A rigid Boolean algebra that admits the elimination of $Q^2_1$

by

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Abstract. Using ♦, we construct a rigid atomless Boolean algebra that has no uncountable antichain and that admits the elimination of the Malitz quantifier $Q^2_1$.

1. Introduction. Malitz quantifiers are introduced in [Mag-Mal]. Let us recall the semantics of $Q^n_\alpha$, $n \geq 1$, $\alpha \in \text{ORD}$: $\mathfrak{A} \models Q^n_\alpha \phi(\vec{a}, \vec{x})$ iff there is a subset $H$ of $\mathfrak{A}$ such that $\text{card}(H) \geq \aleph_\alpha$ and $\mathfrak{A} \models \phi(\vec{a}, h)$ for all pairwise different $h_0, h_1, \ldots, h_{n-1} \in H$. Such a set $H$ is called a homogeneous set for $\phi(\vec{a}, \vec{x})$. Baldwin and Kueker [Bal-Ku], Rothmaler and Tuschik [Ro-Tu], Bürger [Bü] and Koepke [Ko] consider the question of elimination of some of these quantifiers in certain theories or structures. [Ro-Tu] shows that any saturated model allows the elimination of all $Q^n_\alpha$, $\alpha \in \text{ORD}$, $n \geq 1$.

Saturated models with two elements of the same type are not rigid. On the other hand, there are $\mathcal{L}_{\omega \omega}(Q^2_1)$-sentences $\phi$ that have only rigid models and that are satisfiable under CH (see [Ot], [Mil]). We consider

$$\phi := \text{“the structure is a Boolean algebra with } 0 \neq 1\text{” \land } \forall x (x \neq 0 \rightarrow Q_1 y y \subseteq x) \land \neg Q^2_1 x y x \not\subseteq y.$$ 

[Ba-Ko, Theorem 5(a)] shows that all models of $\phi$ are rigid. The search for a model of $\phi$ that contains two different elements of the same $\mathcal{L}_{\omega \omega}(Q^2_1)$-type leads, under ♦, to a model of $\phi$ that admits the elimination of $Q^2_1$ and in which therefore any two elements $\neq 0, 1$ have the same $\mathcal{L}_{\omega \omega}(Q^2_1)$-type.

In ZFC + ♦ and even in ZFC + CH there are various constructions of uncountable Boolean algebras with no uncountable antichains and with some other algebraic properties (see [Ba-Ko], [Sh], [Ru], but also [Ba]). In the course of showing that additional tasks may be fulfilled along the way given in [Ba-Ko], we get a partition of all formulas $\phi(z, x, y) \in \mathcal{L}_{\omega \omega}(Q^2_1)$, $r \in \omega$, into two classes $\Phi_1$ and $\Phi_2$ such that
1. The methods of [Ba-Ko] are applicable to any \( \phi(z, x, y) \in \Phi_1 \). They will allow us to show that the homogeneous sets for any \( \phi(z, x, y) \in \Phi_1 \) will grow only during countably many steps in the chain which we build in the next section.

2. For any Boolean algebra \( \mathfrak{A} \) with \( \mathfrak{A} \models \forall x \neq 0 Q_1 y y \subseteq x \) and any \( \phi(z, x, y) \in \Phi_2: \mathfrak{A} \models \exists z Q_1^2 x y \phi(z, x, y) \).

"\( \phi(z, x, y) \in \Phi_1 \)" will be shown to be equivalent under the first order theory of atomless Boolean algebras to a first order formula with its free variables among \( z_0, z_1, \ldots, z_{r-1} \). The consideration of the possible quantifier-free types of \( z \) leads to a procedure for eliminating \( Q_1^2 \).

2. The construction

*Notation.* We will use \( \mathfrak{A}, \mathfrak{B}, \mathfrak{B}_\alpha \) to denote Boolean algebras. Boolean algebras are considered as \( \tau_{BA} \)-structures with \( \tau_{BA} = \{ \cap, \cup, -, 0, 1 \} \). \( x \subseteq y \) is written for \( x \cap y = x \subseteq \cap \) means strict inclusion, \( x \setminus y \) is used for \( x \cap (-y) \). \( \mathcal{P}(\omega) \) denotes the powerset algebra of \( \omega \). For \( \mathfrak{A} \subseteq \mathcal{P}(\omega) \) we often write \( A \) for \( \mathfrak{A} \). The interpretations of the \( \tau_{BA} \)-symbols in \( \mathcal{P}(\omega) \) are denoted by the symbols themselves.

\( a, b \in A \) are **comparable** (in \( \mathfrak{A} \)) iff \( a \subseteq^\mathfrak{A} b \) or \( b \subseteq^\mathfrak{A} a \). \( C \subseteq \mathfrak{A} \) is a **chain** (an antichain) iff any two distinct elements of \( C \) are comparable (not comparable). For \( a \subseteq^\mathfrak{A} b \in A \) let \( (a, b)_A := \{ c \in A \mid a \subseteq^\mathfrak{A} c \subseteq^\mathfrak{A} b \} \).

Using \( \phi \), we shall construct a Boolean algebra \( \mathfrak{B} \) such that \( \mathfrak{B} \) is a model of the sentence \( \phi \) from the introduction and \( \mathfrak{B} \) admits the elimination of \( Q_1^2 \). As the construction of our Boolean algebra \( \mathfrak{B} \) follows the pattern of [Ba-Ko], we restrict ourselves to a short description, heavily referring to [Ba-Ko].

Inductively on \( \alpha \in \omega_1 \), we shall build a chain \( (\mathfrak{B}_\alpha, M_\alpha)_{\alpha \in \omega_1} \), where the \( \mathfrak{B}_\alpha \) are countable atomless subalgebras of \( \mathcal{P}(\omega) \) and each \( M_{\alpha+1} \) is a countable collection of pairs \( (M, \phi(\bar{t}, x, y)) \), where \( M \subseteq B_\alpha \) and \( \phi(\bar{t}, x, y) \) is a quantifierfree (qf) \( L_{\omega_1}[\tau_{BA}] \)-formula with a property that will be defined later on, and \( \bar{t} \) are elements of \( B_\alpha \). At limit steps we take unions. \( \mathfrak{B}_{\alpha+1} \) will be the Boolean algebra that is generated by \( B_\alpha \cup \{ x_\alpha \} \) in \( \mathcal{P}(\omega) \), where the \( x_\alpha \) is chosen by the same forcing \( P(B_\alpha) \) as in [Ba-Ko], namely: \( P(B_\alpha) = \{(a, b)_{B_\alpha} \mid a \subset b \in B_\alpha \}, (a', b')_{B_\alpha} \leq^{P(B_\alpha)} (a, b)_{B_\alpha} \) iff \( a \subseteq a' \subset b' \subseteq b \).

We shall define \( D_A(M, \phi(\bar{t}, x, y), e, f) \) and \( M_{\alpha+1} \). Then we take a \( \{ D_A(M, \phi(\bar{t}, x, y), e, f) \mid e, f \in B_\alpha, (M, \phi(\bar{t}, x, y)) \in M_{\alpha+1} \} \)-generic subset \( \{ (a_n, b_n) \mid n \in \omega \} \) of \( P(B_\alpha) \) such that \( \{ (a_n, b_n) \mid n \in \omega \} \) additionally satisfies the properties described in [Ba-Ko] and set \( x_\alpha = \bigcup \{ a_n \mid n \in \omega \} \). In [Ba-Ko], \( M_{\alpha+1} \) is chosen so that chains and antichains are countable. Our \( M_{\alpha+1} \) differs from that of [Ba-Ko], because we also want all homogeneous sets for
any $\phi(z, x, y) \in \Phi_1$ to be countable. The next items are the generalizations of the corresponding points of [Ba-Ko].

