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CREATURES ON ω_1 AND WEAK DIAMONDS

HEIKE MILDENBERGER

Abstract. We specialise Aronszajn trees by an ω^ω-bounding forcing that adds reals. We work with creature forcings on uncountable spaces.

As an application of these notions of forcing, we answer a question of Moore, Hrušák and Džamonja whether $\diamondsuit(\mathfrak{b})$ implies the existence of a Souslin tree in a negative way by showing that "$\diamondsuit(\mathfrak{d})$ and every Aronszajn tree is special" is consistent relative to ZFC.

§0. Introduction. We specialise Aronszajn trees by forcing with countable trees of finite partial specialisations. We modify the forcings with creatures on $\aleph_1$ from [9] by dropping one coordinate of the entries at the nodes of the trees that serve as forcing conditions and by dropping the growth restraints on the nodes. Our forcings do not have the halving property (for the interested reader: [12, Def. 2.2.7] or [9, Def. 3.4]), whereas a version of a Ramsey property (see Lemma 3.3) still holds. A very useful property of our forcings it that they allow continuous reading of names (Theorem 3.4).

As an application of these notions of forcing, we answer a question of Moore, Hrušák and Džamonja (Question 6.3 of [10]), whether $\diamondsuit(\mathfrak{b})$ implies the existence of a Souslin tree, in a strong negative way by showing that "$\diamondsuit(\mathfrak{d})$ and every Aronszajn tree is special" is consistent.

The paper is almost self-contained in the part specific to the creatures. However, we will cite known preservation theorems for proper notions of forcing with certain additional properties. The reader interested in other examples of creature forcing may consult [12, 13, 11, 14].

In Section 1 we introduce the creatures and prove several lemmata about finite compositions. In Section 2 we introduce the notion of forcing and show that it specialises a given Aronszajn tree. In Section 3 we prove continuous reading of names. In Section 4 we give some applications: Certain weak diamonds are consistent together with "all Aronszajn trees are special".

§1. Normed tree creatures. Given an Aronszajn tree $(\boldsymbol{T}, <_{\boldsymbol{T}})$, a specialisation function is a function $f \colon \boldsymbol{T} \to \omega$ such that for all $s, t \in \boldsymbol{T}$, if $s <_{\boldsymbol{T}} t$, then

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$f(s) \neq f(t)$. In the nomenclature of [6, p. 244] this is a regularisation function. Now we first consider finite partial specialisation functions.

Definition 1.1. For $u \subseteq \boldsymbol{T}$, u finite, we let

$$\mathrm{spec}(u) = \{\eta \mid \eta \colon u \to \omega \ \wedge\ (\forall x, y \in u)(\eta(x) = \eta(y) \to \neg(x <_{\boldsymbol{T}} y))\}.$$

$\mathrm{spec}^{\boldsymbol{T}} = \bigcup\{\mathrm{spec}(u) \ :\ u \subset \boldsymbol{T}, u \text{ finite}\}$.

Definition 1.2. A creature is a tuple $\boldsymbol{c} = (\eta(\boldsymbol{c}), \mathrm{pos}(\boldsymbol{c}))$ with the following properties: The first component $\eta(\boldsymbol{c})$ is called the base of $\boldsymbol{c}$ and is in $\mathrm{spec}^{\boldsymbol{T}}$. The second component $\mathrm{pos}(\boldsymbol{c})$ is a non-empty subset of $\{\eta \in \mathrm{spec}^{\boldsymbol{T}} \ :\ \eta(\boldsymbol{c}) \subsetneq \eta\}$.

In the language of [12, Remark 1.3.4(3)], the set $\mathrm{pos}(\boldsymbol{c})$ is $\mathrm{range}(\mathrm{val}(\boldsymbol{c}))$, and "pos" stands for possibilities. For creatures with $\mathrm{nor}^0(\boldsymbol{c}) \geq 1$ we have by Lemma 1.4 that $\eta(\boldsymbol{c}) = \bigcap\{\eta \ :\ \eta \in \mathrm{pos}(\boldsymbol{c}))\}$, and hence $\eta(\boldsymbol{c})$ is determined by $\mathrm{pos}(\boldsymbol{c})$. The norm defined in the next definition is a simplification of norms used in [9].

Definition 1.3. (1) For a creature $\boldsymbol{c}$ we define $\mathrm{nor}^0(\boldsymbol{c})$ as the maximal natural number k such that: If $a \subseteq \omega$ and $|a| \leq k$ and $B_0, \dots, B_{k-1}$ are branches of $\boldsymbol{T}$, then there is $\eta \in \mathrm{pos}(\boldsymbol{c})$ such that

$$(\forall x \in (\bigcup_{\ell<k} B_\ell \cap \mathrm{dom}(\eta)) \setminus \mathrm{dom}(\eta(\boldsymbol{c})))(\eta(x) \notin a).$$

(2) For $\boldsymbol{c}$ with $\mathrm{nor}^0(\boldsymbol{c}) \geq 1$ we define $\mathrm{nor}^1(\boldsymbol{c}) = \log_2(\mathrm{nor}^0(\boldsymbol{c})) \in \mathbb{R}$.

The norm is well-defined, because $\mathrm{nor}^0(\boldsymbol{c}) < |\mathrm{pos}(\boldsymbol{c})|$. For a real r, we use the upper angles $\lceil r \rceil$ to denote the smallest integer larger than or equal to r.

Lemma 1.4. *If* $\mathrm{nor}^0(\boldsymbol{c}) \geq 1$, *then* $\bigcap \mathrm{pos}(\boldsymbol{c}) = \eta(\boldsymbol{c})$.

Proof. By Definition 1.2 the left-hand side is a superset of $\eta(\boldsymbol{c})$. Suppose that $(x, y) \in \bigcap \mathrm{pos}(\boldsymbol{c}) \smallsetminus \eta(\boldsymbol{c})$. Then take $a = \{y\}$ and B_0 containing x and since $\mathrm{nor}^0(\boldsymbol{c}) \geq 1$ there is some $\eta \in \mathrm{pos}(\boldsymbol{c})$ such that either $x \notin \mathrm{dom}(\eta)$ or $\eta(x) \neq y$. So $(x, y) \notin \bigcap \mathrm{pos}(\boldsymbol{c})$. Contradiction. ⊣

The following lemma shows how to make one creature out of a creature $\boldsymbol{c} = (\eta, \{\eta \cup \rho_k \ :\ k < k^*\})$ and creatures $\boldsymbol{c}_k = (\eta \cup \rho_k, \{\eta_{k,j} \ :\ j < n(k)\})$, $k < k^*$, under suitable conditions. The new creature will have norm not too much smaller than the minimum of the norms of $\boldsymbol{c}_0, \dots, \boldsymbol{c}_{k^*-1}$. The lemma will be used to construct some condition witnessing continuous reading of names in Theorem 3.4. Note that the norm of $\boldsymbol{c}$ need not be large: The pairwise disjointness of the $\mathrm{dom}(\rho_k)$ as in (b) suffices. If the images of the ρ_k coincide, the norm of $\boldsymbol{c}$ can be 1.

Lemma 1.5. *Assume that* $k^* \geq 1$.

(*a*) $\boldsymbol{c} = (\eta, \{\eta \cup \rho_k \ :\ k < k^*\})$ *is a creature,*

(*b*) ρ_k, $k < k^*$, *are partial specialisation functions such that for each* $k < k' < k^*$ *for each* $x \in \mathrm{dom}(\rho_k)$, $y \in \mathrm{dom}(\rho_{k'})$, x *and* y *are incomparable in* $\boldsymbol{T}$,

(*c*) *for every* $k < k^*$, $\eta \cup \rho_k$ *is the base of a creature* $\boldsymbol{c}_k = (\eta \cup \rho_k, \{\eta_{k,j} \ :\ j < n(k)\})$.

Then $\boldsymbol{d} = (\eta, \{\eta_{k,j} \ :\ k < k^*, j < n(k)\})$ *is a creature and*

$$\mathrm{nor}^0(\boldsymbol{d}) \geq m_0 = \min\left(\{\mathrm{nor}^0(\boldsymbol{c}_k) \ :\ k < k^*\} \cup \{k^* - 1\}\right). \tag{1.1}$$

PROOF. Let branches $B_0, \dots, B_{m_0-1}$ of $\boldsymbol{T}$ and a set $a \subseteq \omega$ be given, $|a| \leq m_0$. Now for each $\ell < m_0$, we let

$$w_\ell = \{k < k^* \,:\, B_\ell \cap \operatorname{dom}(\rho_k) \setminus \operatorname{dom}(\eta) \neq \emptyset\}.$$

Now we have that $|w_\ell| \leq 1$ because otherwise we would have $k_1 < k_2 < k^*$ in w_ℓ and $x_i \in B_\ell \cap \operatorname{dom}(\rho_{k_i}) \setminus \operatorname{dom}(\eta)$. So x_1 and x_2 are $<_{\boldsymbol{T}}$-comparable, and this contradicts the premise (b). Since $m_0 < k^*$, there is some $k \in k^* \setminus \bigcup_{\ell<m_0} w_\ell$. We fix such a k. Now we use that $\operatorname{nor}^0(\boldsymbol{c}_k) \geq m_0$ and find $\mu \in \operatorname{pos}(\boldsymbol{c}_k)$ such that for all $\ell < m_0$ and for all $x \in B_\ell \cap \operatorname{dom}(\mu) \setminus \operatorname{dom}(\eta \cup \rho_k)$, $\mu(x) \notin a$. Then, by the choice of k, for all $\ell < m_0$, we have $(\forall x \in B_\ell \cap \operatorname{dom}(\mu) \setminus \operatorname{dom}(\eta))(\mu(x) \notin a)$. ⊣

The following two lemmata will be used in the next section for density arguments. The aim is to show that the union of the roots of the forcing conditions (which will be ω-trees) in the generic filter is a total specialisation function for $\boldsymbol{T}$. The first lemma is used later only for $m = 1$.

