\trianglelefteq$ preserving and incomparability preserving manner into the tree 2^n and if k is minimal with this property. We take $\mathbb{S}$ with the Axiom A structure $p \leq_n q$ iff $p \leq q$ and p and q agree up to the n-th splitting levels of q. There is a body of work by Baumgartner and Laver and by Hrušák on the Sacks forcing, which we will generalise to other Axiom A tree forcings with the finiteness condition. The set of conditions such that for every n, the n-th splitting level consists just of the 2^n nodes that are the leaves of the 2^n tree witnessing that it is the n-th splitting level, are dense in the Sacks forcing. We set $f(p, n) = (n$-th splitting level of $p)$ and the equivalence in (2.1) holds.

Miller forcing. Conditions are subtrees $p \subseteq \omega^{<\omega}$ such that $(\forall s \in p)(\exists t \trianglerighteq s)|\{n : t^\frown n \in p\}| = \omega$. The latter is called "$t$ is infinitely splitting." Stronger conditions are subtrees. We restrict the poset to the dense subset of trees all of whose splitting nodes are infinitely splitting. We take a linear ordering $\leq_{1,k}$ on $\omega^{<\omega}$ of type ω. Then we let $p \leq_{n+1} q$ if the $\leq_1$-first splitting nodes of p with n splitting predecessors s in p with the first n immediate successors of each such s are present in q and they are infinitely splitting in q. $f(p, 0) = 0$, for $n \geq 1$, $f(p, n) = \max$(height of the n-th splitting level of p, n-th successor of any of the splitting nodes witnessing the n splitting levels).

Similarly we can do with *Laver forcing*, which increases $\mathfrak{b}$ and $\mathfrak{r}$. We think of a Laver tree of strictly increasing sequences. $f(p, n) = \max$(n-th successor of any of the nodes witnessing for the n-th splitting level).

Cohen forcing $\mathrm{Fn}_{<\omega}(\omega, 2)$. Since this is a c.c.c. forcing, Axiom A holds by letting $\leq_n$ being the equality for all $n \geq 1$. For $n \in \omega$ and a condition p, we let $f(p, n+1) = \max(\mathrm{dom}(p))$. Then f witnesses the finiteness property.

And finally we have that *random forcing* is a counterexample to the finiteness property of $(\leq_n)_{n<\omega}$ for whatever $\leq_n$-relations, $n \in \omega$, we define on it. This follows from [18] and our results.

We write iterations as $\mathbb{P}_\gamma = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma, \alpha \leq \gamma \rangle$ with $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_\alpha$. In our iterations, we have $\mathbb{Q}_\beta = \varphi^{\mathbf{V}^{\mathbb{P}_\beta}}$, for some definition φ of some Axiom A forcing with the finiteness condition that is evaluated in the respective intermediate stage. All limits in the iterations are taken with countable supports. Different kinds of iterands could be mixed in the iterations, and still the club holds in the final model. However, we do not (yet) have an application for this.

Now let $\mathbb{Q}_\beta$ be an iterand, an Axiom A forcing with the finiteness property. Let $\mathbb{P}_\alpha$ denote a countable support iteration of $\mathbb{Q}_\beta$ of length α. If $p, q \in \mathbb{P}_\alpha$, $n \in \omega$, $F \subseteq \alpha$, F finite, we write $q \geq_{(F,n)} p$ iff $(\forall \beta \in F)((q \restriction \beta) \Vdash_{\mathbb{P}_\beta} q(\beta) \geq_n p(\beta))$. We recall the fusion lemma for $\mathbb{P}_\alpha$:

Lemma 2.3. *(See* [2, Lemma 1.2]*.) Let n_i, $i \in \omega$, be a strictly increasing sequence and let $F_i \subseteq \alpha$ be finite. Assume that p_i, $i \in \omega$, is such that $F_{i+1} \supseteq F_i$ and $\bigcup_{i\in\omega} F_i = \bigcup_{i\in\omega} \mathrm{supp}(p_i)$, and $p_{i+1} \geq_{(F_i,n_i)} p_i$. Then we define p so that $\mathrm{supp}(p) = \bigcup_{i\in\omega} \mathrm{supp}(p_i)$ and $\forall \beta \in \mathrm{supp}(p)$, $p(\beta)$ is a name for the fusion of $\{p_i(\beta) : i \in \omega, \beta \in \mathrm{supp}(p_i)\}$, then $p \in \mathbb{P}_\alpha$ and $p \geq_{(F_i,n_i)} p_i$.*

Now we introduce a finer notion of $\leq_{F,n}$ in order to describe constructions in which the $\leq$-relation is improved at only one coordinate.

Definition 2.4. *If $p, q \in \mathbb{P}_\beta$, $F \subseteq \beta$, F finite, $\vec{n} \in {}^F\omega$, we write $q \geq_{(F,\vec{n})} p$ iff $F = \{\beta_0, \dots, \beta_r\}$, $\vec{n} = (n(\beta_0), \dots, n(\beta_r))$ and $(\forall i \leq r)((q \restriction \beta_i) \Vdash_{\mathbb{P}_{\beta_i}} q(\beta_i) \geq_{n(\beta_i)} p(\beta_i))$.*

We write for $\sigma \in {}^k k \cap p$, $p_\sigma = \{\tau \in p : \tau \vartriangleleft \sigma \vee \tau \trianglerighteq \sigma\}$. If p_σ is defined, i.e., if $\sigma \in p$, we say that σ *is consistent with* p. Now we generalise this notion of subtrees to iterated forcings:

Definition 2.5. *(See* [2]*.) Let $p \in \mathbb{P}_\alpha$, $F = \{\beta_0, \dots, \beta_r\} \in [\mathrm{supp}(p)]^{<\omega}$, $\vec{k} = (k(\beta_0), \dots, k(\beta_r)) \in {}^F\omega$, and $\vec{\sigma} = (\sigma(\beta_0), \dots, \sigma(\beta_r))$, $\sigma(\beta_i) \in k(\beta_i)^{k(\beta_i)}$. By* *induction on* $\beta \le \alpha$ *we define when* $\vec{\sigma} \restriction \beta+1$ is consistent with $p_{\vec{\sigma}\restriction\beta}$ is consistent with p *and then we define* $p_{\vec{\sigma}\restriction\beta+1}$.