**Definition 2.1.** Let $A \subseteq \mathcal{P}(\omega)$ and $\bar{c}, e, f \in A$. Let $\phi(\bar{z}, x, y)$ be qf.

(i) $D_A(M, \phi(\bar{c}, x, y), e, f) := \{(a, b)_A \in \mathcal{P}(A) \mid \text{for any } u \in (a, b)_{\mathcal{P}(\omega)} \text{ one of the following points is true:}

1. $(u \cap e) \cup (f \setminus u) \in M$.
2. There is some $y \in M$ such that
   $\mathcal{P}(\omega) \vdash \neg \phi(\bar{c}, (u \cap e) \cup (f \setminus u), y) \lor \neg \phi(\bar{c}, y, (u \cap e) \cup (f \setminus u))$.}

(ii) $M$ is called **maximally homogeneous for $\phi(\bar{c}, x, y)$** in $\mathcal{A}$ iff $M \subseteq A$ is homogeneous for $\phi(\bar{c}, x, y)$ and for all $a \in A \setminus M$ there is some $b \in M$ such that $\mathcal{A} \vdash \neg \phi(\bar{c}, a, b) \lor \neg \phi(\bar{c}, b, a)$.

(iii) $\phi(\bar{c}, x, y)$ is **small** in $\mathcal{A}$ iff for any $\emptyset \neq M \subseteq A$ that is maximally homogeneous for $\phi(\bar{c}, x, y)$ in $\mathcal{A}$, $D_A(M, \phi(\bar{c}, x, y), e, f)$ is dense in $P(A)$ for any $e, f$ in $A$.

**Proof.** [Ba-Ko, Lemmas 2.3 and 2.4].

Also the proof of the next lemma can be carried out as in [Ba-Ko]: just take a $u$ for $\mathcal{A}$ and $\mathcal{M}$ in the same way as they take $x_\alpha$ for $\mathcal{B}_\alpha$ and $M_{\alpha+1}$.

**Lemma 2.3.** Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be atomless and countable and let $\mathcal{M}$ be a countable subset of

$$\{(M, \phi(\bar{c}, x, y)) \mid \bar{c} \in A^{<\omega}, \phi(\bar{c}, x, y) \in \mathcal{L}_{\omega}^{\mathcal{M}_{\mathcal{B}_A}} \text{ qf, } \phi(\bar{c}, x, y) \text{ small in } A \text{ and } M \text{ is maximally homogeneous for } \phi(\bar{c}, x, y) \text{ in } A\}.$$

Then for any $(a, b)_A \in \mathcal{P}(A)$ there is a $u \in (a, b)_{\mathcal{P}(\omega)}$ such that:

1. $u \not\in A$.
2. $[A \cup \{u\}]_{\mathcal{P}(\omega)}$, the subalgebra generated by $A \cup \{u\}$ in $\mathcal{P}(\omega)$, is atomless.
3. For any $(M, \phi(\bar{c}, x, y)) \in \mathcal{M}$ the set $M$ is maximally homogeneous for $\phi(\bar{c}, x, y)$ also in $[A \cup \{u\}]_{\mathcal{P}(\omega)}$.

Now using Lemma 2.3 and $\Diamond$, we can construct our $\mathcal{B}$. Let $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ be a $\Diamond$-sequence. Let $\langle a_\xi \mid \xi \in \omega_1 \rangle$ be an enumeration of $\mathcal{P}(\omega)$ in which each element of $\mathcal{P}(\omega)$ appears $\omega_1$ times.

In step $\alpha + 1$, let $M_{\alpha+1} = M_\alpha \cup \{(a_\xi \mid \xi \in S_\alpha), \phi(\bar{c}, x, y)\} \setminus \{a_\xi \mid \xi \in S_\alpha\}$ is a maximally homogeneous set for $\phi(\bar{c}, x, y)$ in $\mathcal{B}_\alpha$ and $\phi(\bar{c}, x, y)$ is small in $\mathcal{B}_\alpha$ and $\bar{c} \in B_\alpha$. Apply Lemma 2.3 with $\mathcal{A} = \mathcal{B}_\alpha$ and $\mathcal{M} = M_{\alpha+1}$ to get an $x_\alpha$. Define $B_{\alpha+1}$ as $[B_\alpha \cup \{x_\alpha\}]_{\mathcal{P}(\omega)}$. Let $\mathcal{B} = \bigcup \{\mathcal{B}_\alpha \mid \alpha \in \omega_1\}$. Take the $x_\alpha$ so that $\mathcal{B} \models \forall x (x \neq 0 \rightarrow Q_1y y \subseteq x)$. Then it is easy to see that for any $\phi(\bar{c}, x, y)$ which is small in every $\mathcal{B}_\alpha$ with $\bar{c} \in B_\alpha$, we have
3. Large homogeneous sets. The aim of this section is to define a mapping
\[
\text{big} : \bigcup_{r \in \omega} \mathcal{L}_{\omega \omega}[\tau_{BA}](\xi, x, y) \rightarrow \bigcup_{r \in \omega} \mathcal{L}_{\omega \omega}[\tau_{BA}](\zeta),
\]
where \(\phi(\xi, x, y) \mapsto \text{big}(\phi(\xi, x, y))(\zeta),\)
such that for every \(\phi(\xi, x, y) \in \mathcal{L}_{\omega \omega}[\tau_{BA}]\)
\[(*) \quad \mathcal{B} \models \forall \zeta \left( Q^2_1 x y \phi(\xi, x, y) \leftrightarrow \text{big}(\phi(\xi, x, y))(\zeta) \right).
\]
Then \(\Phi_2\) will be
\[
\{ \phi(\zeta, x, y) \mid \text{big}(\phi(\zeta, x, y))(\zeta) \text{ is valid in any atomless Boolean algebra} \}.
\]
In order to simplify the notation we tacitly assume that always the variables \(x\) and \(y\) are intended to be quantified by \(Q^2_1\).

Let \(\mathfrak{A}\) be any atomless Boolean algebra. Since \(\mathfrak{A}\) admits the elimination of \(\exists\) it is enough to define \(\text{big}\) for quantifierfree \(\phi(\zeta, x, y) \in \mathcal{L}_{\omega \omega}[\tau_{BA}].\)

For any \(\bar{c} \in A\) and qf \(\phi(\bar{c}, x, y)\) there is a qf \(\psi(\bar{c}', x, y)\) such that \(\bar{c}'\)
is an (injective) enumeration of the atoms of the subalgebra generated by \(\bar{c},\) and \(\mathfrak{A} \models \forall x y (\psi(\bar{c}', x, y) \leftrightarrow \phi(\bar{c}, x, y)).\) Also if \(\phi(\zeta, x, y)\) is a disjunction \(\bigvee_i (\phi(\zeta, x, y) \land \psi_i(\zeta))\) then knowing \(\chi_i = \text{big}(\phi(\zeta, x, y) \land \psi_i(\zeta))(\zeta)\) we can define \(\text{big}(\phi(\zeta, x, y))(\zeta)\) to be \(\bigvee_i \chi_i.\) Hence it suffices to define \(\text{big}(\phi(\zeta, x, y))(\zeta)\) only for those qf \(\phi(\zeta, x, y)\) that imply that \(\{z_0, \ldots, z_{r-1}\}\) is the set of atoms in the subalgebra generated by \(\{z_0, \ldots, z_{r-1}\}\).

If \(H\) is an uncountable homogeneous set for \(\phi(\bar{c}, x, y)\), then there is an \(\mathcal{L}_{\omega \omega}-1\)-type \(t(\bar{c}, x)\) over \(\bar{c}\) and an uncountable \(H_1 \subseteq H\) such that every element of \(H_1\) has the \(\mathcal{L}_{\omega \omega}-1\)-type \(\text{tp}(x/\bar{c}) = t(\bar{c}, x)\) over \(\bar{c}\). Hence it is enough to define big for the \(\phi(\xi, x, y)\) with the above mentioned property and the additional property that there is an \(\mathcal{L}_{\omega \omega}-1\)-type \(t(\zeta, x)\) over \(\zeta\) (independent of the assignment \(\bar{c}\) of \(\bar{c}\), because we consider only \(\bar{c}\) that are atoms in the subalgebra generated by \(\bar{c}\)) such that
\[
\mathfrak{A} \models \forall x y \zeta \left( \phi(\zeta, x, y) \leftrightarrow (\phi(\xi, x, y) \land t(\zeta, x) = \text{tp}(x/\zeta) \land t(\zeta, y) = \text{tp}(y/\zeta)) \right).
\]
We will call such formulas special. Finally, note that any \(\mathcal{L}_{\omega \omega}-2\)-type \(t(\bar{c}, x, y)\) over \(\bar{c}\) is determined by the corresponding \(r\)-tuple of the quantifierfree types of \(x \cap c_i, y \cap c_i\) in \(\{a \in A \mid a \subseteq c_i\}, i < r\). For any such type there are 15 possibilities, and under the condition \(\text{tp}(x/\zeta) = \text{tp}(y/\zeta)\) there remain the 9 possibilities not marked with an \(\bullet\) in the table below.