LEMMA 1.6. *Suppose that $\boldsymbol{c}$, m, k are as follows*:

(a) *$\boldsymbol{c}$ is a creature*,

(b) $\operatorname{nor}^0(\boldsymbol{c}) = k > m$,

(c) $x_0, \dots, x_{m-1} \in \boldsymbol{T}$, $1 \leq m$.

Then there is some creature $\boldsymbol{d}$ such that

(1) $\eta(\boldsymbol{d}) = \eta(\boldsymbol{c})$,

(2) $\operatorname{pos}(\boldsymbol{d}) \subseteq \{\nu \in \operatorname{spec}^{\boldsymbol{T}} \,:\, (\exists \eta \in \operatorname{pos}(\boldsymbol{c}))(\eta \subseteq \nu \wedge \operatorname{dom}(\nu) = \operatorname{dom}(\eta) \cup \{x_0, \dots, x_{m-1}\})\}$,

(3) $\operatorname{nor}^0(\boldsymbol{d}) \geq k$.

PROOF. For each $\eta \in \operatorname{pos}(\boldsymbol{c})$ we choose $m + k$ elements from $\omega \setminus \operatorname{range}(\eta)$, and put them into a set B_η so that the B_η are pairwise disjoint. For each $a \in [\omega]^k$ choose some set $\{z_{m'} \,:\, m' < m\} \subseteq B_\eta$, $\{z_{m'} \,:\, m' < m\} \cap a = \emptyset$ such that the $z_{m'}$'s are pairwise different. Then we have a specialisation $\nu_{\eta,\bar{z}} = \eta \cup \{(x_{m'}, z_{m'}) \,:\, m' < m\}$. We set

$$\boldsymbol{d} = \{(\eta(\boldsymbol{c}), \nu_{\eta,\bar{z}}) \,:\, \eta \in \operatorname{pos}(\boldsymbol{c}), \bar{z} \in [B_\eta]^m\}.$$

Now we check the norm: Let $B_1, \dots, B_k$ be branches of $\boldsymbol{T}$ and let $a \subseteq \omega$, $|a| \leq k$. We have to find $\nu \in \operatorname{pos}(\boldsymbol{d})$ such that

$$(\forall \ell < k)(\forall y \in \operatorname{dom}(\nu) \cap B_\ell \setminus \operatorname{dom}(\eta(\boldsymbol{c})))(\nu(y) \notin a).$$

By premise (b), we find $\eta \in \operatorname{pos}(\boldsymbol{c})$ such that

$$(\forall \ell < k)(\forall x \in \operatorname{dom}(\eta) \cap B_\ell \setminus \operatorname{dom}(\eta(\boldsymbol{c})))(\eta(x) \notin a).$$

Taking $\bar{z}$ disjoint from a, we have $\nu_{\eta,\bar{z}} \in \operatorname{pos}(\boldsymbol{d})$ such that

$$(\forall \ell < k)(\forall x \in \operatorname{dom}(\nu_{\eta,\bar{z}}) \cap B_\ell \setminus \operatorname{dom}(\eta(\boldsymbol{c})))(\nu_{\eta,\bar{z}}(x) \notin a). \qquad ⊣$$

Suppose we have extended the partial specialisation functions in the possibility set of a creature as in the previous lemma. Then we want that these extended functions can serve as bases for suitable creatures (at one level higher up in a tree built from creatures, see Definition 2.2) as well. Under additional premises, these bases can indeed be extended:

LEMMA 1.7. *Assume that*

(a) $\boldsymbol{c}$ *is a creature with* $0 < \mathrm{nor}^0(\boldsymbol{c})$.

(b) $\eta^* \supseteq \eta(\boldsymbol{c})$, $\eta^* \in \mathrm{spec}^{\boldsymbol{T}}$, *and for* $v \in \mathrm{pos}(\boldsymbol{c})$, $\mathrm{dom}(\eta^*) \cap \mathrm{dom}(v) = \mathrm{dom}(\eta(\boldsymbol{c}))$.

(c) *We set*

$$\ell_2^* = |\,\mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\boldsymbol{c}))|,$$

and

$$\begin{aligned}\ell_1^* = |\{y \,:(\exists v \in \mathrm{pos}(\boldsymbol{c}))(y \in \mathrm{dom}(v) \setminus \mathrm{dom}(\eta(\boldsymbol{c})))\\ \wedge (\exists x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\boldsymbol{c})))(x <_{\boldsymbol{T}} y)\}|,\end{aligned}$$

and we assume that $\ell_1^* + \ell_2^* < \mathrm{nor}^0(\boldsymbol{c})$.

We define

$$\boldsymbol{d} = \{(\eta^*, v \cup \eta^*) \,:\, v \in \mathrm{pos}(\boldsymbol{c}) \ \wedge \ v \cup \eta^* \in \mathrm{spec}^{\boldsymbol{T}} \ \wedge \eta^* \subsetneqq v \cup \eta^*\}.$$

Then

(α) $\boldsymbol{d}$ *is a creature.*

(β) $\mathrm{nor}^0(\boldsymbol{d}) \geq \mathrm{nor}^0(\boldsymbol{c}) - \ell_2^* - \ell_1^*$.

PROOF. Item (α) follows from item (β), where it is also shown that $\mathrm{pos}(\boldsymbol{d}) \neq \emptyset$. For item (β), we set $k = \mathrm{nor}^0(\boldsymbol{c}) - \ell_1^* - \ell_2^*$. We let $B_0, \dots, B_{k-1}$ be branches of $\boldsymbol{T}$ and $a \subseteq \omega$, $|a| \leq k$. We let $\langle y_\ell \,:\, \ell < \ell_1^*\rangle$ list $Y = \{y \,:\, (\exists v \in \mathrm{pos}(\boldsymbol{c}))(y \in \mathrm{dom}(v) \setminus \mathrm{dom}(\eta(\boldsymbol{c}))) \wedge (\exists x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\boldsymbol{c})))(x <_{\boldsymbol{T}} y)\}$ without repetition. Let $B_k, \dots, B_{k+\ell_1^*-1}$ be branches of $\boldsymbol{T}$ such that $y_\ell \in B_{k+\ell}$ for $\ell < \ell_1^*$. Let $\langle x_\ell \,:\, \ell < \ell_2^*\rangle$ list $\mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\boldsymbol{c}))$. Take for $\ell < \ell_2^*$ a branch $B_{k+\ell_1^*+\ell}$ such that $x_\ell \in B_{k+\ell_1^*+\ell}$. We set $a' = a \cup \{\eta^*(x_\ell) \,:\, \ell < \ell_2^*\}$. Since $\mathrm{nor}^0(\boldsymbol{c}) \geq k + \ell_1^* + \ell_2^*$ there is some $v \in \mathrm{pos}(\boldsymbol{c})$ such that

$$(\otimes) \qquad \forall x \in ((\mathrm{dom}(v) \setminus \mathrm{dom}(\eta(\boldsymbol{c}))) \cap \bigcup_{\ell < k + \ell_1^* + \ell_2^*} B_\ell)(v(x) \notin a').$$

Then, if $x \notin \mathrm{dom}(\eta^*)$, $(v \cup \eta^*)(x) \notin a$.

We show that $v \cup \eta^*$ is a partial specialisation: It is a function since $v \supseteq \eta(\boldsymbol{c})$ and $\eta^* \supseteq \eta(\boldsymbol{c})$ and $\mathrm{dom}(v) \cap \mathrm{dom}(\eta^*) = \mathrm{dom}(\eta(\boldsymbol{c}))$. Since η^* and v are specialisation maps, we have to consider only the case $x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\boldsymbol{c}))$ and ($y \in Y$ or ($y \in \mathrm{dom}(v) \setminus \mathrm{dom}(\eta^*)$ and $y <_{\boldsymbol{T}} x$)). If $y \in Y$, then we have $v(y) \neq \eta^*(x_\ell)$ for all $\ell < \ell_2^*$ by Equation $(\otimes)$. If $y \in \mathrm{dom}(v) \setminus \mathrm{dom}(\eta^*)$ and $y <_{\boldsymbol{T}} x$, then y is in a branch leading to some x_ℓ, $\ell < \ell_2^*$, and hence again by Equation $(\otimes)$, $v(y) \neq \eta^*(x_\ell)$, $\ell < \ell_2^*$. ⊣

In the applications we can arrange, by filling up slowly step by step, that $\ell_2^* = 1$. We will have $\ell_1^* = |u|$, where u is the set that sticks out of $\boldsymbol{T}_{<\alpha(p)}$ (see Definition 2.2) and this part will increase all the time if we fill up step by step.

The next lemma gives large homogeneous subtrees of the trees built from creatures that will later be used as forcing conditions. In [12, Definition 2.3.2] the following property is called the 2-bigness property.

LEMMA 1.8. *If* $\boldsymbol{c}$ *is a creature with* $\mathrm{nor}^0(\boldsymbol{c}) \geq 2$ *and* $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ *are creatures such that* $\mathrm{pos}(\boldsymbol{c}) = \mathrm{pos}(\boldsymbol{c}_1) \cup \mathrm{pos}(\boldsymbol{c}_2)$ *and* $\eta(\boldsymbol{c}) = \eta(\boldsymbol{c}_1) = \eta(\boldsymbol{c}_2)$, *then* $\max(\mathrm{nor}^1(\boldsymbol{c}_1), \mathrm{nor}^1(\boldsymbol{c}_2)) \geq \mathrm{nor}^1(\boldsymbol{c}) - 1$. (*This means for at least one of them* nor^1 *is defined.*)

PROOF. Let $j = \mathrm{nor}^1(\boldsymbol{c}) - 1 \geq 0$. We suppose that $\mathrm{nor}^0(\boldsymbol{c}_1) < 2^j$ and $\mathrm{nor}^0(\boldsymbol{c}_2) < 2^j \in \mathbb{N}$ and derive a contradiction. Let $J = 2^j$. For $\ell = 1, 2$ let branches $B^\ell_0, \dots, B^\ell_{J-1}$ and sets $a^\ell \subseteq \omega$, $|a^\ell| = J$, exemplify this.