Suppose that $p_{\vec{\sigma}\restriction\beta}$ *is defined. If* $\beta \in F$ *and if* $p_{\vec{\sigma}\restriction\beta} \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$\sigma(\beta) \in p_{\vec{\sigma}\restriction\beta}(\beta)(= p(\beta))$", *then we say that* $\vec{\sigma} \restriction (\beta+1)$ *is consistent with* p *and we define the condition* $p_{\vec{\sigma}\restriction\beta+1}$ *by*

$$\begin{aligned} p_{\vec{\sigma}\restriction\beta+1}(\alpha) &= p_{\vec{\sigma}\restriction\beta}(\alpha) \text{ for } \alpha < \beta, \\ p_{\vec{\sigma}\restriction\beta+1}(\beta) &= (p_{\vec{\sigma}\restriction\beta}(\beta))_{\sigma(\beta)}, \\ p_{\vec{\sigma}\restriction\beta+1}(\gamma) &= p(\gamma) \text{ for } \gamma > \beta. \end{aligned}$$

If $\beta \in F$ *and if* $p_{\vec{\sigma}\restriction\beta} \restriction \beta \not\Vdash$ "$\sigma(\beta) \in p(\beta)$", *then* $p_{\vec{\sigma}\restriction\beta+1}$ *is not defined and* $\vec{\sigma} \restriction \beta+1$ *is not consistent with* p. *If* $\beta \notin F$, *then* $p_{\vec{\sigma}\restriction\beta+1} = p_{\vec{\sigma}\restriction\beta}$. *For a limit ordinal* α, $p_{\vec{\sigma}\restriction\alpha} = p_{\vec{\sigma}\restriction\max(F\cap\alpha)}$.

The idea behind the next definition is: We want to get rid of the gap between $p_{\vec{\sigma}\restriction\beta} \restriction \beta \not\Vdash$ "$\sigma(\beta) \in p(\beta)$" and $p_{\vec{\sigma}\restriction\beta} \restriction \beta \Vdash$ "$\sigma(\beta) \notin p(\beta)$" for sufficiently many $\vec{\sigma}$. Let $f_{\beta_i} \in \mathbf{V}^{\mathbb{P}_{\beta_i}}$ belong to $\mathbb{Q}_{\beta_i}^{\mathbf{V}^{\mathbb{P}_{\beta_i}}}$ as in the finiteness property. $k(\beta_i)$ shall be so large that $p \restriction \beta_i \Vdash_{\mathbb{P}_{\beta_i}} k(\beta_i) \ge f_{\beta_i}(p(\beta_i), n_i)$ for all $\beta_i \in F$. Since in an iteration $p(\beta)$ is only a name for a condition in $\mathbb{Q}^{\mathbf{V}^{\mathbb{P}_\beta}}$, we need to find sufficiently large finite sets such that pinning down all their members guarantees that all interpretations of the name of the condition are in the desired $\le_{(F,\vec{m})}$-relation.

Definition 2.6. *(Compare with* [2].*) Let* $F = \{\beta_0, \dots, \beta_r\}$ *be a finite subset of* α *and* $\vec{k} \in {}^F\omega$. *A condition* $p \in \mathbb{P}_\alpha$ *is said to be* $(F, \vec{k})$-determined *if for every* $\vec{\sigma} = (\sigma(\beta_0), \dots, \sigma(\beta_r))$ *with* $\sigma(\beta_i)\colon k(\beta_i) \to k(\beta_i)$ *the following holds: either* $\vec{\sigma}$ *is consistent with* p *or* $\exists \beta \in F$ *so that* $\vec{\sigma} \restriction (F \cap \beta)$ *is consistent with* p *and* $p_{\vec{\sigma}\restriction(F\cap\beta)} \restriction \beta \Vdash_{\mathbb{P}_\beta} \sigma(\beta) \notin p(\beta)$.

Let $F_n \in [\alpha]^{<\omega}$ and $m_n \in \omega$. Now we want that $(F_n, \vec{m}_n)$, $n \in \omega$, grows in such a way that we have a fusion lemma again. The following definition helps to describe the growth behaviour of the $\vec{m}_n$.

Definition 2.7. *Let* $F, G \in [\alpha]^{<\omega}$ *and* $F \subseteq G$. *Let* $\vec{m} \in {}^F(\omega \setminus \{0\})$ *and let* $\vec{n} \in {}^G(\omega \setminus \{0\})$. *Then we write* $(F, \vec{m}) \le (G, \vec{n})$ *iff* $F \subseteq G$ *and* $m(\beta) \le n(\beta)$ *for all* $\beta \in F$ *and we write* $(F, \vec{m}) < (G, \vec{n})$ *iff there is* $\beta \in F$ *such that* $m(\beta) < n(\beta)$ *or if* $G \supsetneq F$.

Lemma 2.8. *(Compare to* [2],[12, Lemma II.4].*) For* $\beta < \alpha$, *let* $f_\beta \in \mathbf{V}[\mathbb{P}_\beta]$ *be a function for* $\mathbb{Q}_\beta$ *as in Definition 2.2. Let* $p \in \mathbb{P}_\alpha$, $F \in [\alpha]^{<\omega}$ *and* $\vec{m} \in {}^F\omega$. *There are* $q \ge_{(F,\vec{m})} p$ *and* $\vec{k}(p, F, \vec{m}) = \vec{k} \in {}^F\omega$ *such that* q *is* $(F, \vec{k})$*-determined and for all* $\beta \in F$, $q \restriction \beta \Vdash_{\mathbb{P}_\beta} k(\beta) \ge f_\beta(q(\beta), m(\beta))$.

Proof. The lemma is proved by induction on α simultaneously for all $F \in [\alpha]^{<\omega}$, $\vec{m}$. $\alpha = 1$: this is true since every $p \in \mathbb{P}_1$ is $(\{0\}, k)$-determined for every $k \in \omega$. $\alpha = \beta + 1$: Only the case when $\beta \in F$ needs to be considered. There are a $\mathbb{P}_\beta$-name $\underset{\sim}{q}$ and a name $\underset{\sim}{k}$ such that $p \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{q} \ge_{m(\beta)} p(\beta)$ and $\underset{\sim}{k} = f_\beta(p(\beta), m(\beta))$". By the inductive hypothesis, there are q' and $\vec{k}'$ such that $q' \ge_{F\setminus\{\beta\}, \vec{m}\restriction\beta} p \restriction \beta$, q' is $(F \setminus \{\beta\}, \vec{k}')$-determined and for all $\gamma \in F \cap \beta$, $q' \restriction \gamma \Vdash_{\mathbb{P}_\gamma} k'(\gamma) \ge f_\gamma(q'(\gamma), m(\gamma))$.