The possibilities for the quantifierfree types of \( x \cap c_i, y \cap c_i, i < r \), in \( \{ a \in A \mid a \subseteq c_i \} \)

<table>
<thead>
<tr>
<th>No.</th>
<th>( x \cap y \cap z_i )</th>
<th>( (\neg x) \cap (\neg y) \cap z_i )</th>
<th>( x \cap (\neg y) \cap z_i )</th>
<th>( (\neg x) \cap y \cap z_i )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( x \cap z_i = y \cap z_i \neq 0, z_i )</td>
</tr>
<tr>
<td>3</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>( x \cap z_i = y \cap z_i \neq 0, z_i )</td>
</tr>
<tr>
<td>4</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>( x \cap z_i = y \cap z_i = z_i )</td>
</tr>
<tr>
<td>6</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>( y \cap z_i = z_i )</td>
</tr>
<tr>
<td>7</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( x \cap z_i = y \cap z_i = z_i )</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
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</tr>
<tr>
<td>9</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( x \cap z_i \neq 0, y \cap z_i = z_i )</td>
</tr>
<tr>
<td>10</td>
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<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( x \cap z_i = 0, y \cap z_i = 0 )</td>
</tr>
<tr>
<td>11</td>
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<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( x \cap z_i = y \cap z_i = 0 )</td>
</tr>
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<td>12</td>
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<td>( \neq 0 )</td>
<td>( x \cap z_i \neq 0, y \cap z_i = (-x) \cap z_i )</td>
</tr>
<tr>
<td>13</td>
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<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( x \cap z_i = z_i, y \cap z_i = 0 )</td>
</tr>
<tr>
<td>14</td>
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<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( x \cap z_i = 0, y \cap z_i = z_i )</td>
</tr>
</tbody>
</table>

Let \( \phi^k(z_i, x \cap z_i, y \cap z_i) \) say “the \( \mathcal{L}_{\omega\omega} \)-type of \( x \cap c_i, y \cap c_i \) over \( c_i \) has number \( k \), \( k = 0, \ldots, 14 \). The disjunction \( \phi^{012}(u, v, w) := \phi^0(u, v, w) \lor \phi^1(u, v, w) \lor \phi^2(u, v, w) \) will play an important role in the following.

**Definition 3.1.** Let \( \phi(z, x, y) \in \mathcal{L}_{\omega\omega}[\tau_{BA}] \) be quantifierfree and be of the special form as described above.

\[
\text{big}(\phi(z, x, y))(z) = \exists a \in b \forall xy \left( (a \subseteq x, y \subseteq b \land \bigwedge_{i<r} ((b \setminus a) \cap z_i \neq 0 \rightarrow \phi^{012}(z_i, x \cap z_i, y \cap z_i)) \right) \\
\rightarrow \phi^r(z, x, y).
\]

Equivalent to \( \text{big}(\phi(z, x, y))(z) \) is the formula

\[
\bigvee_{I_0 \cup I_1 \cup I_2 \cup I_3 = \{0, \ldots, r-1\}, I_0 \neq 0} \forall xy \left( \bigwedge_{i \in I_0} \phi^{012}(z_i, x \cap z_i, y \cap z_i) \land \bigwedge_{i \in I_1} x \cap z_i = y \cap z_i \neq 0, z_i \right)
\]
(\cup denotes the disjoint union) which will be useful for the easy direction of (*):

**Lemma 3.2.** Let \( \mathfrak{A} \) be an atomless Boolean algebra. Let \( \mathfrak{A} \models \forall x \neq 0 Q_1 y \subseteq x \), and \( \phi(\overline{z}, x, y) \) be as above. Then \( \mathfrak{A} \models \forall \overline{z} (\big(\phi(\overline{z}, x, y)) \rightarrow Q_2 x y \phi(\overline{z}, x, y)) \).

**Proof.** Let \( \mathfrak{A} \models \big(\phi(\overline{z}, x, y)) \). For \( i \in I_0 \) take an uncountable set \( H_i \subseteq (0, c_i)_{\mathfrak{A}} \) such that for any \( x \in H_i \) the relative complement \( c_i \setminus x \notin H_i \). Let \( \langle h_{i, \alpha} \mid \alpha \in \omega_1 \rangle \) be an injective enumeration of a subset of \( H_i \). Finally, for \( i \in I_1 \) let \( H_i = \{d_i\} \) for some \( 0 < d_i \leq c_i \), for \( i \in I_2 \) let \( H_i = \{0\} \), and for \( i \in I_3 \) let \( H_i = \{c_i\} \). Then

\[
H := \left\{ \bigcup \{h_{i, \alpha} \mid i \in I_0\} \cup \bigcup \{d_i \mid i \in I_1\} \cup \bigcup \{c_i \mid i \in I_3\} \mid \alpha \in \omega_1 \right\}
\]

is an uncountable homogeneous set for \( \phi(\overline{c}, x, y) \).

Now for \( \mathfrak{B} \) as in Section 2, we shall prove the other direction of (*). By the construction, it would suffice to show:

\[
(**) \quad \text{For any enumeration } \overline{c} \text{ of the atoms in the subalgebra of } \mathfrak{B} \text{ generated by } \overline{z}, \text{ if } \mathfrak{B} \models \neg \big(\phi(\overline{z}, x, y)) \text{, then } \phi(\overline{c}, x, y) \text{ is small in every } \mathfrak{B}_\alpha \text{ with } \overline{c} \in B_\alpha.
\]

Unfortunately, this is true only for \( \phi(\overline{c}, x, y) \) that do not forbid certain equalities of Boolean terms. We introduce some notation and then give a sketch of our proof of the hard direction of (*).

We say briefly “\( \phi(\overline{z}, x, y) \) is valid” or just “\( \phi \)” for “\( \phi(\overline{z}, x, y) \) is valid in all atomless Boolean algebras if the assignment of \( \overline{z} \) is an enumeration of the atoms in the subalgebra generated by \( \overline{z} \).” \( \phi(\overline{z}, x, y) \) is satisfiable or consistent if \( \neg \phi(\overline{z}, x, y) \) is not valid.

For a given special \( \phi(\overline{z}, x, y) \) set

\[
R(\phi) := \{ i < r \mid \phi \rightarrow x \cap z_i = y \cap z_i \text{ is not valid} \}.
\]

We will define two mappings \( s \) and \( \text{enl} \) from the set of all special \( \phi(\overline{z}, x, y) \) into itself. The mapping \( s \) is a technical means used to prove \( \text{enl}(\text{enl}(s(\phi))) \rightarrow \text{enl}(s(\phi)) \) (Lemma 3.7) and \( \neg \big(\phi(\overline{z}, x, y)) \rightarrow \neg \big(\phi(\overline{z}, x, y)) \) (Lemma 3.8). Lemma 3.9 says that \( (***) \) is true for formulas of the form \( \text{enl}(s(\phi)) \) for some
special \( \phi \). Hence we get from the construction and from 3.8

\[
\mathfrak{B} \models \neg \text{big}(s(\phi))(\xi) \rightarrow \neg Q_1^2 xy \ \text{enl}(s(\phi))(\xi, x, y),
\]

whence \( s(\phi) \rightarrow \text{enl}(s(\phi)) \) and the monotonicity of the quantifier \( Q_1^2 \) imply

\[
\mathfrak{B} \models \neg \text{big}(s(\phi))(\xi) \rightarrow \neg Q_1^2 xy \ s(\phi)(\xi, x, y)
\]

(Theorem 3.10). Using this result we prove by induction on \( \text{card}(R(\phi)) \), simultaneously for all special formulas \( \phi \),

\[
\mathfrak{B} \models \neg \text{big}(\phi)(\xi) \rightarrow \neg Q_1^2 xy \ \phi(\xi, x, y),
\]

which will finish the proof of \((*)\).