Let $a = a^1 \cup a^2$ and let, by $\mathrm{nor}^0(\boldsymbol{c}) \geq 2^{j+1}$, $\eta \in \mathrm{pos}(\boldsymbol{c})$ be such that for all $x \in (\mathrm{dom}(\eta) \cap \bigcup_{\ell=1,2} \bigcup_{i=0}^{J-1} B^\ell_i) \setminus \mathrm{dom}(\eta(\boldsymbol{c}))$ we have $\eta(x) \notin a$. But then for that $\ell \in \{1, 2\}$ such that $\eta \in \mathrm{pos}(\boldsymbol{c}_\ell)$, we get a contradiction to a^ℓ being an witness for $\mathrm{nor}^0(\boldsymbol{c}_\ell) < 2^j$. We apply the logarithm function and get the desired results for nor^1. $\dashv$

§2. Trees of creatures. Now we define a notion of forcing with conditions that contain the same information as $\langle \boldsymbol{c}_{p,\eta} : \eta \in \mathrm{dom}(p)\rangle$. The conditions $p = (\mathrm{dom}(p), \lhd_p)$ are certain ω-trees whose nodes η are in $\mathrm{spec}^{\boldsymbol{T}}$ such that each node together with its immediate successors is a creature $\boldsymbol{c}_{p,\eta}$ from Definition 1.2.

First we collect some general notation about trees. The trees here are not the Aronszajn trees $\boldsymbol{T}$ of the first section, but trees $\mathscr{S}$ with domain $S \subseteq \mathrm{spec}^{\boldsymbol{T}}$, ordered by $\lhd_{\mathscr{S}}$ which is a subrelation of $\subsetneqq$. Some of these trees will serve as forcing conditions.

DEFINITION 2.1. (1) A $\boldsymbol{T}$-tree $\mathscr{S} = (S, \lhd_{\mathscr{S}})$ is a structure with domain $S \subseteq \mathrm{spec}^{\boldsymbol{T}}$, such that for any $\eta \in S$, $(\{\nu : \nu \lhd_{\mathscr{S}} \eta\}, \lhd_{\mathscr{S}})$ is a finite linear order and such that in S there is one $\lhd_{\mathscr{S}}$-minimal element, called the root, $\mathrm{rt}(\mathscr{S})$. If $\eta \lhd_{\mathscr{S}} \nu$ then $\eta \subsetneqq \nu$. We write $\eta \trianglelefteq_{\mathscr{S}} \zeta$ for $\eta \lhd_{\mathscr{S}} \zeta$ or $\eta = \zeta$. We shall only work with finitely branching trees. In addition we require that every $\eta \in S \setminus \{\mathrm{rt}(\mathscr{S})\}$ appears in exactly one partial branch $\langle \mathrm{rt}(\mathscr{S}), \eta_1, \dots, \eta_n\rangle$ as $\eta = \eta_n$.

(2) We define the successors of η in $\mathscr{S}$, the restriction of $\mathscr{S}$ to η, and the maximal points of $\mathscr{S}$ by

$$\mathrm{suc}_{\mathscr{S}}(\eta) = \{\nu \in S : \eta \lhd_{\mathscr{S}} \nu \wedge \neg(\exists \rho \in S)(\eta \lhd_{\mathscr{S}} \rho \lhd_{\mathscr{S}} \nu)\},$$
$$\mathscr{S}^{\langle \eta \rangle} = (\{\nu \in S : \eta \trianglelefteq_{\mathscr{S}} \nu\}, \lhd_{\mathscr{S}}),$$
$$\max(\mathscr{S}) = \{\nu \in S : \neg(\exists \rho \in S)(\nu \lhd_{\mathscr{S}} \rho\}.$$

(3) The n-th level of $\mathscr{S}$ is

$$\mathscr{S}^{[n]} = \{\eta \in S : \eta \text{ has } n \lhd_{\mathscr{S}}\text{-predecessors}\}.$$

The height of the tree is the minimal level which is empty, and all trees we shall consider here have height ω. The set of all branches through $\mathscr{S}$ is

$$\begin{aligned}\lim(\mathscr{S}) = \{\langle \eta_k : k < \ell\rangle : \; & \ell \leq \omega \wedge (\forall k < \ell)(\eta_k \in \mathscr{S}^{[k]}) \\ & \wedge (\forall k < \ell - 1)(\eta_k \lhd_{\mathscr{S}} \eta_{k+1}) \\ & \wedge \neg(\exists \eta_\ell \in S)(\forall k < \ell)(\eta_k \lhd_{\mathscr{S}} \eta_\ell)\}.\end{aligned}$$

A tree is well-founded if there are no infinite branches through it.

(4) A subset F of S is called a front of $\mathscr{S}$ if every branch of $\mathscr{S}$ passes through this set, and the set consists of $\lhd_{\mathscr{S}}$-incomparable elements.

Since $\mathscr{S}$ is finitely branching and has height ω, all fronts of $\mathscr{S}$ are finite.

DEFINITION 2.2. We define a notion of forcing $Q = Q_{\boldsymbol{T}}$ by $p \in Q$ iff

(i) $p = (\mathrm{dom}(p), \lhd_p)$ is a $\boldsymbol{T}$-tree of height ω with no maximal points.

(ii) $\mathrm{rt}(p)$ is the unique element of level 0 in the tree p. The ℓ-th level of p is denoted by $p^{[\ell]}$.

(iii) For every $\ell < \omega$ and $\eta \in p^{[\ell]}$ the set

$$\mathrm{suc}_p(\eta) = \{\nu \in p^{[\ell+1]} : \eta \lhd_p \nu\}$$

is $\mathrm{pos}(\boldsymbol{c})$ for a creature $\boldsymbol{c}$ with base η. We denote this creature by $\boldsymbol{c}_{p,\eta}$.

(iv) There is some $k < \omega$ and there is some $\alpha < \omega_1$ such that for every $\eta \in p^{[k]}$ there is a finite $u_\eta \subseteq \boldsymbol{T} \setminus \boldsymbol{T}_{<\alpha}$ such that for every ω-branch $\langle \eta_\ell : \ell < \omega\rangle$ of p satisfying $\eta_k = \eta$ we have $\bigcup_{\ell\in\omega} \mathrm{dom}(\eta_\ell) = \boldsymbol{T}_{<\alpha} \cup u_\eta$. We let $\alpha(p)$ be the minimal such α.

(v) For every ω-branch $\langle \eta_\ell : \ell \in \omega\rangle$ of p we have $\lim_{\ell\to\omega} \mathrm{nor}(\boldsymbol{c}_{p,\eta_\ell}) = \omega$.

The order $\leq = \leq_Q$ is given by letting $p \leq q$ (q is stronger than p, we follow the Jerusalem convention) iff there is a projection $\mathrm{pr}_{q,p}$ which satisfies

(a) $\mathrm{pr}_{q,p}$ is a function from $\mathrm{dom}(q)$ to $\mathrm{dom}(p)$.

(b) For every $\eta \in \mathrm{dom}(q)$ we have $\mathrm{pr}_{q,p}(\eta) \subseteq \eta$.

(c) There is some $d \in \omega$ such that for all ℓ, for all $\eta \in q^{[\ell]}$, $\mathrm{pr}_{q,p}(\eta) \in p^{[\ell+d]}$.

(d) If η_1, η_2 are both in q and if $\eta_1 \unlhd_q \eta_2$, then $\mathrm{pr}_{q,p}(\eta_1) \unlhd_p \mathrm{pr}_{q,p}(\eta_2)$.

(e) If $\nu, \rho \in \mathrm{dom}(q)$ and $\nu \lhd_q \rho$, $\mathrm{pr}_{q,p}(\nu) = \eta$, $\mathrm{pr}_{q,p}(\rho) = \tau$, then $\mathrm{dom}(\tau) \cap \mathrm{dom}(\nu) = \mathrm{dom}(\eta)$.

The partial order Q is not empty: We specialise $\boldsymbol{T}_{<\omega}$ by a bijection to $e[b]$ for the left-most branch b of the tree $\{s \in \omega^{<\omega} : (\forall k < |s|)(s(k) \leq k)\}$ that is mapped by e bijectively to ω. The e-images of other branches give alternative specialisations of $\boldsymbol{T}_{<\omega}$. It is easy to break them up into finite specialisations and arrange them in a tree such that the creatures $\boldsymbol{c}$ beginning in level n have $\mathrm{nor}^0(\boldsymbol{c}) = n$.

The relation $\leq_Q$ is indeed a partial order: Suppose $p \leq q$ and $q \leq r$. We take two projections $\mathrm{pr}_{r,q}$ and $\mathrm{pr}_{q,p}$ witnessing this and show that $\mathrm{pr}_{r,p} = \mathrm{pr}_{q,p} \circ \mathrm{pr}_{r,q}$ is as desired: For property (e), let $\nu', \rho' \in \mathrm{dom}(r)$ and $\nu' \lhd_r \rho'$, $\mathrm{pr}_{r,q}(\nu') = \nu$, $\mathrm{pr}_{r,q}(\rho') = \rho$, $\mathrm{pr}_{q,p}(\nu) = \eta$, $\mathrm{pr}_{q,p}(\rho) = \tau$, then $\mathrm{dom}(\tau) \cap \mathrm{dom}(\nu') = \mathrm{dom}(\tau) \cap \mathrm{dom}(\nu') \cap \mathrm{dom}(\rho)$. Since $\mathrm{dom}(\rho) \supseteq \mathrm{dom}(\tau)$, we get from the latter $\mathrm{dom}(\tau) \cap \mathrm{dom}(\nu') = \mathrm{dom}(\tau) \cap \mathrm{dom}(\nu) = \mathrm{dom}(\eta)$.