For every $\vec{\sigma} = (\sigma(\beta_0), \dots, \sigma(\beta_{r-1}))$, $F \setminus \{\beta\} = \{\beta_0, \dots, \beta_{r-1}\}$, $\sigma(\beta_i) \colon k'(\beta_i) \to k'(\beta_i)$, consistent with q' we take $q''_{\vec{\sigma}} \geq q'_{\vec{\sigma}}$ and $k_{\vec{\sigma}}$ such that $q''_{\vec{\sigma}} \Vdash_{\mathbb{P}_\beta} \underset{\sim}{k} = k_{\vec{\sigma}} \geq f_\beta(q'(\beta), m(\beta))$ and such that $q''_{\vec{\sigma}}$ decides $\underset{\sim}{q}(\beta) \cap k_{\vec{\sigma}}^{k_{\vec{\sigma}}}$, say to $q''_{\vec{\sigma}}(\beta)$. We let $\vec{k}(p, F, \vec{m}) \restriction \beta = \vec{k}'$ and we let $\vec{k}(p, F, \vec{m})(\beta) = \max\{k_{\vec{\sigma}} : \vec{\sigma}$ consistent with $q'\}+1$. Now we let $q = \bigcup_{\vec{\sigma}} q''_{\vec{\sigma}} {}^\frown q''_{\vec{\sigma}}(\beta)$ and have that $q \geq_{F,\vec{m}} p$, q is $(F, \vec{k})$-determined and $\forall \gamma \in F$, $q \restriction \gamma \Vdash k(\gamma) \geq f_\gamma(q(\gamma), m(\gamma))$.

For limit α, we choose β such that $\max(F) < \beta < \alpha$. By induction hypothesis there are $q' \in \mathbb{P}_\beta$ $\vec{k} = \vec{k}(p \restriction \beta, F, \vec{m})$ such that q' is $(F, \vec{k})$-determined and $q' \geq_{(F,\vec{m})} p \restriction \beta$. Then we let $p_{\beta,\alpha}$ be such that $p = p \restriction \beta {}^\frown p_{\beta,\alpha}$ and put $q = q' {}^\frown p_{\beta,\alpha}$. $\dashv$

We denote by $\vec{k}(p, F, \vec{m})$ the (lexicographically) minimal $\vec{k}$ as in the previous lemma. From Axiom A we get

Lemma 2.9. [2, Lemma 2.3 (iii)] *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma, \alpha \leq \gamma \rangle$ be a countable support iteration of Axiom A iterands and let $\{p_i : i < \kappa\}$ be an antichain above $p \in \mathbb{P}$. Further, let $G \in [\gamma]^{<\omega}$, $\vec{n} \in {}^G\omega$. Then there is some $q \geq_{(G,\vec{n})} p$ such that $\{p_i : i < \kappa, p_i \not\leq q\}$ is countable.*

The following slight generalisation of a notion introduced in [12] will be very important for showing the club principle:

Definition 2.10. *Given a $\mathbb{P}_\alpha$-name $\underset{\sim}{X}$ for an uncountable subset of ω_1, a condition $p \in \mathbb{P}_\alpha$, $F \in [\alpha]^{<\omega}$, and $\vec{m} \in {}^F\omega$ we let*

$$A_{F,\vec{m}}(p, \underset{\sim}{X}) = \{\gamma \in \omega_1 : (\exists q \in \mathbb{P}_\alpha)(q \geq_{(F,\vec{m})} p \wedge q \Vdash \gamma \in \underset{\sim}{X})\}.$$

Definition 2.11. *A condition $p \in \mathbb{P}_\alpha$ is said to be* $(\underset{\sim}{X}, F, \vec{m})$-good *if p is $(F, \vec{k}(F, p, \vec{m}))$-determined and $\forall q \geq_{(F,\vec{m})} p$, $|A_{F,\vec{m}}(q, \underset{\sim}{x})| = \aleph_1$.*

In the following section we show how one can step up from $(\underset{\sim}{X}, F, \vec{m})$-*good* to $(\underset{\sim}{X}, G, \vec{n})$-*good* conditions.

3. Stepping up from good to better conditions

For one iterand $\mathbb{Q}$, we have the following [12, Claim IV.2] that is stated and proved there for Sacks forcing:

Lemma 3.1. *If $p \in \mathbb{S}$ is $(\underset{\sim}{X}, \{0\}, m)$-good then there is a $q \geq_m p$ that is $(\underset{\sim}{X}, \{0\}, m+1)$-good.*

In order prove an analogous stepping up lemma for the countable support iteration of arbitrary tree forcings with the finiteness condition, we shall carry out an induction over the heights of initial segments of trees such that all possibilities of extending a condition are covered in the end. This version is related to the stated but not proved result [12, Claim IV.3] on product forcing of Sacks factors with countable support. In each step, initial finite parts of the tree are arranged and then frozen. These parts need to be actual finite sets and not just names for parts of the conditions. In the following, let $\mathbb{P}_\gamma = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma, \beta \leq \gamma \rangle$ be a countable support iteration of tree iterands that are tree forcings with the finiteness property.

Definition 3.2. *Let $p \in \mathbb{P}_\gamma$, $G = \{\beta_0, \dots, \beta_r\} \subseteq [\gamma]^{<\omega}$, $\vec{n} \in {}^G\omega$, $\vec{k} \geq \vec{k}(p, G, \vec{n})$ and let p be $(G, \vec{k})$-determined. For*

$$p \restriction \beta_0 \Vdash (a(\beta_0) \subseteq p(\beta_0) \cap k(\beta_0)^{k(\beta_0)} \text{ and } a(\beta_0) \text{ is a tree}) \tag{3.1}$$

we define $p\langle a(\beta_0)\rangle(\gamma) = p(\gamma)$ for $\gamma \neq \beta_0$ and $p\langle a(\beta_0)\rangle(\beta_0) = \bigcup\{p(\beta_0)_\tau \,:\, \tau \in a(\beta_0)\}$.

If for every $\tau \in a(\beta_0)$,

$$p\langle a(\beta_0)\rangle \restriction \beta_1 \Vdash (a(\beta_1, \tau) \subseteq p(\beta_1) \cap k(\beta_1)^{k(\beta_1)} \text{ and } a(\beta_1, \tau) \text{ is a tree}), \tag{3.2}$$

we let

$$\begin{aligned} p\langle a(\beta_0), (a(\beta_1, \tau) \,:\, \tau \in a(\beta_0))\rangle(\gamma) &= p\langle a(\beta_0)\rangle(\gamma) \text{ for } \gamma < \beta_1, \text{ and} \\ p\langle a(\beta_0), (a(\beta_1, \tau) \,:\, \tau \in a(\beta_0))\rangle(\beta_1) &= \bigcup\{p(\beta_1)_{\tau'} \,:\, \tau \in a(\beta_0), \tau' \in a(\beta_1, \tau)\} \\ p\langle a(\beta_0), (a(\beta_1, \tau) \,:\, \tau \in a(\beta_0))\rangle(\gamma) &= p(\gamma) \text{ for } \gamma > \beta_1. \end{aligned}$$

If for every $\tau \in a(\beta_0)$, $\tau' \in a(\beta_1, \tau)$,

$$\begin{aligned} p\langle a(\beta_0), (a(\beta_1, \tau) \,:\, \tau \in a(\beta_0))\rangle \restriction \beta_2 \Vdash_{\mathbb{P}_{\beta_2}} &(a(\beta_2, \tau, \tau') \subseteq k(\beta_2)^{k(\beta_2)} \\ &\text{and } a(\beta_2, \tau, \tau') \text{ is a tree}), \end{aligned} \tag{3.3}$$

we continue the inductive definition in the obvious way.