In order to simplify the notation, we often suppress the free variables \((z, x, y)\) or \((z_i, x \cap z_i, y \cap z_i)\).

**Definition 3.3** (The mapping \( s \)). For \( R \subseteq r = \{0, 1, \ldots, r - 1\} \) and for \( \chi(z_i, x \cap z_i, y \cap z_i) \in L_{\omega w}[\tau_{BA}] \) we define

\[
s_R(\chi(z_i, x \cap z_i, y \cap z_i)) := \begin{cases} 
\chi(z_i, x \cap z_i, y \cap z_i) & \text{if } i \notin R \text{ or } \\
\phi^{012}(z_i, x \cap z_i, y \cap z_i) \rightarrow \chi(z_i, x \cap z_i, y \cap z_i) & \text{is valid; } \\
\chi(z_i, x \cap z_i, y \cap z_i) \land x \cap z_i \neq y \cap z_i & \text{else.}
\end{cases}
\]

Let \( S = \{ \Lambda_{i < r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \mid w \in W \} \) be a finite set such that for all \( w \in W \) the conjunction \( \Lambda_{i < r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \) is satisfiable and \( \Lambda_{i < r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \rightarrow \phi(\xi, x, y) \) is valid, and such that for any satisfiable conjunction \( \delta = \Lambda_{i < r} \chi_i'(z_i, x \cap z_i, y \cap z_i) \) such that \( \delta \rightarrow \phi(\xi, x, y) \) is valid there is a \( w \in W \) with \( \Lambda_{i < r} \chi_i'(z_i, x \cap z_i, y \cap z_i) \rightarrow \Lambda_{i < r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \). We will call such a set \( S \) a set of representatives for \( \phi \). Given such a set, let \( R = R(\phi) \) and define

\[
s(\phi(\xi, x, y)) = \bigvee_{w \in W} \bigwedge_{i < r} s_R(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i)) \).
\]

If \( \models \neg \exists xy \phi(\xi, x, y) \), then let \( s(\phi(\xi, x, y)) \) be any inconsistent formula.

A brief reflection shows that \( s(\phi) \) is well defined up to logical equivalence: Let \( S' = \{ \Lambda_{i < r} \chi'_{w,i}(z_i, x \cap z_i, y \cap z_i) \mid w' \in W' \} \) be another set of representatives for \( \phi \).

For \( \bigvee_{w' \in W'} \bigwedge_{i < r} s_R(\chi'_{w',i}) \rightarrow \bigvee_{w \in W} \bigwedge_{i < r} s_R(\chi_{w,i}) \), it suffices to show that for each \( w' \in W' \) there is some \( w \in W \) such that \( \bigwedge_{i < r} s_R(\chi'_{w',i}) \rightarrow \bigwedge_{i < r} s_R(\chi_{w,i}) \). Let \( w' \in W' \) be given. Since \( S \) is a set of representatives for \( \phi \) there is a \( w \in W \) such that \( \bigwedge_{i < r} \chi'_{w',i} \rightarrow \bigwedge_{i < r} \chi_{w,i} \), which is equivalent to
\( \chi'_{w,i} \rightarrow \chi_{w,i} \) for \( i < r \). Immediately from the definition of \( s_R \), if \( \chi'_{w,i} \rightarrow \chi_{w,i} \), then \( s_R(\chi'_{w,i}) \rightarrow s_R(\chi_{w,i}) \). Hence \( \bigwedge_{i<r} s_R(\chi'_{w,i}) \rightarrow \bigwedge_{i<r} s_R(\chi_{w,i}) \).

The other direction follows by symmetry.

Remark. \( s(\phi) \) may be unsatisfiable, e.g. for \( \phi = (x \cap z_0 = y \cap z_0 \land x \cap z_1 \subseteq y \cap z_1) \lor (x \cap z_0 \subseteq y \cap z_0 \land x \cap z_1 = y \cap z_1) \land \bigwedge_{i=0,1} x \cap z_i \neq z_i, 0 \land \bigwedge_{i=0,1} y \cap z_i \neq z_i, 0 \land z_0 \cap z_1 = 0 \land z_0 \cup z_1 = 1 \).

Definition 3.4 (The mapping \( \text{enl} \)). For \( \chi(z_i, x \cap z_i, y \cap z_i) \in \mathcal{L}_{\omega}[\tau_{BA}] \) we define

\[
\text{enl}(\chi(z_i, x \cap z_i, y \cap z_i)) := \begin{cases} \\
\chi(z_i, x \cap z_i, y \cap z_i) \\
\lor (x \cap z_i = (-y) \cap z_i \land \exists x \chi(z_i, x \cap z_i, y \cap z_i) \\
\land \exists y \chi(z_i, x \cap z_i, y \cap z_i) \\
\text{if } \phi^{\text{enl}}(z_i, x \cap z_i, y \cap z_i) \rightarrow \chi(z_i, x \cap z_i, y \cap z_i) \text{ is not valid; } \\\n\chi(z_i, x \cap z_i, y \cap z_i) \lor ((x \cap z_i = (-y) \cap z_i) \\
\lor x \cap z_i = y \cap z_i \land \exists x \chi(z_i, x \cap z_i, y \cap z_i) \\
\land \exists y \chi(z_i, x \cap z_i, y \cap z_i) \text{ otherwise.}
\end{cases}
\]

Let \( \{ \bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \mid w \in W \} \) be a set of representatives for \( \phi \). Then set

\[
\text{enl}(\phi^R(z, x, y)) = \bigvee_{w \in W \mid i<r} \text{enl}(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i)).
\]

If \( \models \exists x y \phi^R(z, x, y) \), then let \( \text{enl}(\phi^R(z, x, y)) \) be any inconsistent formula.

From the fact that \( \chi'_{w,i} \rightarrow \chi_{w,i} \) implies \( \text{enl}(\chi'_{w,i}) \rightarrow \text{enl}(\chi_{w,i}) \), we conclude by an analogous consideration as above that \( \text{enl}(\phi) \) is well-defined.

In order to apply Lemmas 2.2 and 2.3 we may replace \( \text{enl}(\phi^R(z, x, y)) \) by an equivalent (with respect to the theory of atomless Boolean algebras) qf formula.

The next two lemmas collect some properties of \( s \) and \( \text{enl} \) that will be useful in the proofs of 3.7 and of 3.8.

Lemma 3.5. Let \( \chi_s(z_i, x \cap z_i, y \cap z_i), s = 0, 1 \), be qf and \( R \subseteq r \).

(i) \( (\text{enl}(\chi_0) \lor \text{enl}(\chi_1)) \rightarrow \text{enl}(\chi_0 \lor \chi_1) \).
(ii) \( (s_R(\chi_0) \lor s_R(\chi_1)) \rightarrow s_R(\chi_0 \lor \chi_1) \).

For (iii), (iv) and (v), assume additionally that \( \chi_s(z_i, x \cap z_i, y \cap z_i), s = 0, 1 \), determine the same 1-type \( t(z_i, x \cap z_i) \) of \( x \cap z_i \) over \( z_i \) and of \( y \cap z_i \) over \( z_i \).
(iii) Assume that, for $s = 0, 1$, if not $t(z, x \cap z_i, y \cap z_i) \rightarrow \chi_s(z, x \cap z_i, y \cap z_i)$, then $\chi_s(z, x \cap z_i, y \cap z_i) \rightarrow x \cap z_i \neq y \cap z_i$. Then $(\text{enl}(\chi_0) \land \text{enl}(\chi_1)) \rightarrow \text{enl}(\chi_0 \land \chi_1)$.

(iv) $(s_R(\chi_0) \land s_R(\chi_1)) \rightarrow s_R(\chi_0 \land \chi_1)$.

(v) Assume that $\chi_s \rightarrow x \cap z_i = y \cap z_i$ for $s = 0, 1$ if $i \notin R$. Then for any $i < r$ the formula

$$(\text{enl}(s_R(\chi_0))(z, x \cap z_i, y \cap z_i) \land \text{enl}(s_R(\chi_1))(z, x \cap z_i, y \cap z_i))$$

is valid.