We shall also use $p^{[\leq\ell]}$, $p^{[\geq\ell]}$ with the obvious meanings. For $\eta \in \mathrm{dom}(p)$ we have $p^{\langle\eta\rangle} \geq p$. Let us give some informal description of the $\leq$-relation in Q: The stronger condition's domain is via $\mathrm{pr}_{q,p}$ mapped homomorphically w.r.t. the tree orders into a subtree of $p^{\langle \mathrm{pr}_{q,p}(\mathrm{rt}(q))\rangle}$. The projection $\mathrm{pr}_{q,p}$ need not be unique. If we restrict to the dense set of conditions that have incompatible immediate successors (that means the union of two such successors is not a partial specialisation anymore) at each node then the projections are unique. According to (b), the projection preserves the levels in the trees but for one jump in heights (the ℓ's in $p^{[\ell]}$), due to a possible lengthening of the root. The partial specialisation functions sitting on the nodes $\mathrm{pr}_{q,p}(\eta)$ of the tree p are extended (possibly by more than one extension per function) into $\eta \in \mathrm{dom}(q)$. According to (e) the new part of the domain of the extension is disjoint from the domains of the old partial specialisation functions living higher up in the projection of the new tree to the old tree.

In [12] the $\leq$-relation of the forcing is based on a sub-composition function (whose definition is not used here, because we just deal with one particular $\leq$ for notions of forcing) whose inputs are well-founded subtrees of the weaker condition. This well-foundedness condition [12, 1.3.5] is not fulfilled: we have to look at all the branches of p through $\mathrm{pr}_{q,p}(\nu)$ that are in the range of $\mathrm{pr}_{q,p}$ in order to see whether for all ρ the pair $\nu, \rho \in \mathrm{dom}(q)$ fulfil (e) of the definition of $p \leq q$. On the other hand, the projections shift all the levels by the same amount d, and are not arbitrary finite contractions as in a good part of the forcings in the book [12].

Definition 2.3. (1) $p \in Q$ is called smooth iff in clause (iv) of Definition 2.2 the number k is 0 and u is empty.

(2) $p \in Q$ is called weakly smooth iff in clause (iv) of Definition 2.2 the number k is 0.

If $p \in Q$ is smooth then there is some $\alpha < \omega_1$ such that for every ω-branch $\langle \eta_\ell : \ell \in \omega \rangle$ of p we have $\bigcup_{\ell<\omega} \mathrm{dom}(\eta_\ell) = \boldsymbol{T}_{<\alpha}$.

Lemma 2.4. *If $p \leq q$, p is weakly smooth, witnessed by $T_{<\alpha(p)} \cup u$, and $\mathrm{pr}_{q,p}$ is a projection from q onto p, then we have for all $\nu \in q$ that $\mathrm{dom}(\nu) \cap (\boldsymbol{T}_{<\alpha(p)} \cup u) = \mathrm{dom}(\mathrm{pr}_{q,p}(\nu))$.*

Proof. If p is weakly smooth, then all branches of p have the same union of domains, and hence it is immaterial whether τ from Definition 2.2(e) is in the range of $\mathrm{pr}_{q,p}$ or not. $\dashv$

Definition 2.5. For $0 \leq n < \omega$ we define the partial order $\leq_n$ on Q by letting $p \leq_n q$ iff

(i) $p \leq q$,

(ii) $\mathrm{rt}(p) = \mathrm{rt}(q)$,

(iii) there is some projection $\mathrm{pr}_{q,p}$ such that, if $\mathrm{pr}_{q,p}(\eta) = \nu$, then

- $\eta = \nu$ and $\boldsymbol{c}_{q,\eta} = \boldsymbol{c}_{p,\nu}$
- or $\mathrm{nor}^1(\boldsymbol{c}_{p,\nu}) \geq n$ and $\mathrm{nor}^1(\boldsymbol{c}_{q,\eta}) \geq n$.

So $q \leq_0 p$ implies that p and q have the same root. We state and prove some basic properties of the notions defined above.

Lemma 2.6. (1) *$(Q, \leq_n)$ is a partial order.*

(2) $p \leq_{n+1} q \rightarrow p \leq_n q \rightarrow p \leq q$.

The next lemma states that the smooth conditions in Q fulfil some fusion property:

Lemma 2.7. *Let $\langle n_i : i \in \omega \rangle$ be a strictly increasing sequence of natural numbers. We assume that for every i, q_i is weakly smooth with $\bigcup_{\eta \in \mathrm{dom}(q_i)} \mathrm{dom}(\eta) = \boldsymbol{T}_{<\alpha(q_i)} \cup u_{q_i}$ and $q_i \leq_{n_i} q_{i+1}$. We set $n_{-1} = 0$. If there is some α such that $\bigcup_{i \in \omega} \boldsymbol{T}_{<\alpha(q_i)} \cup u_{q_i} = \boldsymbol{T}_{<\alpha}$, then the fusion $q = \bigcup_{i<\omega} (q_i \restriction \{\eta \in \mathrm{dom}(q_i) : n_{i-1} \leq \mathrm{nor}^1(\boldsymbol{c}_{q_i,\eta}) < n_i\})$ of $\langle q_i, n_i : i \in \omega \rangle$ has the following properties: $q \in Q$, for all i, $q \geq_{n_i} q_i$, $\alpha = \alpha(q) \geq \sup_i \alpha(q_i)$ and q is smooth.*

Proof. The domains of the $\eta \in \bigcup \mathrm{dom}(q_i)$ combine along each branch of q to the same union, because of the weak smoothness and the fact that the partial specialisations in the trees q_i form along each branch of q_i an ascending chain of partial specialisations. $\dashv$

Now we want to fill up the domains of the partial specialisation functions and to show that the smooth conditions are dense. First we need some gluing procedures:

LEMMA 2.8. (1) *If $p \in Q$ and $\{\eta_1, \dots, \eta_n\}$ is a front of p, then $\{p^{\langle \eta_1 \rangle}, \dots, p^{\langle \eta_n \rangle}\}$ is predense above p.*

(2) *If $\{\eta_1, \dots, \eta_n\}$ is a front of p and $p^{\langle \eta_\ell \rangle} \leq_0 q_\ell \in Q$ for each ℓ, then there is a unique $q \geq p$ with $\{\eta : \eta \in \mathrm{dom}(p) \wedge \exists i \in \{1, \dots, n\} \eta \trianglelefteq \eta_i\} \subseteq \mathrm{dom}(q)$ such that for all ℓ we have that $q^{\langle \eta_\ell \rangle} = q_\ell$.*

(3) *Let q be as in (2). Then we have for $\eta \in \mathrm{dom}(q)$:*

$$\mathrm{nor}^0(\boldsymbol{c}_{q,\eta}) = \begin{cases} \mathrm{nor}^0(\boldsymbol{c}_{p,\eta}) & \text{if } \exists \ell (\eta \triangleleft \eta_\ell \text{ and } \eta \text{ is not a direct predecessor of } \eta_\ell), \\ \mathrm{nor}^0(\boldsymbol{c}_{q_\ell,\eta}) & \text{if } \eta \trianglerighteq \eta_\ell. \end{cases}$$

(4) *If we strengthen the conditions in item (2) to $p^{\langle \eta_\ell \rangle} \leq_0 q_\ell \in Q$, then*

$$\mathrm{nor}^0(\boldsymbol{c}_{q,\eta}) = \begin{cases} \mathrm{nor}^0(\boldsymbol{c}_{p,\eta}) & \text{if } \exists \ell (\eta \triangleleft \eta_\ell), \\ \mathrm{nor}^0(\boldsymbol{c}_{q_\ell,\eta}) & \text{if } \eta \trianglerighteq \eta_\ell. \end{cases}$$

(5) *For every densely many $p \in Q$ we have that* $\lim_{n \to \omega} \min\{\mathrm{nor}^0(\boldsymbol{c}_{p,\eta}) : \eta \in p^{[n]}\} = \infty$.

PROOF. Only item (5) is not so obvious. By Lemma 1.8, applied to the partition $\{\eta \in \mathrm{dom}(p) : \mathrm{nor}^0(\boldsymbol{c}_{p,\eta}) \geq k\}$ and its complement, for $n \in \omega$, there is $q \geq_n p$ such that for some $m \in \omega$ for all $\eta \in q^{[\geq m]}$, $\mathrm{nor}^0(\boldsymbol{c}_{q,\eta}) \geq k$. Now we apply fusion. ⊣

Fusion and any combination of infinitely many conditions work better with smooth conditions, because otherwise the fusion of the conditions may not be a condition any more because it needs infinitely many u_η to describe the union of the domains along the branches or because the union of the u_{q_i}-parts of the unions of the domains will be the same $\triangleleft_q$-above any η, but might differ from every $\boldsymbol{T}_{<\alpha}$ in infinitely many points. We can restrict our work to smooth conditions because they are $\leq_m$-dense in the forcing:

LEMMA 2.9. *If $p \in Q$ and $m < \omega$ then there is some smooth $q \in Q$ such that $p \leq_m q$. Moreover, if $\bigcup\{\mathrm{dom}(\eta) : \eta \in \mathrm{dom}(p)\} \subseteq \boldsymbol{T}_{<\alpha}$ then we can demand that $\bigcup\{\mathrm{dom}(\eta) : \eta \in \mathrm{dom}(q)\} = \boldsymbol{T}_{<\alpha}$.*

PROOF. We fix k as in item (iv) of Definition 2.2. There are u_η, $\alpha(\eta)$, $\eta \in p^{[k]}$, such that for every ω-branch $\langle \eta_\ell : \ell < \omega \rangle$ of p we have $\bigcup_{\ell \in \omega} \mathrm{dom}(\eta_\ell) = \boldsymbol{T}_{<\alpha(\eta_k)} \cup u_{\eta_k}$. We take α such that $\bigcup\{\mathrm{dom}(\eta) : \eta \in \mathrm{dom}(p)\} \subseteq T_{<\alpha}$.