If p is $(G, \vec{k})$-determined then $\bar{a} = (a(\beta_0), (a(\beta_1, \tau) \,:\, \tau \in a(\beta_0)), \dots,$ $(a(\beta_r, \tau_0, \dots, \tau_{r-1}) \,:\, \tau_0 \in a(\beta_0), \tau_1 \in a(\beta_1, \tau_0), \dots, \tau_{r-1} \in a(\beta_{r-1}, \tau_0, \dots, \tau_{r-2})))$ can be chosen as a tree in $\mathbf{V}$ (or rather a tree of trees, also indexed by $\beta_0, \dots, \beta_r$) in the ground model and not a name. Note that $p\langle a(\beta_0), (a(\beta_1, \tau) \,:\, \tau \in a(\beta_0))\rangle_{\{(\beta_0, \tau), (\beta_1, \tau')\}} \restriction \beta_2 = (p_{\{(\beta_0, \tau), (\beta_1, \tau')\}}) \restriction \beta_2$ if $\tau \in a(\beta_0)$ and $\tau' \in a(\beta_0, \tau)$.

Definition 3.3. *Let $(F, m) \leq (G', \vec{n}') \leq (G, \vec{n})$ and let p, G, $\vec{n}$, $\bar{a}$ be as above. If p is $(G, \vec{k}(p, G, \vec{n}))$-determined and if there is a $(G, \vec{k}(p, G, \vec{n}))$-determined $q \geq_{(F, \vec{m})} p$ such that $q \geq_{(G', \vec{n}')} p\langle \bar{a}\rangle \geq_{(F, \vec{m})} p$ and all $\tau \in a_i$ for any $a_i \in \bar{a}$ and $\tau \in a_i$ fulfil $\tau \in \ell^\ell$, then a sequence $\bar{a}$ like this is called* relevant for p, $\vec{k}(p, G, \vec{n})$ of strength $(\ell, (G', \vec{n}'))$.

Note that $\vec{k}(p, G, \vec{n})$ can be much bigger than $\vec{k}(p, F, \vec{m})$ and then there are many relevant sequences. However, the number of relevant sequences is finite for each fixed strength.

The thickness of $\bar{a}$ leads to $p\langle \bar{a}\rangle \leq_{(G, \vec{n})} q$ for larger $(G, \vec{n})$: If the sets $\bar{a} = (a(\beta_0), (a(\beta_1, \tau_0) \,:\, \tau_0 \in a(\beta_0)), \dots, (a(\beta_i, \tau_0, \dots, \tau_i) \,:\, \tau_0 \in a(\beta_0), \dots, \tau_i \in a(\beta_i, \tau_0, \dots, \tau_{i-1})))$, $i \leq r$, are chosen such that for all $i \leq r$,

$$\begin{aligned} p\langle \bar{a}\rangle \restriction \beta_i \Vdash_{\mathbb{P}_{\beta_i}} a(\beta_i, \tau_0, \dots, \tau_{i-1}) \cap q(\beta_i) = f_{\beta_i}(q(\beta_i), n(\beta_i)) \cap q(\beta_i) \wedge \\ a(\beta_i, \tau_0, \dots, \tau_{i-1}) \subseteq q(\beta_i) \subseteq p(\beta_i), \end{aligned}$$

then we get $p\langle \bar{a}\rangle \restriction \beta_i \Vdash p\langle \bar{a}\rangle(\beta_i) \leq_{n(\beta_i)} q(\beta_i)$ and $p\langle a\rangle \leq_{(G, \vec{n})} q$.

Lemma 3.4. *Let $p \leq_{(F, \vec{m})} q$ and assume that p and q are $(G, \vec{k}(G, \vec{n}))$-determined. Then there are ℓ and $\bar{a}$ such that $p \leq_{(F, \vec{m})} p\langle a\rangle \leq_{(G, \vec{n})} q$ and $\bar{a}$ is of strength $(\ell, (G, \vec{n}))$.*

Proof. Just cut $\bar{a}$ out of q with $\subseteq$ replaced by equality in equations (3.1), (3.2), (3.3) of Definition 3.2, and p by q and also the $\vec{k}$-function from q. The resulting ℓ is determined by $f_{\beta_i}(q,\vec{n})$ and not uniform in p, $(G,\vec{n})$. $\dashv$

The following stepping up lemma is an important tool for the proof on the club principle.

Lemma 3.5. *(Cf.* [12, Claim IV.3]*) If $p \in \mathbb{P}_\alpha$, $F \subseteq G$ be finite subsets of α, $\underset{\sim}{X}$ a $\mathbb{P}_\alpha$-name for an uncountable subset of ω_1, and let $\vec{m} \in {}^F(\omega \setminus \{0\})$, $\vec{n} \in {}^G(\omega \setminus \{0\})$, $(F,\vec{m}) \leq (G,\vec{n})$. If p is $(\underset{\sim}{x}, F, \vec{m})$-good then there is $q \in \mathbb{P}_\alpha$, $q \geq_{(F,\vec{m})} p$ and q is $(\underset{\sim}{X}, G, \vec{n})$-good.*

Proof. The lemma can be proved by successive applications of steps of the following two basic kinds: First with $G \setminus F = \{\alpha_0\}$ and $n(\alpha_0) = 1$ for some α_0, or second, with $G = F$ and one coordinate is increased by one when going from $\vec{m}$ to $\vec{n}$. Hence we only need to consider these two cases. (This cutting down to elementary steps is not necessary for the proof. Working along $\bar{a}$ that grow higher and higher in just one coordinate β and otherwise are the same and at the same time freezing higher and higher subtrees of the conditions in more and more coordinates in the iteration length merely seems to be easier than working with more (still finitely many) $\bar{a}$ that grow in all coordinates of $\mathrm{dom}(\vec{n})$ simultaneously.)