Proof. (i), (ii) $\chi_s \rightarrow \chi_0 \lor \chi_1$ implies $\text{enl}(\chi_s) \rightarrow \text{enl}(\chi_0 \lor \chi_1)$ and $s_R(\chi_s) \rightarrow s_R(\chi_0 \lor \chi_1)$.

(iii) Define

$$\phi_\varepsilon(z, x \cap z_i, y \cap z_i) := x \cap z_i = y \cap z_i \land t(z, x \cap z_i)$$

and

$$\phi_\varepsilon(z, x \cap z_i, y \cap z_i) := x \cap z_i = (-y) \cap z_i \land t(z, x \cap z_i).$$

Case 1: $\phi^{012} \rightarrow \chi_s$ for $s = 0, 1$. Then $\phi^{012} \rightarrow \chi_0 \lor \chi_1$ and $\text{enl}(\chi_0) \land \text{enl}(\chi_1) = (\chi_0 \lor \phi_{\varepsilon} \lor \phi_{\varepsilon}) \land (\chi_1 \lor \phi_{\varepsilon} \lor \phi_{\varepsilon}) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_{\varepsilon} \lor \phi_{\varepsilon} = \text{enl}(\chi_0 \land \chi_1)$.

Case 2: Not $\phi^{012} \rightarrow \chi_s$ for $s = 0, 1$. Then not $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $\text{enl}(\chi_0) \land \text{enl}(\chi_1) = (\chi_0 \lor \phi_{\varepsilon} \lor \phi_{\varepsilon}) \land (\chi_1 \lor \phi_{\varepsilon} \lor \phi_{\varepsilon}) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_{\varepsilon} \lor \phi_{\varepsilon} = \text{enl}(\chi_0 \land \chi_1)$.

Case 3: $\phi^{012} \rightarrow \chi_0$ and not $\phi^{012} \rightarrow \chi_1$. Then not $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $\text{enl}(\chi_0) \land \text{enl}(\chi_1) = (\chi_0 \lor \phi_{\varepsilon} \lor \phi_{\varepsilon}) \land (\chi_1 \lor \phi_{\varepsilon} \lor \phi_{\varepsilon}) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_{\varepsilon} \lor \phi_{\varepsilon} = \text{enl}(\chi_0 \land \chi_1)$.

Since by the assumption of (iii), $\phi_{\varepsilon} \land \chi_1$ is not satisfiable, the latter formula is equivalent to $(\chi_0 \land \chi_1) \lor \phi_{\varepsilon} = \text{enl}(\chi_0 \land \chi_1)$.

(iv) Assume $i \in R$, otherwise $s_R$ does not change $\chi_0, \chi_1, \chi_0 \lor \chi_1$.

Case 1: $\phi^{012} \rightarrow \chi_s$ for $s = 0, 1$. Then $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $s_R(\chi_0) \land s_R(\chi_1) = \chi_0 \land \chi_1 = s_R(\chi_0 \land \chi_1)$.

Case 2: E.g. not $\phi^{012} \rightarrow \chi_0$. Then not $\phi^{012} \rightarrow \chi_0 \land \chi_1$ and $s_R(\chi_0) \land s_R(\chi_1) = (\chi_0 \land x \cap z_i \neq y \cap z_i) \land s_R(\chi_1) \leftrightarrow (\chi_0 \land \chi_1) \land x \cap z_i \neq y \cap z_i = s_R(\chi_0 \land \chi_1)$.

(v) For $i \in R$, the assumptions for (iii) are true for $\psi_s = s_R(\chi_s)$. Hence by (iii) and (iv),

$$(\text{enl}(s_R(\chi_0))(z, x \cap z_i, y \cap z_i) \land \text{enl}(s_R(\chi_1))(z, x \cap z_i, y \cap z_i))$$

$$\rightarrow \text{enl}(s_R(\chi_0 \land \chi_1))(z, x \cap z_i, y \cap z_i).$$

For $i \notin R$, we have $\chi_s \rightarrow x \cap z_i = y \cap z_i$ for $s = 0, 1$ and hence $\text{enl}(s_R(\chi_0)) \land \text{enl}(s_R(\chi_1)) = (\chi_0 \lor \phi_{\varepsilon}) \land (\chi_1 \lor \phi_{\varepsilon}) \leftrightarrow (\chi_0 \land \chi_1) \lor \phi_{\varepsilon} = \text{enl}(s_R(\chi_0 \land \chi_1))$.

Lemma 3.6. Let $\phi$ be special and satisfiable, $R = R(\phi)$, and let $\{\bigwedge_{i < r} \chi_{w, i} \mid w \in W\}$ be a set of representatives for $\phi$.

(i) For any $\bigwedge_{i < r} \chi_i \rightarrow \bigvee_{w \in W} \bigwedge_{i < r} s_R(\chi_{w, i})$, there is a $w \in W$ such that $\bigwedge_{i < r} \chi_i \rightarrow \bigwedge_{i < r} s_R(\chi_{w, i})$. 
(ii) \( \text{enl}(s(\phi)) \leftrightarrow \bigvee_{w \in W} \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i})) \).

(iii) For any \( \bigwedge_{i < r} \chi'_i \rightarrow \bigvee_{w \in W} \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i})) \), there is a \( w \in W \) such that \( \bigwedge_{i < r} \chi'_i \rightarrow \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i})) \).

**Proof.** We will first prove (iii). Then the proof of (i) which is similar but easier will be clear. Let \( \bigwedge_{i < r} \chi'_i(z_i, x \cap z_i, y \cap z_i) \) be consistent, otherwise one can take any \( w \in W \).

For \( i < r \) there is an \( n_i, 0 < n_i < 15 \), and there are \( \hat{\chi}_i, 0, ..., \hat{\chi}_i, n_i - 1 \in \{0, ..., 1\} \) such that

\[
\bigwedge_{i < r} \chi'_i(z_i, x \cap z_i, y \cap z_i) \leftrightarrow \bigwedge_{i < r} \left( \bigvee_{i < r} \hat{\chi}_i, 0 \bigvee ... \bigvee \hat{\chi}_i, n_i - 1 \right) (z_i, x \cap z_i, y \cap z_i).
\]

We will show the claim by induction on \( \prod_{i < r} n_i \).

**Case** \( \prod_{i < r} n_i = 1 \). Take an atomless Boolean algebra \( \mathfrak{A} \) and \( \mathfrak{c} \in A \) such that \( \mathfrak{c} \) is an enumeration of all the atoms in the generated subalgebra. Take \( a, b \in A \) such that \( \mathfrak{A} \models \bigwedge_{i < r} \chi'_i(c_i, a \cap c_i, b \cap c_i) \). Then there is some \( w \in W \) with \( \mathfrak{A} \models \bigwedge_{i < r} \text{enl}(s_R(\chi_{w,i}(c_i, a \cap c_i, b \cap c_i))) \). Since \( \bigwedge_{i < r} \chi'_i(z_i, x \cap z_i, y \cap z_i) \) defines an \( \mathcal{L}_{\omega \omega} \)-2-type of \( (x, y) \) over \( \mathfrak{c} \), we have \( \bigwedge_{i < r} \chi'_i(z_i, x \cap z_i, y \cap z_i) \rightarrow \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i))). \)

**Induction step.** We consider the step from \( \prod_{i < r} n_i \) to \( (n_0 + 1) \times \prod_{0 < i < r} n_i \), the other cases are similar.

\[
\bigvee_{0 < i < r} \hat{\chi}_i, 0 \bigvee ... \bigvee \hat{\chi}_i, n_0 - 1) \wedge \bigwedge_{0 < i < r} \chi'_i \leftrightarrow \left( \bigvee_{0 < i < r} \hat{\chi}_i, 0 \bigwedge \bigwedge_{0 < i < r} \chi'_i \bigvee (\bigvee_{0 < i < r} \hat{\chi}_i, 1 \bigwedge \bigwedge_{0 < i < r} \chi'_i) \right).
\]

By induction hypothesis there are \( w', w'' \in W \) such that

\[
\bigvee_{0 < i < r} \hat{\chi}_i, 0 \bigwedge \bigwedge_{0 < i < r} \chi'_i \rightarrow \bigwedge_{i < r} \text{enl}(s_R(\chi_{w',i})) \bigwedge \bigwedge_{0 < i < r} \chi'_i \rightarrow \bigwedge_{i < r} \text{enl}(s_R(\chi_{w'',i})) \bigwedge \bigwedge_{0 < i < r} \chi'_i.
\]