We work separately for each $\eta \in p^{[k]}$. We set $q_{0,\eta} = p^{\langle \eta \rangle}$ and $\ell_1^* = |u_\eta|$. For $\eta \in p^{[k]}$ let $\{x_\ell^\eta : \ell < \omega\}$ enumerate $\boldsymbol{T}_{<\alpha} \setminus (\boldsymbol{T}_{<\alpha(\eta)} \cup u_\eta)$ without repetition. We will in the first step $\leq_m$-strengthen $p^{\langle \eta \rangle} = q_{0,\eta}$ to a condition $q_{\eta,1}$ that has x_0^η in each $\mathrm{dom}(\zeta)$ for a front of ζ's and that looks to a certain height like p. Then we will repeat this process for $\leq_{m+1}$, x_1^η, $\ell_1^1 = \ell_1^* + 1 = |u_\eta| + 1$, and $q_{\eta,1}$ at a higher level of the intermediate condition $q_{\eta,1}$ and so on until all x_ℓ^η are built in and in such a way that the outcome q_η is a fusion of the $q_{\eta,i}$ (with suitable n_i) and q_η is still a condition. Putting the $q_\eta \geq_m p^{\langle \eta \rangle}$ for $\eta \in p^{[k]}$ together as in Lemma 2.8 we thus get and $p \leq_m q$. The premise of the fusion lemma about the domains is fulfilled: Above each $\eta \in p^{[k]}$, p and all intermediate conditions are weakly smooth, and the union of the domains along each branch will be $\boldsymbol{T}_{<\alpha}$, by our choice of the set $\{x_\ell^\eta : \ell \in \omega\}$.

So now we go on for each η separately: We show that for every n there are $q_{\eta,n+1} \geq_n q_{\eta,n}$ and $h(n+1)$ such that set $\{x^\eta_{\ell'} : \ell' \leq n\}$ is a subset of the union of the domains along every branch of $q_{\eta,n+1}$ and such that

$$q_\eta := \bigcup_{n<\omega} \bigcup_{h(n)\leq h<h(n+1)} (q_{n,\eta} \restriction q^{[h]}_{n,\eta}) \in Q.$$

We let $h(0) = 0$. $q_{\eta,0} = p^{\langle\eta\rangle}$. Suppose by induction hypothesis we have $q_{\eta,n} \geq_{n-1+m} q_{\eta,n-1} \geq_{n-2+m} \cdots \geq_m q_{\eta,0}$ such that for all $i \leq n$ for all $\zeta \in \operatorname{pos}(\operatorname{rt}(q_{\eta,i}))$, $x^\eta_i \in \operatorname{dom}(\zeta)$, and that $h(n) \in \omega$ is chosen.

We find by Lemma 2.6 some $h(n+1)$ such that

(1) $h(n) < h(n+1) < \omega$,

(2) for every $\nu \in q^{[\geq h(n+1)]}_{\eta,n}$, we have $\operatorname{nor}^0(\boldsymbol{c}_{q_{\eta,n},\nu}) \geq n + \ell^*_1 + 2 + 2^{n+m}$.

We let $q^{[\leq h(n+1)]}_{\eta,n+1} = q^{[\leq h(n+1)]}_{\eta,n}$. For each $\nu \in q^{[h(n+1)]}_{\eta,n}$ we change $\boldsymbol{c}_{q_n,\nu}$ into $\boldsymbol{d} =: \boldsymbol{c}_{q_{n+1},\nu}$ as in Lemma 1.6 (with $m = 1$) thus having x^η_n in the domain in each $\zeta \in \operatorname{pos}(\boldsymbol{d})$. We do not loose in the norm, as $\operatorname{nor}^0(\boldsymbol{d}) \geq \operatorname{nor}^0(\boldsymbol{c}_{q_n,\nu})$. The $\rho' \in \operatorname{pos}(\boldsymbol{c}_{q_n,\nu})$ that do not have an extension $\rho = \rho' \cup \{(x_n, \rho(x_n))\}$ in $\operatorname{pos}(\boldsymbol{d})$ are dropped. For the ρ' that stay and extended to ρ we have $\operatorname{pr}_{q_{\eta,n+1},q_{\eta,n}}(\rho' \dot\cup \{(x_n, \rho(x_n))\}) = \rho'$.

Suppose we are at the coordinate of a staying $\rho = \rho' \cup \{(x_n, \rho(x_n))\}$. Then we change $\boldsymbol{c}_{q_{\eta,n},\rho'}$ into $\boldsymbol{c}_{q_{\eta,n+1},\rho}$ as in Lemma 1.7 (at most members of $u_\eta \cup \{x^\eta_0, \dots, x^\eta_{n-1}\}$ are $\lhd_{q_{\eta,n}}$-above x^η_n) now getting ρ as the base of the new creature and having in the positions of the new creature only supersets of ρ. By this we loose at most $\ell^*_1 + n + 1$ off the nor^0, so that in the end $\operatorname{nor}^0(\boldsymbol{c}_{q_{\eta,n+1},\rho}) \geq 2^{n+m}$. Now we repeat this procedure through the $q_{\eta,n}$ tree, thus we get $q_{\eta,n+1} \geq_{n+m} q_{\eta,n}$.

Finally we set $q_\eta = \bigcup_{n<\omega} \bigcup_{h(n)\leq h<h(n+1)} (q_{\eta,n} \restriction q^{[h]}_{\eta,n})$. ⊣

Corollary 2.10. *Forcing with Q specialises $\boldsymbol{T}$.*

Proof. In Lemma 2.9 we can demand arbitrarily high $\alpha < \omega_1$. Let G be generic. Then $s_G = \bigcup\{\operatorname{rt}(p) : p \in G\}$ specialises $\boldsymbol{T}$. ⊣

§3. Continuous reading of names.

In this section we prove that Q has "continuous reading of names", that is, for each m and for each name for an ordinal and for each p there is some $q \geq_m p$ which forces that the evaluation of the name is in a finite set in the ground model. This property implies Axiom A (see [1]) and properness and ω^ω-bounding (for definitions and proofs see [12, Chapter 2.3]).

As a preparation we look for large homogeneous subconditions:

Lemma 3.1. *If $p \in Q$ and for all $\eta \in \operatorname{dom}(p)$, $\operatorname{nor}^1(\boldsymbol{c}_{p,\eta}) \geq 1$ and $X \subseteq \operatorname{dom}(p)$ is upward closed in $\lhd_p$, then there is some q such that*

(a) $\operatorname{dom}(q) \subseteq \operatorname{dom}(p)$ and $p \leq_0 q$, and this is witnessed by $\operatorname{pr}_{q,p} = \operatorname{id} \restriction \operatorname{dom}(q)$,

(b) either $(\exists \ell) q^{[\geq \ell]} \subseteq X$ or $q \cap X = \emptyset$,

(c) for every $\nu \in q$, if $\boldsymbol{c}_{q,\nu} \neq \boldsymbol{c}_{p,\nu}$, then $\operatorname{nor}^1(\boldsymbol{c}_{q,\nu}) \geq \operatorname{nor}^1(\boldsymbol{c}_{p,\nu}) - 1$.

Proof. We will choose $\operatorname{dom}(q) \subseteq \operatorname{dom}(p)$. For each $\ell \in \omega$ we first choose by downward induction on $j \leq \ell$ subsets $X_{\ell,j} \subseteq p^{[\leq \ell]}$ and a colouring $f_{\ell,j}$ of $X_{\ell,j} \cap p^{[j]}$ with two colours, 0 and 1. The choice is performed in such a way that $X_{\ell,j-1} \subseteq X_{\ell,j}$ and such that $p^{[i]} \subseteq X_{\ell,j}$ for $i \leq j$.

We choose $X_{\ell,\ell} = p^{[\leq \ell]}$ and for $v \in p^{[\ell]}$ we set $f_{\ell,\ell}(v) = 0$ iff $(\exists \ell')(p^{\langle v \rangle})^{[\geq \ell']} \subseteq X$ and $f_{\ell,\ell}(v) = 1$ otherwise.

Suppose that $X_{\ell,j}$ and $f_{\ell,j}$ are chosen. For $\eta \in p^{[j-1]} \cap X_{\ell,j}$ we have

$$\begin{aligned}\operatorname{pos}(\boldsymbol{c}_{p,\eta}) =&\{v \in \operatorname{pos}(\boldsymbol{c}_{p,\eta}) \cap p^{[j]} \,:\, f_{\ell,j}(v) = 0\} \cup \\ &\{v \in \operatorname{pos}(\boldsymbol{c}_{p,\eta}) \cap p^{[j]} \,:\, f_{\ell,j}(v) = 1\}.\end{aligned}$$

Note that the sets would be all the same if we intersect with $X_{\ell,j}$, because $p^{[j]} \subseteq X_{\ell,j}$. By Lemma 1.8 at least one of the two sets gives a creature $\boldsymbol{c}$ with $\operatorname{nor}^1(\boldsymbol{c}) \geq \operatorname{nor}^1(\boldsymbol{c}_{p,\eta}) - 1$.

So we keep in $X_{\ell,j-1} \cap p^{[j]}$ only those v of the majority colour and close this set downwards in p. (If both colours give creatures of large norms, the choice is arbitrary.) This is $X_{\ell,j-1}$. We colour the points on $p^{[j-1]} \cap X_{\ell,j-1}$ with $f_{\ell,j-1}$ according to these majority colours, i.e., $f_{\ell,j-1}(\eta) = i$ iff $\{v \in \operatorname{pos}(\boldsymbol{c}_{p,\eta}) \,:\, f_{\ell,j}(v) = i\} \subseteq X_{\ell,j-1}$. We work downwards until we come to the root of p and keep $f_{\ell,0}(\operatorname{rt}(p))$ in our memory.