Suppose that the lemma fails. We construct a sequence $\langle p_i, G_i, \vec{k}_i, \vec{n}_i, : i \in \omega\rangle$ with the following properties:

(1) $\bigcup_{i\in\omega} G_i = \bigcup_{i\in\omega} \mathrm{supp}(p_i)$, $(G_0,\vec{n}_0) \geq (G,\vec{n})$, $\vec{n}_i \in {}^{G_i}(\omega \setminus \{0\})$,

(2) $p_i \in \mathbb{P}_\alpha$, $p_0 = p$,

(3) $p_{i+1} \geq_{(G_i,\vec{n}_i)} p_i$,

(4) p_i is $(G_i, \vec{k}_i)$-determined for $\vec{k}_i = \vec{k}(p_i, G_i, \vec{n}_i)$,

(5) for all $\vec{\sigma} \in \prod_{\beta\in G_i} k_i(\beta)^{k_i(\beta)}$, if $\vec{\sigma}$ is consistent with p_i then $\vec{\sigma}$ is consistent with p_{i+1},

(6) $\lim_{i\to\infty} \vec{n}_i(\beta) = \infty$ for all $\beta \in \bigcup_{i\in\omega} \mathrm{supp}(p_i)$, and for all i, $(G_{i+1}, \vec{n}_{i+1}) > (G_i, \vec{n}_i)$,

(7) for all i, for all $\bar{a}$ relevant for p_i, $\vec{k}(p_i, G_i, \vec{n}_i)$ of strengths $(i,(G,\vec{n}))$ we have $|A_{G,\vec{n}}(p_{i+1}\langle\bar{a}\rangle, \underset{\sim}{X})| < \aleph_1$.

To do this, suppose that p_i, G_i, $\vec{k}_i$, $\vec{n}_i$ have been constructed. Enumerate all $\bar{a}$ as in (7) as $\{\bar{a}_r : r < R\}$. We construct $\{p_i^r : r \leq R\}$ so that

(a) $p_r^0 = p_i$,

(b) $p_i^{r+1} \geq_{G_i,\vec{n}_i} p_i^r$,

(c) $|A_{G,\vec{n}}(p_i^{r+1}\langle\bar{a}_r\rangle, \underset{\sim}{X})| < \aleph_1$.

At step r we find $\bar{p}_i^r \geq p_i^r\langle\bar{a}_r\rangle$, such that $\bar{a}_r \subseteq \bar{p}_i^r$, so $\bar{p}_i^r \geq_{G,\vec{n}} p_i^r\langle\bar{a}_r\rangle$, and $|A_{G,\vec{n}}(\bar{p}_i^r, \underset{\sim}{X})| < \aleph_1$. If this were not possible then the lemma holds, as $p_i^r\langle\bar{a}_r\rangle \geq_{F,\vec{m}} p$ and as then $p_i^r\langle\bar{a}_r\rangle$ is $(\underset{\sim}{X}, G, \vec{n})$-good. We take a maximal antichain in the set of these counterexamples $\bar{p}_i^r$. By Lemma 2.9 there is a $\vec{k}(p_i^r\langle\bar{a}_r\rangle, G_i, \vec{n}_i)$-determined $q_i^{r+1} \geq_{(G_i,\vec{n}_i)} p_i^r(\langle\bar{a}_r\rangle)$ such that $q_i^{r+1} = q_i^{r+1}\langle\bar{a}_r\rangle$ is compatible with at most countably many of the members of the antichain and hence has $|A_{G,\vec{n}}(q_i^{r+1}\langle\bar{a}_r\rangle, \underset{\sim}{X})| < \aleph_1$. We define p_i^{r+1}. For $\vec{\tau} \in \bar{a}_r$ we let $(p_i^{r+1})_{\vec{\tau}} = (q_i^{r+1})_{\vec{\tau}}$. In the parts not above a stem in $\bar{a}_r$, we let $(p_i^{r+1})_{\vec{\tau}} = (p_i^r)_{\vec{\tau}}$. Thus we have $p_i^{r+1} \geq_{(G_i,\vec{n}_i)} p_i^r$ and $|A_{G,\vec{n}}(p_i^{r+1}\langle\bar{a}_r\rangle, \underset{\sim}{X})| < \aleph_1$. Now also property (5) is true for p_i^r and for p_i^{r+1}. Now we go over to $\bar{a}_{r+1}$, and work on it if it is still a subset of p_i^{r+1}. (Otherwise $\bar{a}_{r+1}$ will be erased from the list of tasks.) So we go on until we reach $r = R$ and set $p_i^R = p_{i+1}$. Then we choose G_{i+1} and $\vec{n}_{i+1}$ with an eye towards (6) and choose $\vec{k}_{i+1} = \vec{k}(p_{i+1}, G_{i+1}, \vec{n}_{i+1}) \geq \vec{k}_i$ and a $p'_{i+1} \geq_{G_i,\vec{n}_i} p_{i+1}$ such that p'_{i+1} is $(G_{i+1}, \vec{k}_{i+1})$-determined. We rename such a p'_{i+1} to p_{i+1} and finish the inductive step.

Now let p_ω be the fusion of the sequence p_i and let

$$A = \bigcup\{A_{G,\vec{n}}(p_\omega\langle\bar{a}\rangle, \underset{\sim}{X}) \,:\, (\exists i, r < \omega)(\bar{a} \text{ that is relevant for } p_i^r,\, G_i,\, \vec{k}(p, G_i, \vec{n}_i) \text{ and of strength } (i, (G, \vec{n}))))\}.$$

Since there are countably many $\bar{a}$'s and since $A_{G,\vec{n}}(p_\omega\langle\bar{a}\rangle, \underset{\sim}{X}) \subseteq A_{G,\vec{n}}(p_r^i\langle\bar{a}\rangle, \underset{\sim}{X})$ for the construction stage i, r when $\bar{a}$ is relevant and of strength $(i, (G, \vec{n}))$ for the first time, we have that A is countable. The set $A_{F,\vec{m}}(p_\omega, \underset{\sim}{X})$ is uncountable since $p_\omega \geq_{G,\vec{n}} p$ and p is $(F, \vec{m})$-good and $(G, \vec{n}) \geq (F, \vec{m})$. Now we choose $\gamma \in A_{F,\vec{m}}(p_\omega, \underset{\sim}{X}) \setminus A$. Then there is a $(G, \vec{k}(p_\omega, G, \vec{n}))$-determined $p' \geq_{F,\vec{m}} p_\omega$ such that $p' \Vdash \gamma \in \underset{\sim}{X}$. We choose

$$\bar{a} = \{\vec{\sigma} \in \prod_{\beta \in G} \vec{k}(p_\omega, G, \vec{n})(\beta) \,:\, \vec{\sigma} \text{ is consistent with } p'\}.$$