Thus we have

\[
\left( \bigvee_{0 < i < r} \hat{\chi}_i, 0 \bigwedge \bigwedge_{0 < i < r} \chi'_i \bigvee (\bigvee_{0 < i < r} \hat{\chi}_i, 1 \bigwedge \bigwedge_{0 < i < r} \chi'_i) \right) \rightarrow
\left( \bigvee_{0 < i < r} \text{enl}(s(s_R(\chi_{w',i}))) \bigwedge \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w'',i})) \bigwedge \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w'',i})) \right).
\]

Note that in the last conjunction we get “and” and not only “or”, because

\[
\bigwedge_{0 < i < r} \chi'_i \rightarrow \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w',i})) \bigwedge \bigwedge_{0 < i < r} \text{enl}(s_R(\chi_{w'',i}))
\]

as the situation below any \( z_i \) is independent of the situation below the other \( z_j \).
From 3.5(i), (ii) and (v) we get
\[
\left( \hat{\chi}_{0,0} \land \bigwedge_{0<i<r} \chi^i \right) \lor \left( (\hat{\chi}_{0,1} \lor \ldots \lor \hat{\chi}_{0,n_0}) \land \bigwedge_{0<i<r} \chi^i \right)
\]
\[
\rightarrow \text{enl}(s_R(\chi_{w',0} \lor \chi_{w'',0})) \land \bigwedge_{0<i<r} \text{enl}(s_R(\chi_{w',i} \land \chi_{w'',i})).
\]
Since \(\{\chi_{w,i} \mid w \in W\}\) is a set of representatives for \(\phi(z,x,y)\) and since \(w', w'' \in W\), we have \((\chi_{w',0} \lor \chi_{w'',0}) \land \bigwedge_{0<i<r} (\chi_{w',i} \land \chi_{w'',i}) \rightarrow \phi\) and there is a \(w \in W\) such that
\[
(\chi_{w',0} \lor \chi_{w'',0}) \land \bigwedge_{0<i<r} (\chi_{w',i} \land \chi_{w'',i}) \rightarrow \bigwedge_{i<r} \chi_{w,i}.
\]
For such a \(w\) we have
\[
\text{enl}(s_R(\chi_{w',0} \lor \chi_{w'',0})) \land \bigwedge_{0<i<r} \text{enl}(s_R(\chi_{w',i} \land \chi_{w'',i})) \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i})),
\]
and thus the induction step is complete and (iii) is shown.

(ii) Assume \(s(\phi)\) is satisfiable, otherwise both sides are not satisfiable. Let \(S = \{\chi_{w,i} \mid w \in W\}\) be a set of representatives for \(\phi\), and \(S' = \{\chi_{w',i} \mid w' \in W'\}\) be a set of representatives for \(s(\phi) = \bigvee_{w \in W} \chi_{w,i} \land \text{enl}(s_R(\chi_{w,i}))\) such that \(W' \supseteq W' := \{w \in W \mid \chi_{w,i} \land \text{enl}(s_R(\chi_{w,i}))\text{ is satisfiable}\}\) and \(s_{w'} = s_R(\chi_{w,i})\) for \(w \in W'\).

By definition, \(\text{enl}(s_R(\phi)) = \bigvee_{w' \in W'} \bigwedge_{i<r} \text{enl}(\chi_{w',i})\). By (i), for any \(w' \in W'\) there is some \(w \in W\) such that \(\bigwedge_{i<r} \chi_{w',i} \rightarrow \bigwedge_{i<r} \text{enl}(\chi_{w,i})\) and hence \(\bigwedge_{i<r} \text{enl}(\chi_{w',i}) \rightarrow \bigwedge_{i<r} \text{enl}(\phi(\chi_{w,i}))\). Thus \(\text{enl}(s(\phi)) = \bigvee_{w \in W} \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\). The other direction follows immediately from the choice of \(S'\) and the definition of \(\text{enl}\).

**Lemma 3.7.** Let \(\phi\) be a special formula. Then \(\text{enl}(\text{enl}(s(\phi))) \leftrightarrow \text{enl}(s(\phi))\).

**Proof.** Assume \(s(\phi)\) is satisfiable, otherwise both sides are not satisfiable. Let \(S, W\) be as above and \(S'' = \{\chi_{w'',i} \mid w'' \in W''\}\) be a set of representatives for \(\text{enl}(s(\phi))\). By definition, \(\text{enl}(\text{enl}(s(\phi))) = \bigvee_{w'' \in W''} \bigwedge_{i<r} \text{enl}(\chi_{w'',i})\). For \(w'' \in W''\) we have \(\bigwedge_{i<r} \chi_{w'',i} \rightarrow \text{enl}(s(\phi))\), hence by 3.6(ii), \(\bigwedge_{i<r} \chi_{w'',i} \rightarrow \bigvee_{w \in W} \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\). By 3.6(iii) there is some \(w \in W\) such that \(\bigwedge_{i<r} \chi_{w'',i} \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i}))\), whence \(\bigwedge_{i<r} \text{enl}(\chi_{w'',i}) \rightarrow \bigwedge_{i<r} \text{enl}(\text{enl}(s_R(\chi_{w,i})))\). It is easy to check that for qf \(\chi(z_i, x \cap z_i, y \cap z_i)\) by definition
\[
\text{enl}(\text{enl}(\chi(z_i, x \cap z_i, y \cap z_i))) \rightarrow \text{enl}(\chi(z_i, x \cap z_i, y \cap z_i)).
\]
Therefore \(\bigwedge_{i<r} \text{enl}(\chi_{w'',i}) \rightarrow \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i})),\) and putting things together yields \(\bigvee_{w'' \in W''} \bigwedge_{i<r} \text{enl}(\chi_{w'',i}) \rightarrow \bigvee_{w \in W} \bigwedge_{i<r} \text{enl}(s_R(\chi_{w,i})),\) and, by 3.6(ii), \(\bigvee_{w'' \in W''} \bigwedge_{i<r} \text{enl}(\chi_{w'',i}) \rightarrow \text{enl}(s(\phi))\).
The other direction is obvious.

**Lemma 3.8.** \(-\text{big}(s(\phi)) \rightarrow \text{big}(\text{enl}(s(\phi)))\) is valid for special \(\phi\).

**Proof.** Let \(\mathfrak{A}\) be any atomless Boolean algebra. Assume \(\mathfrak{A} \models \text{big}(\text{enl}(s(\phi(z, x, y))))(\vec{c})\). We show that \(\mathfrak{A} \models \text{big}(s(\phi(z, x, y)))(\vec{c})\). Since the 1-types of \(x\) and of \(y\) over \(\vec{c}\) are determined by \(\mathfrak{A} \models \exists y \text{enl}(s(\phi(\vec{c}, x, y)))\) and \(\mathfrak{A} \models \exists x \text{enl}(s(\phi(\vec{c}, x, y)))\), there is just one pair \((I_2, I_3)\) such that

\[
\mathfrak{A} \models \bigvee_{\{I_0, I_1\}|I_0 \cup I_1 \cup I_2 \cup I_3 = \{0, ..., r-1\}, I_0 \neq 0} (\bigwedge_{i \in I_0} \phi^{012}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \\
\land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i) \rightarrow \text{enl}(s(\phi(\vec{c}, x, y)))).
\]

Take \(I_0 \subseteq \text{-maximal such that}
\[
\mathfrak{A} \models \forall x y \left(\bigwedge_{i \in I_0} \phi^{012}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \\
\land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i \rightarrow \text{enl}(s(\phi(\vec{c}, x, y))))\right)
\]

Let \(R = R(\phi)\) and \(\{\bigwedge_{i < r} \chi_{w,i} \mid w \in W\}\) be a set of representatives for \(\phi\). By 3.6(ii) and (iii) there is a \(w \in W\) such that

\[
\mathfrak{A} \models \forall x y \left(\bigwedge_{i \in I_0} \phi^{012}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \\
\land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i \rightarrow \text{enl}(s_R(\chi_{w,i}(c_i, x \cap c_i, y \cap c_i))))\right).
\]