We repeat the procedure of the downwards induction on $j \leq \ell$ for larger and larger ℓ.

If there is one ℓ where the root got colour 0 after the downwards induction, then since X is upwards closed, $q^{[\geq \ell']} \subseteq X$ for some $\ell' \geq \ell$. If for all ℓ the root got colour 1, we have for all ℓ finite subtrees t such that for all nodes $v \in t$ the thinner tree $p^{\langle v \rangle} \cap t$ has at "least original nor^1 -1" at its root. By König's lemma (initial segments of trees are taking from finitely many possibilities) we build a condition $q \geq_0 p$ such that all of its nodes are not in X, and thus (b) is proved. By Lemma 1.8 the choices in König's lemma can be performed such that also requirement (c) is fulfilled. $\dashv$

Now we want to find a homogeneous $q \geq_m p$, and therefore we have to weaken the homogeneity property in item (b) of Lemma 3.1.

Lemma 3.2. *Suppose p is a condition and $k \in \omega$. There is a front $\{v_0, \dots v_s\}$ of p such that for all $q \geq p$: If $v_i \in \operatorname{dom}(q)$ for $i = 0, \dots, s$, and $q^{\langle v_i \rangle} \geq_k p^{\langle v_i \rangle}$ for $i \leq s$, then $p \leq_k q$.*

Proof. We take a straight front $\{v_0, \dots, v_s\} = p^{[\ell]}$ such that for every $\eta \in p^{[\geq \ell]}$, $\operatorname{nor}^1(\boldsymbol{c}_{p,\eta}) \geq k$, and then we apply Lemma 2.8. Note that since we glue with $q^{\langle v_i \rangle} \geq_k p^{\langle v_i \rangle}$ and $k \geq 0$, the norms of the creatures in the predecessor level of the front $p^{[\ell]}$ do not drop. $\dashv$

Lemma 3.3. *If $p \in Q$, $m \in \omega$, and $X \subseteq \operatorname{dom}(p)$ is upwards closed. Then there is some q such that*

(*a*) *there is a front $\{v_0, \dots v_s\}$ of p as in the previous lemma, such that $\{v_0, \dots v_s\} \subseteq \operatorname{dom}(q)$ and $q^{\langle v_i \rangle} \geq_m p^{\langle v_i \rangle}$ for $i \leq s$,*

(*b*) *for all v_i we have that either $(\exists \ell)(q^{\langle v_i \rangle})^{[\geq \ell]} \subseteq X$ or $q^{\langle v_i \rangle} \cap X = \emptyset$,*

(*c*) *for every $v \in q$, if $\boldsymbol{c}_{q,v} \neq \boldsymbol{c}_{p,v}$, then $\operatorname{nor}^1(\boldsymbol{c}_{q,v}) \geq \operatorname{nor}^1(\boldsymbol{c}_{p,v}) - 1$.*

Proof. By Lemma 2.6 there is some k such that $\forall v \in p^{[\geq k]}$, $\operatorname{nor}^1(\boldsymbol{c}_{p,v}) \geq m + 1$. Item (c) of Lemma 3.1 gives indeed some homogeneous q such that $p \leq_m q$ for $m = \max(0, \min\{\operatorname{nor}^1(\boldsymbol{c}_{p,\eta}) - 1 \,:\, \eta \in p^{[\geq k]}\})$. We use this stronger formulation of Lemma 3.1 for each $p^{\langle v \rangle}$, $v \in p^{[k]}$, and then we use the previous lemma. $\dashv$

Finally we can state a version of continuous reading of names.

THEOREM 3.4. *Suppose that $p \in Q$ is smooth and that $m < \omega$ and that τ is a Q-name of an ordinal. Then there is some smooth $q \in Q$ such that*

(*a*) $p \leq_m q$,

(*b*) *for some $\ell \in \omega$ we have that for every $\eta \in q^{[\ell]}$ the condition $q^{\langle \eta \rangle}$ forces a value to τ.*

PROOF. First we try to build some $q \geq_{m+1} p$ with the desired properties by a fusion argument. Let $\chi > 2^{\aleph_1}$ be a regular cardinal and let H_χ be the set of set with hereditary cardinality $< \chi$. Let $<_\chi$ be a well-order on H_χ. By induction on $n \in \omega$ we choose q_n and countable elementary subsets $N_n \prec (H(\chi), \in, <_\chi)$ such that

(1) $q_0 = p$,

(2) $Q, p, \tau \in N_0$,

(3) $q_n \in N_n$, q_n is smooth, $\alpha(q_{n+1}) \geq N_n \cap \omega_1$,

(4) $N_n \in N_{n+1}$,

(5) $q_{n+1} \geq_{m+1} q_n$, and on each branch of q_{n+1} from some point ν on, $q_{n+1}^{\langle \nu \rangle} \geq_{m+n+1} q_n^{\langle \mathrm{pr}_{q_{n+1}, q_n}(\nu) \rangle}$,

(6) for each $\nu \in q_n^{[h(n)]}$ if there is some $r \geq_{m+1} q_n^{\langle \nu \rangle}$ with property (b) for r and for no $k < n$ there was a $\nu' \in q_k^{[h(k)]}$ with an $r' \geq_{m+1} q_k^{\langle \nu' \rangle}$ and property (b) and $\nu' \lhd_{q_n} \nu$, then we take $q_{n+1}^{\langle \nu \rangle} = r$. If there is no such ν, then we just fill up the domains in order to fulfil $\alpha(q_{n+1}) \geq N_n \cap \omega_1$ and in order to build some $q_{n+1}^{\langle \nu \rangle} \geq_{m+n+1} q_n^{\langle \nu \rangle}$,

(7) $h(n+1) > h(n)$ is such that $(\forall \nu \in q_{n+1}^{[\geq h(n+1)]})(\mathrm{nor}^1(\boldsymbol{c}_{q_{n+1}, \nu}) \geq m + n + 1)$.

We take a fusion $q = \bigcup_{n<\omega} \bigcup_{h(n) \leq h < h(n+1)} (q_{n+1} \restriction q_{n+1}^{[h]})$. Thus $q \in Q$, as there are only finitely many ν as in (6) with a possible drop of the norm to $m + 1$, because otherwise there would be two $\lhd_q$-comparable such ν's. Since $q_n \in N_n$, $\alpha(q_n) < N_n \cap \omega_1$. We let $N = \bigcup N_n$. Then we have $\alpha(q) = \bigcup \alpha(q_n)$. Then $N \cap \omega_1 = \alpha(q)$. If there is at some stage some q_n with a front of ν such that $q_n^{\langle \nu \rangle}$ has property (b), then the proof is finished. Hence assume that not.

We apply Lemma 3.3 with $\leq_m$, q and

$$X = \{\nu' \in q^{[\geq h(0)]} : (\exists \rho \trianglelefteq_q \nu')(\rho \text{ is as in (6) for } q_{n(\rho)} \text{ and } \leq_m)\}$$

above the front $q^{[h(0)]}$. Here, $n(\rho) = \min\{n : \rho \in q_n^{[\leq h(n)]}\}$. It is easy to see that X is upwards closed.

So by Lemma 3.3 we get some $r \geq_m q$, $\mathrm{dom}(r) \subseteq \mathrm{dom}(q)$ such that each $r^{\langle \nu_i \rangle}$ is homogeneous for X. If from the disjunction in (b) of Lemma 3.3 for each $i = 0, \ldots, s$ the first case is true, then $q \restriction X = r$ is the desired object. We assume that there is some $\nu_i = \nu$ such that $r^{\langle \nu \rangle} \cap X = \emptyset$ and we shall derive a contradiction.

We choose $s \geq r^{\langle \nu \rangle}$ such that s forces a value to τ and such that $(\forall \eta \in \mathrm{dom}(s))\, \mathrm{nor}^1(\boldsymbol{c}_{s,\eta}) \geq m$. There is some n such that $\mathrm{rt}(s) \supseteq \mathrm{pr}_{s,q_n}(\mathrm{rt}(s)) = \rho \trianglerighteq_{q_n} \nu$ and $n(\rho) = n$. By strengthening s we can assume that $\rho \in q_n^{[h(n)]}$. So $\mathrm{dom}(\mathrm{rt}(s)) = \mathrm{dom}(\rho) \cup \{x_0, \ldots, x_{\tilde{s}-1}\}$. We assume that $\tilde{s} > 0$, otherwise we can jump with

$s = t$ to the third but last line of this proof. Since $q_n \in N$ and since $x_i \notin N$ there are $\aleph_1$ pairwise different (otherwise they would be in N) x with the same property $(\exists s \geq q_n)(\exists x \in [\omega_1 \setminus \alpha]^{<\omega})(\mathrm{pr}_{s,q_n}(\mathrm{rt}(s)) = \rho \cup \eta_x$, s forces a value to τ, $(\forall \eta \in \mathrm{dom}(s)(\mathrm{nor}^1(\boldsymbol{c}_{s,\eta}) \geq m)$ and $\mathrm{dom}(\eta_x) = x)$. If some Δ-system of some uncountable set of such x's had a non-empty root, the $<_\chi$ minimal such system would be definable with parameters from N and hence its root would be a definable non-empty finite set and be in N, which contradicts $\alpha(q) = N \cap \omega_1$. Hence after further thinning out we have $\aleph_1$ pairwise disjoint x all of cardinality $\tilde{s}$. We enumerate them as x^α, $\alpha < \omega_1$, and let $\bar{x}^\alpha = \langle x_0^\alpha, \dots, x_{\tilde{s}-1}^\alpha \rangle$. We set

$$Y = \{\bar{x}^\alpha \,:\, \alpha < \aleph_1\}.$$

Now we choose $k^* = 2^m + 1$. We thin out Y in $\aleph_1$ times $K = \lceil \log_2(k^*) \rceil$ steps, named by (k, β), $k < K$, $\beta < \aleph_1$, in the following manner: In the first step, step $(0,0)$, we use a fact on Aronszajn trees ([6, Lemma 24.2] or [15, III, 5.4] we find two $\bar{y}^{0,0}, \bar{y}^{0,1} \in X$ such that

$$(\forall \ell_1, \ell_2 < \tilde{s})(y^{0,0}_{\ell_1} \text{ and } y^{0,1}_{\ell_2} \text{ are incompatible in } <_{\boldsymbol{T}}). \tag{3.1}$$

Now in step $(0,1)$ we repeat this procedure with $Y_{0,1} = Y \setminus \{\bar{y}^{0,0}, \bar{y}^{0,1}\}$ and we find another two $\bar{y}^{0,2}, \bar{y}^{0,3} \in Y_{0,1}$ that have the same property from equation (3.1). We set $Y_{0,3} = Y_{0,1} \setminus \{\bar{y}^{0,2}, \bar{y}^{0,3}\}$. So we do $\aleph_1$ times and thus get $\bar{y}^\alpha$, $\alpha < \omega_1$.