Then there are i, r such that $\bar{a}$ appeared in the step from i to $i+1$ and hence there are G_i, $\vec{k}_i$ showing that $\bar{a}$ relevant for p_i^r and $(G_i, \bar{n}_i)$ that is of strength $(i, (G, \vec{n}))$. This yields $p'\langle\bar{a}\rangle \geq_{(G,\vec{n})} p_\omega\langle\bar{a}\rangle$. Then, however, we get $p'\langle a\rangle = p'$ and hence $\gamma \in A_{G,\vec{n}}(p_\omega\langle\bar{a}\rangle, \underset{\sim}{X})$, which is impossible. ⊣

4. The finiteness condition for creature forcings

There are many useful notions of forcings that can be described in the following (strongly simplified, see [16]) creature framework: Let $n_0 = 0$, and let n_i, $i < \omega$, be a a strictly increasing sequence of natural numbers. A condition has the form $p = (a, c_0, c_1, \dots)$, with $a \subseteq n_0$ and $c_i \subseteq [n_i, n_{i+1})$ or even $c_i \subseteq 2^{n_i} \times 2^{n_{i+1}}$ (see [5] or [1, page 370]) for some strictly increasing sequence n_i, $i < \omega$, and possibly some limit condition of the kind $\lim_{n\to\infty} \mathrm{norm}(c_n) = \infty$ or some lim sup condition. The relation $p \leq q = (b, d_0, d_1, \dots)$ is defined by $b \supseteq a$ and $b \setminus a$ as well as every d_i is a combination (that is, union of some of them, composition of some of them when conceived as relations, or something similar, called subcomposition in [16]) of finitely many of the c_i, say of $\{c_j \,:\, j \in [m_i, m_{i+1})\}$, such that given finitely many c_i there are only finitely many combinations d allowed. Now we can translate p to a tree $t(p) = \{b \in 2^{<\omega} \,:\, \exists q = (b, \bar{d}) \geq p\}$ of trunks of possible extensions, and get $q \geq p$ implies that $t(q) \subseteq t(p)$ and moreover, if we restrict the part of the tree that is gotten from $\{c_j \,:\, j \leq m_0\}$ by the combination operation then it is finite (though there might be infinitely many trunks of some fixed lengths, just gotten from later and later c_j) It is equivalent to force with the translated trees and the translated $\leq$-relation (which is a sharpening of $\subseteq$). The translation $p \mapsto t(p)$ in injective, and the outcome are somewhat uniform trees.

Then $(\mathbb{P}, \leq_{\mathbb{P}})$ (or rather its translation) has the finiteness property for $(\leq_n)_{n\in\omega}$ if there is a function $f \colon \mathbb{P} \times \omega \to \omega$ such that for every n, p, q:

$$\begin{aligned} p = (s, \bar{c}) \leq_n q = (t, \bar{d}) \text{ iff} \\ & p \leq_{\mathbb{P}} q \text{ and} \\ & (s \cap f(p,n), c_0 \cap f(p,n), c_1 \cap f(p,n), \dots) = \\ & (t \cap f(p,n), d_0 \cap f(p,n), d_1 \cap f(p,n), \dots). \end{aligned}$$

Proof: We look at the subtrees built from c_i for i such that $c_i \subseteq f(p,n)$ or $c_i \subseteq 2^{\leq f(p,n)} \times 2^{\leq f(p,n)}$ if the c_i are relations. These are finitely many i, and the subtree is a subset of $2^{\leq f(p,n)}$.

We give examples. For *Mathias forcing* $\mathbb{M}$, we let $(s, C) \in \mathbb{M}$ if $s \in [\omega]^{<\omega}$, $C \in [\omega]^\omega$ and $\max(s) < \min(C)$. $(s, C) \leq (t, D)$ iff $t \supseteq s$, For $n \in \omega$, $(s, C) \leq_{n+1} (t, D)$ iff $(s, C) \leq (t, D)$ and $s = t$, and the first n elements of D are the first n elements of C. It is well known that an iteration of $\mathbb{M}$ of length ω_2 with countable supports gives a model of $\mathfrak{h} = \aleph_2$, for a proof see e.g. [10, Lemma 3.2]. $f((s, C), 1) = \max(s) + 1$ and $f((s, C), n + 1) = (n\text{-th element of } C) + 1$ for $n \geq 1$. Mathias forcing is used to show the consistency of $\mathfrak{h} = \aleph_2$ together with the club principle.

Matet forcing: (See [3].) Conditions are of the form $p = (s, \bar{c})$, where $\bar{c} = c_0 < c_1 \dots$ and $c_i \in [\omega]^{<\omega}$. Here we write $c < d$ to denote $\max(c) < \min(d)$. We let $(s, \bar{c}) \leq (t, \bar{d})$ iff $t \setminus s$ is the union of some members of $\bar{c}$ and every member of $\bar{d}$ is the union of some members of $\bar{c}$. We let $(s, \bar{c}) \leq_{n+1} (t, \bar{d})$ iff $s = t$ and $c_i = d_i$ for $i < n$ and $c_n = d_n \cap (\max(c_n) + 1)$. $f((s, \bar{c}), n + 1) = \max(c_n) + 1$ with $c_{-1} = s$.

Blass-Shelah forcing (See [1, page 370 ff.] or [5]) also fulfils this condition. $f(p, n)$ is $\max\{x : x \in c_k\}$ where $p = (a, c_0, c_1, \dots)$ and k is such that from k onwards the norm of c_i is above n. Blass-Shelah forcing is translated into a forcing with labelled trees in [1, pages 370–372]. This procedure given their differs slightly from the simple translation given above and allows for a very elegant proof of item (4) in the definition of Axiom A.

5. The effect of Jensen's diamond in the ground model

It is well known [5, Lemma 5.10] that in countable support iterations of proper iterands $\mathbb{Q} \subseteq \omega^\omega$, starting from a ground model with CH, names for subsets of ω_1 have in a certain sense equivalent names $\subseteq H(\omega_1)$. We recall the proof of this fact, and we strengthen the notion of equivalence slightly. In this sitation, names for subsets of ω_1 can be guessed if Jensen's diamond holds in the ground model. Then we find countable subsets of the guessed names, in the ground model, and this will be our club sequence. In the following let $\mathbb{P}_\alpha = \langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\delta : \gamma \leq \alpha, \delta < \alpha \rangle$ be a countable support iteration of proper iterands such that for all $\gamma < \alpha$, $\mathbb{P}_\gamma \Vdash \underset{\sim}{\mathbb{Q}}_\gamma \subseteq \omega^\omega$. Here, we identify each tree

$p \in \mathbb{Q}$, where $\mathbb{Q}$ is an iterand as in the second or in the fourth section, with a real. The guessing procedure to establish the club principle in $\mathbf{V}^{\mathbb{P}_\alpha}$ works for any iteration length α. For application we set $\alpha = \omega_2$.