We claim that also

\[
\mathfrak{A} \models \forall x y \left(\bigwedge_{i \in I_0} \phi^{012}(c_i, x \cap c_i, y \cap c_i) \land \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i \\
\land \bigwedge_{i \in I_2} x \cap c_i = y \cap c_i = 0 \land \bigwedge_{i \in I_3} x \cap c_i = y \cap c_i = c_i \rightarrow \text{s}_R(\chi_{w,i}(c_i, x \cap c_i, y \cap c_i))))\right).
\]

Indeed, by the definition of \text{enl} we have for any \(s_R(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i))\): For \(i \in I_0\), if \(\phi^{012} \rightarrow \text{enl}(s_R(\chi_{w,i})))\), then \(\phi^{012} \rightarrow s_R(\chi_{w,i})\). For \(i \in I_2\), if \(x \cap z_i = y \cap z_i = 0 \rightarrow \text{enl}(s_R(\chi_{w,i})))\), then \(x \cap z_i = y \cap z_i = 0 \rightarrow s_R(\chi_{w,i})\).
For $i \in I_3$, if $x \cap z_i = y \cap z_i = z_i \to \text{enl}(s_R(\chi_{w,i}))$, then $x \cap z_i = y \cap z_i = z_i \to s_R(\chi_{w,i})$.

For $i \in I_1$ the formula $x \cap z_i = y \cap z_i \neq 0$, $z_i \land \text{enl}(s_R(\chi_{w,i})) \land -s_R(\chi_{w,i})$ is consistent only if $\phi^012 \to s_R(\chi_{w,i})$. But then we could take $I'_0 := I_0 \cup \{i\}$ and $I'_1 = I_1 \setminus \{i\}$ and replace $(I'_0, I'_1)$ by $(I_0, I_1)$, which contradicts the maximality of $I_0$.

Now we are ready to prove (***) for special formulas of the form $s(\phi)$.

**Lemma 3.9.** Let $\phi$ be special and $\vec{c} \in B$ be an $r$-tuple that consists of atoms in the generated subalgebra.

(i) If $\neg \text{big}(\phi)$ and $\text{enl}(\phi) \to \phi$ are valid, then for any $\alpha$ with $\vec{c} \in B_\alpha$ the relation $\phi(\vec{c}, x, y)$ is small in $B_\alpha$.

(ii) If $\neg \text{big}(s(\phi))$ is valid, then for any $\alpha$ with $\vec{c} \in B_\alpha$ the relation $\text{enl}(s(\phi(\vec{c}, x, y)))$ is small in $B_\alpha$.

**Proof.** (i) Let $B \models \neg \text{big}(\phi)(\vec{c})$ and $\vec{c} \in B_\alpha$ be atoms in the generated subalgebra. Set $B_\alpha := A$, and let $M \neq \emptyset$ be a maximally homogeneous set for $\phi(\vec{c}, x, y)$ in $A$, and $(a, b)_A \in P(A)$, i.e. $(a, b)_A$ is an interval in $A$. Take $(a', b')_A \leq (a, b)_A$ such that there is just one $i \in r$, say $i_0$, with $(b' \setminus a') \subseteq c_i$ and $c_i \cap a' \neq 0$ and $b' \cap c_i \neq c_i$. We assume $B$ (and also $A$ and $P(\omega)$) satisfy

$$\forall x \in (a', b')_A \exists y \phi(\vec{c}, x, y) \land \exists y \phi(\vec{c}, y, x)(\vec{c}),$$

for otherwise $(a', b')_A \in D_A(M, (\phi(\vec{c}, x, y), 1, 0)$.

Since $B \models \neg \text{big}(\phi)(\vec{c})$, we have $(a', b')_A \cap M \neq (a', b')_A$. We fix $d \in (a', b')_A \setminus M$ and an $m \in M$ such that $A \models \neg \phi(\vec{c}, d, m) \lor \neg \phi(\vec{c}, m, d)$, say $A \models \neg \phi(\vec{c}, d, m)$, and show that there is an $(a'', b'')_A \leq (a', b')_A$ such that for any $x \in (a'', b'')_A$ we have $x \in M$ or $P(\omega) \models \neg \phi(\vec{c}, x, m)$.

Then (i) will be proved, because such an $(a'', b'')_A$ is in $D_A(M, \phi(\vec{c}, x, y), 1, 0)$. Fix a set $\big\{ \wedge_{i \in r} \chi_{w,i} \mid w \in W \big\}$ of representatives for $\phi$.

**Claim.** $d \cap c_{i_0} \neq c_{i_0} \setminus m$.

**Proof.** $\phi(\vec{c}, x, y) = \bigvee_{w \in W} \wedge_{i \in r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i)$, w.l.o.g. $W = \{0, 1, \ldots, s - 1\}$. Hence $A \models \bigwedge_{w \in W} \bigvee_{i \in r} \neg \chi_{w,i}(c_i, d \cap c_i, m \cap c_i)$, say for $w = 0, 1, \ldots, s' - 1$

$$A \models \bigwedge_{i \in r, i \neq i_0} \neg \chi_{w,i}(c_i, d \cap c_i, m \cap c_i),$$

and for $w = s', s' + 1, \ldots, s - 1$

$$A \models \bigwedge_{i \in r, i \neq i_0} \chi_{w,i}(c_i, d \cap c_i, m \cap c_i) \land \neg \chi_{w,i_0}(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0}).$$
We may assume $s > 0$ and $s' \leq s - 1$, because otherwise $(a', b')_A \in D_A(M, \phi(c, x, y), 1, 0)$. Since

$$\mathfrak{A} \models \forall x y \left( \bigwedge_{s' \leq w < s} \bigwedge_{i < r, i \neq i_0} \chi_{w, i}(c_i, x \cap c_i, y \cap c_i) \wedge \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \rightarrow \phi(c, y, x) \right),$$

we have

$$\mathfrak{A} \models \forall x y \left( \bigwedge_{s' \leq w < s} \bigwedge_{i < r, i \neq i_0} \chi_{w, i}(c_i, x \cap c_i, y \cap c_i) \wedge \bigwedge_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \wedge \exists x \chi_{w, i}(c_i, x \cap c_i, y \cap c_i) \wedge \exists y \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right) \rightarrow \text{enl}(\phi(c, x, y)).$$

By the assumptions on $\phi(\tilde{c}, x, y)$ and on $\tilde{c}$ there is just one 1-type of $x \cap c_{i_0}$ over $c_{i_0}$ consistent with $\phi(\tilde{c}, x, y)$ such that for every $w \in W$ the formula $\exists y \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$ is implied by this type. The same holds for the 1-type of $y \cap c_{i_0}$ over $c_{i_0}$, which coincides with the 1-type of $x \cap c_{i_0}$ over $c_{i_0}$, and the formula $\exists x \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$. Since $m \cap c_{i_0}$ and $d \cap c_{i_0}$ have this 1-type, we get

$$\mathfrak{A} \models \exists x \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \wedge \exists y \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, d \cap c_{i_0}, y \cap c_{i_0}).$$

Note that $\mathfrak{A} \models \neg \phi(\tilde{c}, d, m)$ and $\phi$ is equivalent to $\text{enl}(\phi)$. Therefore $d \cap c_{i_0} \neq c_{i_0} \setminus m$ and the claim is proved.

We now give $(a'', b'')_A$ case by case.

Case 1: $d \cap c_{i_0} \neq m \cap c_{i_0}$. Then

$$\mathfrak{A} \models \bigvee_{i = 0, 1, 2, 4, 8} \phi^i(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0}).$$

Assume that $\mathfrak{A} \models \phi^i(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0})$.

If $i = 0$ or $i = 2$, take an $e'$ such that $0 \subset e' \subset c_{i_0} \cap m \cap (\neg d)$, and $(a'', b'')_A = (d, b' \setminus e')_A$. If $i = 1$ or $i = 8$, take $(a'', b'')_A = (a', d)_A$. Finally, if $i = 4$, take $(a'', b'')_A = (d, b')_A$.
Then, in each subcase, for any $x \in (a'', b'')_{P(\omega)}$ we have

$$\mathcal{P}(\omega) \models \text{tp}(x, m, c') = \text{tp}(d, m, c')$$

and hence $\mathcal{P}(\omega) \models \neg \phi(x, m, c')$.