From each pair of results we take the concatenations $\bar{y}^{0,\alpha\cdot 2}{}^\frown\bar{y}^{0,\alpha\cdot 2+1} =: \bar{y}^{1,\alpha}$, $\alpha < \aleph_1$, and put these into a set that we call $Y_{1,0}$. Now we repeat the $\aleph_1$ steps we did on Y with $Y_{1,0}$ and thus get $Y_{2,0}$. We iterate these $\aleph_1$ substeps K times. After all the steps we break the found concatenations of $\tilde{s}$-tuples $\bar{y}^{K,\alpha}$, $\alpha < \aleph_1$, into their $\tilde{s}$ long original parts and thus get $\langle y^j_\ell \,:\, \ell < \tilde{s}, j < k^* \rangle$ such that

$$\text{if } j_1 \neq j_2 < k^* \text{ then } (\forall \ell_1, \ell_2 < \tilde{s})(y^{j_1}_{\ell_1} \text{ and } y^{j_2}_{\ell_2} \text{ are incompatible in } <_{\boldsymbol{T}}).$$

For $j < k^*$, we let $\bar{y}^j = \langle y^j_\ell \,:\, \ell < \tilde{s} \rangle$. Now we recall the conditions s^α, such that $\bar{y}^j = \bar{x}^\alpha$ and call them s_j, $j < k^*$. Note that $\mathrm{rt}(s_j) = \rho \cup \{(y^j_\ell, \mathrm{rt}(s_j)(y^j_\ell)) \,:\, \ell < \tilde{s}\}$.

We apply Lemma 1.5 with the following actors: $\boldsymbol{c}$ from the Lemma is $(\rho, \{\mathrm{rt}(s_j) \,:\, j < k^*\})$. Then for every $j < k^*$ we have that $\mathrm{rt}(s_j) = \rho \cup \{(y^j_\ell, \mathrm{rt}(s_j)(y^j_\ell)) \,:\, \ell < \tilde{s}\}$ is a base for the creature $\boldsymbol{c}_{s_j, \mathrm{rt}\, s_j}$, and Lemma 1.5 gives a creature

$$\boldsymbol{d} = \{(\rho, \zeta) \,:\, \zeta \in s_j^{[1]}, j < k^*\}$$

with $\mathrm{nor}^0(\boldsymbol{d}) \geq \min(\{k^* - 1,\} \cup \{\mathrm{nor}^0(\mathrm{rt}(s_j)) \,:\, j < k^*\}) \geq 2^m$, and hence $\mathrm{nor}^1(\boldsymbol{d}) \geq m$.

By Lemma 2.8 we have that the t that raises from grafting the conditions s_j, $j < k^*$, to the node $\rho \in q_n^{\langle \nu \rangle}$ creates a condition $t^{\langle \rho \rangle} \geq_m q_n^{\langle \rho \rangle}$ such that for all $\zeta \in (t^{\langle \rho \rangle})^{[1]}$, $t^{\langle \zeta \rangle}$ forces a value to τ. Hence $\rho \in r^{\langle \nu \rangle} \cap X$ contradicts the assumption $r^{\langle \nu \rangle} \cap X = \emptyset$. $\dashv$

Corollary 3.5. *Q is a proper ω^ω-bounding forcing adding reals that specialises a given Aronszajn tree.*

Proof. For a proof that continuous reading of names implies properness and ω^ω-bounding, see Sections 2.3 and 3.1 in [12].

We show that forcing with Q adds a new real: Let $x_n \in \boldsymbol{T}$, $n \in \omega$, be pairwise different arbitrary nodes of the Aronszajn tree $\boldsymbol{T}$. Let $\underset{\sim}{f}$ be a name for the generic specification function. By Corollary 2.10, $\underset{\sim}{f}$ is defined on the whole $\boldsymbol{T}$. If a condition p determines $\underset{\sim}{f}(x_n)$ then $x_n \in \bigcup\{\mathrm{dom}(\tilde{\eta}) : \mathrm{nor}^0(\boldsymbol{c}_{p,\eta}) = 0\}$ and this set is finite. So for all but finitely many n, there are $q, q' \geq p$ such that q and q' force different values to $\underset{\sim}{f}(x_n)$. Hence by a density argument, $\langle (n, \underset{\sim}{f}(x_n)) : n \in \omega \rangle$ is a new real. ⊣

The preservation theorems for properness and ω^ω-bounding allow us to iterate forcings $Q = Q_{\boldsymbol{T}}$ with countable support. Starting with CH and $2^{\aleph_1} = \aleph_2$ in the ground model, we choose some book-keeping to enumerate all Aronszajn trees $\boldsymbol{T}$, also the ones arising in intermediate steps of a countable support iteration of length $\aleph_2$. Since every Aronszajn tree is there before the stage $\aleph_2$, we thus get a model where all Aronszajn trees are special and $\mathfrak{d} = \aleph_1$ and $2^\omega = \aleph_2$ and the cardinals from the ground model are preserved.

§**4. Application to the weak diamonds $\diamondsuit(\mathfrak{b})$ and $\diamondsuit(\mathfrak{d})$.** Jensen [7] showed that $\diamondsuit$ implies the existence of a Souslin trees. Since then it has been interesting to investigate which weakenings of $\diamondsuit$ still imply the existence of a Souslin tree. Moore, Hrušák and Džamonja [10] introduce and investigate numerous versions of weak diamonds. Let $Non(\mathscr{M})$ denote the relation (F_σ meagre sets, $\omega^\omega, \not\in$), and let $Non(\mathscr{N})$ denote the relation (G_δ null sets, $\omega^\omega, \not\in$). They show that $\diamondsuit(Non(\mathscr{M}))$ implies the existence of a Souslin tree, and from work by Hirschorn [4] they derive that $\diamondsuit(Non(\mathscr{N}))$ does not imply this. Another model (with larger continuum) for $\diamondsuit(Non(\mathscr{N}))$ and no Souslin tree is given by Laver [8]. A relation located in the Cichoń diagramme below $Non(\mathscr{M})$ and incomparable but close to $Non(\mathscr{N})$ is the unbounding relation. Since the Borel Galois-Tukey connections in the Cichoń diagramme translate into implications of the corresponding weak diamonds [10, Proposition 4.9], the weak diamond of the unbounding relation, $\diamondsuit(\mathfrak{b})$, is weaker than $\diamondsuit(Non(\mathscr{M}))$ and weaker than $\diamondsuit(\mathfrak{d})$. In [10] it is asked, whether $\diamondsuit(\mathfrak{b})$ implies the existence of a Souslin tree. We answer this question in a strong negative form, replacing "there is no Souslin tree" by "every Aronszajn tree is special" and $\diamondsuit(\mathfrak{b})$ by $\diamondsuit(\mathfrak{d})$. First we recall the definition of the weak diamond for a relation (A, B, E), abbreviated by $\diamondsuit(A, B, E)$.

Definition 4.1. (Definition 4.3. of [10]) Suppose A is a Borel subset of 2^ω. A map $F \colon 2^{<\omega_1} \to A$ is Borel if for every $\delta \in \omega_1$, the restriction $F \restriction 2^\delta$ is Borel.

Definition 4.2. (Definition 4.4. of [10]) Let $\diamondsuit(A, B, E)$ be the following statement: For every Borel map $F \colon 2^{<\omega_1} \to A$ there is some $g \colon \omega_1 \to B$ such that for every $f \colon \omega_1 \to 2$ the set

$$\{\alpha \in \omega_1 : F(f \restriction \alpha) E g(\alpha)\}$$

is stationary.

We write $\langle A, B, E\rangle$ for the norm of the relation (A, B, E), and, abusing notation, we write $\diamondsuit(\mathfrak{d})$ for $\diamondsuit(\omega^\omega, \omega^\omega, \leq^*)$ and $\diamondsuit(\mathfrak{b})$ for $\diamondsuit(\omega^\omega, \omega^\omega, \not\geq^*)$. In the following we work with continuous functions F or with Borel functions F such that the Borel rank of $F \restriction 2^\alpha$ is $< \alpha$ for stationarily many $\alpha \in \omega_1$.