Our proof of the existence of hereditarily countable names is a modification of Section 5 in [5]. Since it is a slight strengthening, we carry it out.

We consider $\mathbb{P}_\alpha$-names x for subsets of ω (i.e., reals) as being specified by giving, for each $n \in \omega$, a maximal antichain $A(x,n)$ in $\mathbb{P}_\alpha$ and for each $p \in A(x,n)$ a value $v(x,n,p) \in 2$ such that $p \Vdash x(n) = v(x,n,p)$. It is well known that every name of a real is equivalent, in the sense of equality forced by all conditions, to one of this sort. When we are interested only in conditions extending a particular p, then the antichain $A(x,n)$ need to be maximal only in the weaker sense that no extension of p can be added to them, we then refer to x as a name for a real relative to p. We call such a name x *hereditarily countable relative to p* if, for each n, the subset of $A(x,n)$ that consists of the conditions in $A(x,n)$ that are compatible with p is countable and all the $\mathbb{P}_\beta$-names of reals occurring in the conditions p_β constituting any $p = \langle p_\beta : \beta < \alpha, \beta \in \mathrm{supp}(p)\rangle \in A(x,n)$ are hereditarily countable relative to p. A (just) hereditary countable name x is an $x \in H(\omega_1)$.

Lemma 5.1. [5, Lemma 5.7] *Let N be a countable elementary submodel of $H(\chi)$ that contains $\mathbb{P}_\alpha$, and let p be an $(N, \mathbb{P}_\alpha)$-generic condition. Then for every $\mathbb{P}_\alpha$-name $x \in N$ for a real there is a hereditarily countable $\mathbb{P}_\alpha$-name y relative to p such that $p \Vdash x = y$.*

Corollary 5.2. [5, Cor. 5.8] *If x is a $\mathbb{P}$-name for a real then the set of conditions that force $x = y$ for some $y \in H(\omega_1)$ is dense.*

We need a slightly improved version of this lemma: The hereditarily countable name relative to p is the same for many p.

Lemma 5.3. *Let N be a countable elementary submodel of $H(\chi)$ that contains $\mathbb{P}_\alpha$. Then for every $\mathbb{P}_\alpha$-name $x \in N$ for a real there is a name y, that is hereditarily countable relative to every $(N, \mathbb{P}_\alpha)$-generic p and every $(N, \mathbb{P}_\alpha)$-generic p forces $x = y$.*

Proof. The $(N, \mathbb{P}_\alpha)$-genericity of p implies that $p \restriction \beta$ is $(N, \mathbb{P}_\beta)$-generic for all $\beta \in \alpha$. Since $p \in N$ and $\mathrm{supp}(p)$ is countable $\mathrm{supp}(p) \subseteq N$. The lemma is proved by induction on α. To obtain y, first replace each of the antichains $A(x,n)$ by its intersection with N. These intersections are maximal relative to p for every $(N, \mathbb{P}_\alpha)$-generic p, just by $(N, \mathbb{P}_\alpha)$-genericity, and the name x' obtained in this way is forced by p to equal x. Since N is countable, the antichains $A(x', n) = A(x,n) \cap N$ are countable. If $q = \langle q_\beta : \beta < \gamma\rangle$ is in one of these antichains, hence in N, then N also contains an enumeration, in an ω-sequence of all the countably many conditions as the iteration has countably many non-trivial components q_β. Thus each of these components q_β is in N and can therefore by induction hypothesis be replaced, for every $(N, \mathbb{P}_\beta)$-generic q by a hereditarily countable relative to q $\mathbb{P}_\beta$-name. Doing this simultaneously for all such q and β we obtain the desired name y. $\dashv$

Now from our lemma we also get a slightly stronger corollary:

Corollary 5.4. *If $x \in N$ is a $\mathbb{P}_\alpha$-name for a real then there is some $y \in H(\omega_1)$ such that the set of conditions that force $x = y$ is dense for conditions in N, i.e., $(\forall p \in N)(\exists q \geq p)(q \Vdash x = y)$.*

For $\delta < \omega_1$, the same applies to all $\mathbb{P}_{\omega_2}$-names for subsets of δ instead of subsets of ω. Every name X for a subset of ω_1 is equivalent to the sequence of names of $\langle X \cap \delta : \delta < \omega_1 \rangle$. Given an increasing sequence of models N_δ, $\delta < \omega_1$, each member X_δ of the sequence can be replaced by a hereditarily countable object that is forced to be the same by densely many conditions in N_δ. The same analysis shows that conditions themselves, being countable functions into iterands whose conditions are reals, are equivalent to hereditarily countable objects.

The next lemma is a modification of [12, IV.4] for $\aleph_1$ names for $X \cap \delta$, $\delta < \omega_1$, and for arbitrary iteration length. We write ω_2 nevertheless.

Lemma 5.5. *Assume $\diamondsuit$ in the ground model, and assume that $\mathbb{P}_{\omega_2}$ is an iteration of axiom A forcings with the stepping up property from Lemma 3.5. There is a sequence $\langle C_\delta : \delta \in \lim(\omega_1) \rangle$ such that C_δ is cofinal in δ and for every $p \in \mathbb{P}_{\omega_2}$ and every $\mathbb{P}_{\omega_2}$-name $\underset{\sim}{X}$ for an uncountable subset of ω_1 there are $q \geq p$ and $\delta \in \lim(\omega_1)$ such that $q \Vdash C_\delta \subseteq \underset{\sim}{X}$.*

Proof. We start with the diamond:

Claim 5.6. *(See the claim within the proof of* [12, IV.4]*) Under $\diamondsuit$, there is a sequence $\langle p_\delta, A_\delta, M_\delta : \delta \in \lim(\omega_1) \rangle$ such that if $p \in \mathbb{P}_{\omega_2}$, $\underset{\sim}{X}$ a $\mathbb{P}_{\omega_2}$-name for an uncountable subset of ω_1, $\underset{\sim}{X} \subseteq H(\omega_1)$ and $C \subseteq [H(\omega_3)]^{\aleph_0}$ is a closed and unbounded set of countable elementary submodels then there are an $M \in C$ such that $\underset{\sim}{X}, p, \mathbb{P}_{\omega_2} \in M$, $\underset{\sim}{X}$ can be written as $\bigcup_{\delta \in \omega_1} \underset{\sim}{X}_\delta$, $\underset{\sim}{X}_\delta \in H(\omega_1)$ such that for every δ, $\underset{\sim}{X}_\delta$ is a hereditarily countable name relative to every M_δ-generic q and is equivalent to $\underset{\sim}{X} \cap \delta$ in the sense of Lemma 5.3, and there is $\delta < \omega_1$ such that p is equivalent to p_δ, $M \cap H(\omega_1) = M_\delta$, $M_\delta \cap \omega_1 = \delta$, $p_\delta \in M_\delta$ and $\underset{\sim}{X}_\delta = A_\delta$.*