Case 2: $d \cap c_{i_0} = m \cap c_{i_0}$.

Subcase 2.1: 

$$\mathfrak{A} \models \exists x \left( \phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \land \bigwedge_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right).$$

Since $\phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$ determines the $L_{\omega\omega}$-type of $y \cap c_{i_0}$ over $c_{i_0}$, and $m$ has the same one, we have

$$\mathfrak{A} \models \exists x \left( \phi^{012}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \land \bigwedge_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \right).$$

There is an example $d'$ for $x$ with $d' \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A$, because $m \cap c_{i_0} = d \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A$ and hence within the given 1-type of $x \cap c_{i_0}$ over $c_{i_0}$ the formula $\phi^{i}(c_{i}, x \cap c_{i}, m \cap c_{i})$ can be realized with some $x \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A$ for $i = 0, 1, 2$. We can argue with $(d' \cap c_{i_0}) \cup (d \cap c_{i_0})$ as with $d$ in case 1 for $i = 0, 1, 2$.

Subcase 2.2: 

$$\mathfrak{A} \models \forall x \left( \phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \rightarrow \bigvee_{s' \leq w < s} \chi_{w, i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right).$$

Again we have

$$\mathfrak{A} \models \forall x \left( \bigwedge_{s' \leq w < s} \bigwedge_{i < r, i \neq i_0} \chi_{w, i}(c_{i}, x \cap c_{i}, y \cap c_{i}) \land \bigvee_{s' \leq w < s} \chi_{w, i}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right).$$

Since

$$\phi^{012}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \rightarrow \bigvee_{s' \leq w < s} \chi_{w, i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}),$$

by the definition of enl we have

$$\forall x \left( \bigwedge_{i < r, i \neq i_0} \text{enl} \left( \bigwedge_{s' \leq w < s} \chi_{w, i}(z_{i}, x \cap z_{i}, y \cap z_{i}) \right) \land \bigvee_{s' \leq w < s} \chi_{w, i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \lor \left( x \cap z_{i_0} = y \cap z_{i_0} \right) \land \exists x \bigvee_{s' \leq w < s} \chi_{w, i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \land \exists y \bigvee_{s' \leq w < s} \chi_{w, i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0})) \rightarrow \text{enl}(\phi^{i}(z, x, y)) \right).$$
In \( \mathfrak{A} \) we get
\[
\mathfrak{A} \models \forall x y \left( \bigwedge_{s' \leq w < s} \bigwedge_{i < r, i \neq i_0} \text{enl}(\chi_{w,i}(c_i, x \cap c_i, y \cap c_i)) \right)
\]
\[
\wedge \left( \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \lor \left( x \cap c_{i_0} = y \cap c_{i_0} \right) \right)
\]
\[
\wedge \exists x \left( \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right)
\]
\[
\wedge \exists y \left( \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \right) \rightarrow \text{enl}(\phi(\bar{c}, x, y)) \right).
\]

As in the first subcase, we get
\[
\mathfrak{A} \models \exists x \left( \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \right)
\]
\[
\wedge \exists y \left( \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, d \cap c_{i_0}, y \cap c_{i_0}) \right) \wedge d \cap c_{i_0} = m \cap c_{i_0} .
\]

Putting things together yields \( \mathfrak{A} \models \text{enl}(\phi(\bar{c}, d, m)) \) and hence \( \mathfrak{A} \models \phi(\bar{c}, d, m) \), a contradiction to the choice of \( d \) and \( m \).

(ii) By 3.8, \( \neg \text{big}(s(\phi)) \rightarrow \neg \text{big}(\text{enl}(s(\phi))) \), and, by 3.7, \( \text{enl}(\text{enl}(s(\phi))) \rightarrow \text{enl}(s(\phi)) \) is valid. Therefore (ii) follows from (i) applied to \( \text{enl}(s(\phi)) \).

Lemma 3.9, the construction and the monotonicity of \( Q^2_1 \) yield:

**Theorem 3.10.** For any special \( \phi \),

\[
\mathfrak{B} \models \forall z \left( (\text{"}z\text{ are the atoms in the generated subalgebra"} \wedge \neg \text{big}(s(\phi))(\bar{z}))) \right)
\]
\[
\rightarrow \neg Q^2_1 x y \ s(\phi(\bar{z}, x, y)).
\]

Finally, we show how to get Theorem 3.10 for \( \phi \) instead of \( s(\phi) \).

**Theorem 3.11.** For any special \( \phi \)

\[
\mathfrak{B} \models \forall z \left( (\text{"}z\text{ are the atoms in the generated subalgebra"} \wedge \neg \text{big}(\phi)(\bar{z}))) \right)
\]
\[
\rightarrow \neg Q^2_1 x y \ \phi(\bar{z}, x, y)).
\]

**Proof** (by induction on \( \text{card}(R(\phi)) \)). If \( R(\phi) = \emptyset \), then \( \phi(\bar{z}, x, y) \rightarrow x = y \), and hence \( \mathfrak{B} \models \neg Q^2_1 x y \phi(\bar{c}, x, y) \).

Now assume \( \mathfrak{B} \models \forall z \left( (\text{"}z\text{ are the atoms in the generated subalgebra"} \wedge \neg \text{big}(\psi)(\bar{z}))) \rightarrow \neg Q^2_1 x y \psi(\bar{z}, x, y) \) for all \( \psi \) with \( R(\psi) \subset R(\phi) \). We show \( \mathfrak{B} \models Q^2_1 x y \phi(\bar{c}, x, y) \rightarrow \text{big}(\phi)(\bar{c}) \) for any \( r \)-tuple \( \bar{c} \) that consists of atoms in the generated subalgebra. Assume \( \mathfrak{B} \models Q^2_1 x y \phi(\bar{c}, x, y) \) and let \( H \) be an uncountable homogeneous set for \( \phi(\bar{c}, x, y) \) in \( \mathfrak{B} \). By recursion on \( i \leq r \) we define uncountable subsets \( H^{(i)}, 0 \leq i \leq r \).
Set $H^{(0)} := H$. Assume $H^{(i)}$ is defined. We distinguish two cases:

**Case 1:** $\{x \cap c_i \mid x \in H^{(i)}\}$ is uncountable. Then take $H^{(i+1)} \subseteq H^{(i)}$ such that $H^{(i+1)}$ is uncountable and for any $x, y \in H^{(i+1)}$, if $x \neq y$ then $x \cap c_i \neq y \cap c_i$.

**Case 2:** $\{x \cap c_i \mid x \in H^{(i)}\}$ is countable. Then there is some $x \in H^{(i)}$ such that $\{y \in H^{(i)} \mid x \cap c_i = y \cap c_i\}$ is uncountable. Let $H^{(i+1)}$ be such a set.

For $i \not\in R$, $\{x \cap c_i \mid x \in H^{(i)}\}$ is a singleton, and we are in case 2. Now consider $H^{(0)}, H^{(1)}, \ldots, H^{(r)}$. If for all $i \in R$ case 1 is true, then $H^{(r)}$ shows $\mathfrak{B} \models Q_1^2 xy \sigma(\phi(c, x, y))$. By 3.10, $\mathfrak{B} \models \text{big}(s(\phi(c)))$. Since $s(\phi) \to \phi$, $\mathfrak{B} \models \text{big}(\phi(c))$.

If there is some $i \in R$ with case 2 being true, fix such an $i$. Then $H^{(i+1)}$ shows $\mathfrak{B} \models Q_1^2 xy (\phi \land x \cap z_i = y \cap z_i)(c, x, y)$. Take $\psi = \phi \land x \cap z_i = y \cap z_i$. Then $\psi$ is also special. Since $\psi \to \phi$ and $i \in R(\phi) \setminus R(\psi)$, we have $R(\psi) \subseteq R(\phi)$. By induction hypothesis, we conclude from $\mathfrak{B} \models Q_1^2 xy (\phi \land x \cap z_i = y \cap z_i)(c, x, y)$ that $\mathfrak{B} \models \text{big}(\psi(c))$ and hence $\mathfrak{B} \models \text{big}(\phi(c))$.

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**References**


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