Theorem 4.3. *$2^{\aleph_0} = \aleph_2$ and $\diamondsuit(\mathfrak{d})$ for continuous F and every Aronszajn tree is special is consistent relative to ZFC.*

Proof. The proof works with the forcing P from the first three sections. Since P is ω^ω-bounding, we have $\mathfrak{d} = \aleph_1$ in V^P. Models for $\mathfrak{d} = \aleph_1 < \mathfrak{c}$ and all Aronszajn trees are special are also given in [4] and [9]. Now we use the Axiom A property and the continuous reading of names of our iterands. This allows us to adapt work by Hrušák [5] on products and iterations of Sacks forcing to the countable support iteration of the iterands $Q_{\boldsymbol{T}}$. $\dashv$

Let P_β denote a countable support iteration of iterands of the form $Q_{\boldsymbol{T}}$ of length α. We have a version of the fusion lemma also for P_β. If $p, q \in P_\beta$, $n \in \omega$ and $F \in [\mathrm{supp}(q)]^{<\omega}$ we will write $q \geq_{F,n} p$ when $q \geq p$ and $(\forall \beta \in F)(q \restriction \beta \Vdash q(\beta) \geq_n q(\beta))$. We have the fusion lemma:

Lemma 4.4. [2] *If (p_i, n_i, F_i) is such that $F_{i+1} \supseteq F_i$ and $\bigcup F_i = \bigcup \mathrm{supp}(p_i)$, and $p_{i+1} \geq_{F_i,n_i} p_i$. Then we define p so that $\mathrm{supp}(p) = \bigcup \mathrm{supp}(p_i)$ and $\forall \beta \in \mathrm{supp}(p)$, $p(\beta)$ is a name for the fusion of $\{p_i(\beta) \,:\, i \in \omega\}$, then $p \in P_\beta$.*

Now we state an iterative version of continuous reading of names. We say that $p \in P_\beta$ is smooth if $(\forall \alpha < \beta)(p \restriction \alpha \Vdash_{P_\alpha}$ "$p(\alpha)$ is smooth").

Lemma 4.5. *Suppose that $p \in P_\beta$ is smooth and that $m < \omega$, $E \in [\mathrm{supp}(p)]^{<\omega}$ and that τ is a P_β-name of an ordinal. Then there is some smooth $q \in P_\beta$ such that*

(a) $p \leq_{E,m} q$,

(b) *for some $\ell \in \omega$ we have that for every $(\eta_\alpha)_{\alpha \in E}$ such that for every $\alpha \in E$, $\eta_\alpha \in q_\alpha^{[\ell]}$, the condition $(q_\alpha^{\langle \eta_\alpha \rangle})_{\alpha \in E}{}^\frown (q_\alpha)_{\alpha \notin E}$ determines the value to τ.*

Proof. By 3.4, all the strengthenings of the conditions in order to force values are determined by thinning out finitely branching trees of at a finite height. So this is like Baumgartner and Laver's analysis of iterated Sacks forcing [2]. $\dashv$

Lemma 4.6. *Assume that $\boldsymbol{V} \models \diamondsuit_S$. There is a sequence $\langle g(\delta) \,:\, \delta \in \lim(\omega_1) \rangle$ of functions $g(\delta) \in \omega^\omega$ (not depending on F) such that: Let $p \in P_{\omega_2}$ be a condition and let $\underset{\sim}{F}$ be a P_{ω_2}-name for a Borel function from $2^{<\omega_1}$ to ω^ω. Let $\underset{\sim}{f}$ be a P_{ω_2}-name for a function from ω_1 to 2. Then P_{ω_2} forces that there is a stationary set S such that for every $\delta \in S$, $g(\delta) \geq^* \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta)$.*

Proof. The following is based on [5]: Given a countable model

$$M \prec (H(\chi), \in, <_\chi)$$

such that $p, P, \underset{\sim}{f}, \underset{\sim}{F}, \underset{\sim}{C} \in M$ and $p \Vdash$ ($\underset{\sim}{F}$ is a Borel function and $\underset{\sim}{f} \in 2^{\omega_1}$). We construct $g(\delta)$ as follows. We collapse p,P,$\underset{\sim}{f}$,$\underset{\sim}{F}$, $\underset{\sim}{C}$, M and name the images p^δ,P^δ,$\underset{\sim}{f}^\delta$,$\underset{\sim}{F}^\delta$, $\underset{\sim}{C}^\delta$, M^δ. We assume that $M^\delta \cap \omega_1 = \delta$. We construct a sequence $\langle q_i, n_i, F_i, \ell_i, m_i \,:\, i \in \omega \rangle$ such that

(1) $F_i \subseteq F_{i+1}$, $\bigcup_{i \in \omega} F_i = \alpha$,

(2) $q_0 \geq p^\delta$,

(3) $q_i \in P_\beta \cap M^\delta$, q_i is smooth,

(4) $q_{i+1} >_{F_i,n_i} q_i$ and ℓ_{i+1}, q_{i+1} and (F_{i+1}, n_{i+1}) are such that such that for every $\bar{\eta} \in ((q_{i+1})_\beta^{[\ell_{i+1}]})_{\beta \in F_i}$ the condition $q_{i+1}^{\langle \bar{\eta} \rangle}$ decides $(F_\alpha(f_\alpha))(i)$ and n_{i+1} is so large that $\leq_{F_{i+1},n_{i+1}}$ freezes level ℓ_{i+1},

(5) for every $\bar{\eta} \in ((q_{i+1})_\beta^{[\ell_{i+1}]})_{\beta \in F_i}$, $q_{i+1}^{\langle \bar{\eta} \rangle} \Vdash (F^\delta(f^\delta))(i) \leq m_i$.

Finally set $g(\delta)(i) = m_i$ and let q^δ be the fusion of the q_i.

We use the $\diamondsuit_S$-sequence $\langle A_\delta : \delta \in S\rangle$ in V in the following manner: By easy integration and coding we have $\langle (N^\delta, \underset{\sim}{f}^\delta, \underset{\sim}{F}^\delta, \underset{\sim}{C}^\delta, P^\delta_{\omega_2}, p^\delta, <^\delta) : \delta \in S\rangle$ such that

(a) N^δ is a transitive collapse of the countable $M \prec H(\chi, \in, <^*_\chi)$, $<^\delta$ is a well-ordering of N^δ.

(b) $N^\delta \models P^\delta_{\omega_2} = \langle P^\delta_\alpha, \underset{\sim}{Q}^\delta_\beta : \alpha \leq \omega_2^{N^\delta}, \beta < \omega_2^{N^\delta}\rangle$ is as in Definition 2.2.

(c) $N^\delta \models (p^\delta \in P^\delta_{\omega_2}, \underset{\sim}{f}^\delta$ is a $P^\delta_{\omega_2}$-name of a member of ${}^{\omega_1}2$ $\underset{\sim}{F}^\delta : 2^{<\omega_1} \to 2^\omega$ is Borel).

(d) If $p \in P_{\omega_2}$,

$$p \Vdash_{P_{\omega_2}} \underset{\sim}{f} \in 2^{\omega_1} \wedge \underset{\sim}{F} : 2^{<\omega_1} \to 2^\omega \text{ is continuous, } \underset{\sim}{C} \subseteq \omega_1 \text{ is club,}$$

and $p, P_{\omega_2}, \underset{\sim}{F}, \underset{\sim}{f}, \underset{\sim}{C} \in H(\chi)$, then

$$\begin{aligned} S(p, \underset{\sim}{F}, \underset{\sim}{f}) := \{\delta \in S :\ & \text{there is a countable } M \prec (H(\chi), \in, <^*_\chi) \\ & \text{such that } \underset{\sim}{f}, \underset{\sim}{F}, \underset{\sim}{C}, P_{\omega_2}, p \in M \\ & \text{and there is an isomorphism } h^\delta \text{ from } N^\delta \text{ onto } M \\ & \text{mapping } P^\delta_{\omega_2} \text{ to } P_{\omega_2}, \underset{\sim}{f}^\delta \text{ to } \underset{\sim}{f}, \\ & \underset{\sim}{F}^\delta \text{ to } \underset{\sim}{F}, \underset{\sim}{C}^\delta \to \underset{\sim}{C}, p^\delta \text{ to } p, <^\delta \text{ to } <^*_\chi \restriction M\} \end{aligned}$$

is a stationary subset of ω_1.

We assume that $p \in G$ and $\underset{\sim}{F}, \underset{\sim}{f}, \underset{\sim}{C}$ are as in (d). Then we construct q^δ and $g(\delta)$.

We assume that G is P_{ω_2}-generic over V and that $p \in G$. $q^\delta \Vdash \delta \in \underset{\sim}{C}^\delta$. We show that g is a diamond function for the dominating relation. Since $\omega \subseteq M, N^\delta$, $h^{\delta\prime\prime}(g(\delta)) = g(\delta)$.

Since P_{ω_2} is proper $S(p, \underset{\sim}{f}, \underset{\sim}{F})$ is also stationary in $V[G]$. Now we have that $q = h^{\delta\prime\prime} q^\delta \geq p$ and for $\delta \in S(p, \underset{\sim}{f}, \underset{\sim}{F}) \cap \underset{\sim}{C}[G]$ we have by the isomorphism property of h^δ that

$$q \Vdash h^{\delta\prime\prime} \underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta) = \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \ \wedge\ \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \leq^* g(\delta) \ \wedge\ \delta \in \underset{\sim}{C}.$$

So we have that p forces that $\{\alpha \in S : F(f \restriction \delta) \leq^* g(\delta)\}$ contains a stationary set, namely a stationary subset of $S(p, \underset{\sim}{f}, \underset{\sim}{F})$. Note that the stationary subset depends on F (and f of course), but the guessing function g does not. So actually we proved a diamond of the kind:

There is some $g : \omega_1 \to \omega^\omega$ such that for every Borel map $F : 2^{<\omega_1} \to \omega^\omega$ and for every $f : \omega_1 \to 2$ the set

$$\{\alpha \in \omega_1 : F(f \restriction \alpha) \leq^* g(\alpha)\}$$

is stationary. $\dashv$

Putting the steps together yields:

THEOREM 4.7. *$\diamondsuit(\mathfrak{d})$ and $2^{\aleph_0} = \aleph_2$ and "all Aronszajn trees are special" is consistent relative to ZFC.*

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UNIVERSITÄT WIEN
KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
WÄHRINGER STR. 25, A-1090 VIENNA, AUSTRIA
E-mail: heike@logic.univie.ac.at