Proof. Fix a $\diamondsuit$-sequence $\langle D_\delta : \delta \in \lim(\omega_1) \rangle$. First, using CH, we construct a sequence $\langle M_\delta : \delta \in C' \rangle$ for some club C' in ω_1 such that

(a) M_δ is an elementary submodel of $H(\omega_1)$,

(b) $M_\beta \preceq M_\delta$ for $\beta < \delta$, and $M_\delta = \bigcup \{M_\beta : \beta < \delta\}$ for limit δ,

(c) $\{M_\delta : \delta \in C'\}$ is a closed and unbounded subset of $[H(\omega_1)]^{\aleph_0}$,

(d) $M_\delta \cap \omega_1 = \delta$ for $\delta \in C'$.

Note that $\bigcup \{M_\delta : \delta \in C'\} = H(\omega_1)$. For every $C \subseteq [H(\omega_3)]^{\aleph_0}$ that is a closed and unbounded set of elementary submodels of $H(\omega_3)$ the set

$$\{\delta \in \omega_1 : (\exists M \in C) M \cap H(\omega_1) = M_\delta\}$$

contains a club subset of C'. We also fix a bijection $\Phi \colon \omega_1 \to H(\omega_1)$ such that $\Phi[\delta] = M_\delta$ for $\delta \in C'$. Now we define p_δ, A_δ. For $\delta \in C'$ if $\Phi[D_\delta] = \{p\} \times (\delta, A)$ we let $p_\delta = p$, $A_\delta = A$. Otherwise we let p_δ and A_δ be arbitrary.

To see that the construction works let p, $\underset{\sim}{X}$, C be as required. Let $D = \Phi^{-1}[\{p\} \times \bigcup_{\delta \in C}(\{\delta\} \times \underset{\sim}{X}_\delta)]$. Let $C'' = \{\delta \in C' : (\exists M \in C)(M \cap H(\omega_1) = M_\delta)\}$ and note that this is a club subset of ω_1. There is an $\delta \in C''$ such that $D_\delta = D \cap \delta$, since $(D_\delta)_{\delta \in \lim(\omega_1)}$ is a $\diamondsuit$-sequence. This implies that p is equivalent to p_δ and $\underset{\sim}{X}_\delta = A_\delta$. As $\delta \in C''$, also $M_\delta \cap \omega_1 = \delta$, and there is an $M \in C$ such that $M \cap H(\omega_1) = M_\delta$. This finishes the proof of the claim.

Having fixed a sequence $\langle p_\delta, A_\delta, M_\delta : \delta \in \lim(\omega_1)\rangle$ like this, construct C_δ as follows: If there is a $p \in \mathbb{P}_{\omega_2}$, $\underset{\sim}{X}$ a name for an uncountable subset of ω_1 of the prescribed form and an elementary submodel M containing p and $\underset{\sim}{X}$ such that $p_\delta = p$, $M_\delta = M \cap H(\omega_1)$, $M_\delta \cap \omega_1 = \delta$, and $A_\delta = \underset{\sim}{X} \cap M_\delta$ then we find a sequences $\langle \alpha_i \in \omega \rangle \nearrow \delta$ and an enumeration I_i, $i \in \omega$, of the dense sets of $\mathbb{P}_{\omega_2}$ that are in M_δ, and according to the stepping up lemma we may construct a sequence $\langle q_i, F_i, \vec{n}_i, \vec{k}_i, \beta_i : i \in \omega\rangle$ such that

(1) $F_i \subseteq F_{i+1}$, $\bigcup_{i\in\omega} F_i = \delta$,

(2) $\alpha_i < \beta_i < \delta$,

(3) $q_0 \geq p_\delta$,

(4) $q_i \in \mathbb{P}_{\omega_2} \cap M_\delta$ is $(F_i, \vec{k}_i)$ determined and $\vec{k}_i = \vec{k}(p_i, F_i, \vec{n}_i)$, w.l.o.g. $\vec{n}_i$ can be the vector constant to i, at least we need that $(\forall \beta \in \mathrm{supp}(p))(\lim_{i\to\omega} \vec{n}_i(\beta) = \infty)$,

(5) $q_{i+1} \geq_{(F_i, \vec{n}_i)} q_i$,

(6) q_i is $(\underset{\sim}{X}_\delta, F_i, \vec{n}_i)$-good in M_δ,

(7) $q_i \Vdash \beta_i \in A_\delta$ and

(8) $q_i \Vdash I_i \cap M_\delta \cap \underset{\sim}{G} \neq \emptyset$.

Finally set $C_\delta = \{\beta_i : i < \omega\}$.

In order to verify that $\langle C_\delta, : \delta \in \omega_1\rangle$ construction works let $p \in \mathbb{P}_{\omega_2}$ and $\underset{\sim}{X}$ be as required. Let C be a closed unbounded set of elementary submodels of $H(\omega_3)$. Take $M \in C$ containing p, $\mathbb{P}_{\omega_2}$ and $\underset{\sim}{X}$. Then there is a $\delta \in \lim(\omega_1)$ such that p is equivalent to p_δ and $\underset{\sim}{X} \cap M_\delta = A_\delta$ and $M \cap H(\omega_1) = M_\delta$. Let q be the fusion of the sequence constructed at stage δ as above. Note that though the model in which q was constructed was probably different from M and the name for an uncountable subset of ω_1 was most likely not $\underset{\sim}{X}$ in the construction we never had to go outside $H(\omega_1)$ on which the two models agree. Since q is $(M_\delta, \mathbb{P})$-generic Cor. 5.4 yields $q \Vdash \underset{\sim}{X} \cap M_\delta = A_\delta$ and hence $q \Vdash_{\mathbb{P}_{\omega_2}} C_\delta \subseteq \underset{\sim}{X}$. $\dashv$

In Lemma 5.5 we just used Cor. 5.4 and the stepping up lemma. The premises to these lemmas are fulfilled by countable support iterations of axiom A forcing with the finiteness property. Putting things together we get: If Jensen's diamond holds in the ground model, and we iterate axiom A forcings that have the finiteness property with countable support, then in the resulting extension the club principle holds. So Theorem 1.4 is proved. Now we take ω_2 as the iteration length. Since in the Mathias model $\mathfrak{h} = \aleph_2$, Theorem 1.2 is now proved. Since in the Matet model $\mathfrak{u} < \mathfrak{g}$, Theorem 1.3 is proved.

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