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Groupwise dense families

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Abstract. We show that the Filter Dichotomy Principle implies that there are exactly four classes of ideals in the set of increasing functions from the natural numbers. We thus answer two open questions on consequences of $\mathfrak{u} < \mathfrak{g}$. We show that $\mathfrak{r} < \mathfrak{g}$ implies that $\mathfrak{u} = \mathfrak{r}$, and that Filter Dichotomy together with $\mathfrak{u} < \mathfrak{s}$ implies $\mathfrak{u} < \mathfrak{g}$. The technical means is the investigation of groupwise dense sets, ideals, filters and ultrafilters. With related techniques we prove the new inequality $\mathfrak{s} \leq \mathrm{cf}(\mathfrak{d})$.

1. Introduction

We are going to consider some cardinal invariants between $\mathfrak{b}$ and cf($\mathfrak{d}$), all variants of the groupwise density number $\mathfrak{g}$. All cardinal invariants and combinatorial principles used in this paper will be explained in the end of this introduction.

We shall show that in the following chain of equivalences and implications from [4,5,15]

$$\begin{aligned} \mathfrak{u} < \mathfrak{g} &\Leftrightarrow \text{ five classes in } \omega^{\uparrow\omega} \Leftrightarrow \text{ trichotomy} \\ &\Rightarrow \text{ only four types of ideals (4I)} \\ &\Rightarrow \text{ only four growth types (4G)} \qquad (*) \\ &\Rightarrow \text{ filter dichotomy (FD)} \\ &\Rightarrow \text{ near coherence of filters (NCF)} \end{aligned}$$

in the third and in the fourth line also the reverse implication holds. Since (4G) is equivalent to the statement that there are exactly four slenderness classes of groups (see [12]), our result shows that the latter, algebraic statement follows from FD.

In the second section we give some estimates for the variants $\mathfrak{g}_i$ of $\mathfrak{g}$.

In the third section we show that $\mathfrak{u} < \mathfrak{g}_i$ is equivalent to the i-th line in $(*)$ and that the middle three principles are equivalent.

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In the fourth section we show that $\mathfrak{u}$ may be replaced by $\mathfrak{r}$ and that $\mathfrak{u} < \mathfrak{g}_4$ implies $\mathfrak{r} = \mathfrak{u}$. We also prove that $\mathfrak{u} < \mathfrak{g}$ follows from FD together with $\mathfrak{s} = \mathfrak{d}$.

In the final section we prove a new inequality for the splitting number.

In the rest of this section, we explain the notation and recall the definitions of the well-known cardinal characteristics $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{r}$, $\mathfrak{s}$ and the six principles on the right hand side of the scheme $(*)$ above, and we define the new variations $\mathfrak{g}_i$. This paper will be self-contained in definitions and in most of the proofs of the new results, though the latter requires to present some parts of [2, 5–7, 15]. In general, we focus on the direction "getting back to the strict inequality" and merely give references where to find the known implication from the strict inequalities to the combinatorial principle.

Notation. The set of all functions from ω to ω is denoted by ω^ω; the set of all increasing functions from ω to ω is denoted by $\omega^{\uparrow\omega}$; and the set of all infinite subsets of ω is written as $[\omega]^\omega$. The quantifier $\forall^\infty$ is interpreted by "for all but finitely many", and its dual quantifier is $\exists^\infty$. On the set ω^ω, the ordering of eventual dominance is defined by $f \leq^* g$ iff $\forall^\infty n\ f(n) \leq g(n)$. Similarly we define eventual inclusion, which is rather called almost inclusion, for two infinite subets X, Y of ω: $X \subseteq^* Y$ iff $X \setminus Y$ is finite.

The bounding number $\mathfrak{b}$ is the smallest cardinality of a subset of ω^ω that is not bounded with respect to $\leq^*$. The dominating number $\mathfrak{d}$ is the smallest cardinality of a dominating subset in the same partial order.

A subset $\mathscr{D}$ of $[\omega]^\omega$ is called dense if for any infinite set X there is an $Y \in \mathscr{D}$ such that $Y \subseteq^* X$. A subset $\mathscr{D}$ of $[\omega]^\omega$ is called open if for any $Y \in \mathscr{D}$ for any $X \subseteq^* Y$ also $X \in \mathscr{D}$. The density number, $\mathfrak{h}$, is the smallest number of open dense sets whose intersection is empty.

A subset $\mathscr{G}$ of $[\omega]^\omega$ is called groupwise dense if

$\mathscr{G}$ is open, and

for every partition of ω into finite intervals $\Pi = \{[\pi_i, \pi_{i+1}) \mid i \in \omega\}$ there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) \mid i \in A\} \in \mathscr{G}$.

The groupwise density number, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection.

A base (pseudo base) of an ultrafilter $\mathscr{U}$ is a subset $\mathscr{B}$ of $\mathscr{U}$ (of $[\omega]^\omega$) such that $\forall U \in \mathscr{U}\, \exists B \in \mathscr{B}\ B \subseteq U$. The smallest cardinality of a base (pseudo base) of $\mathscr{U}$ is called $\chi_\mathscr{U}$ ($\pi\chi_\mathscr{U}$). The cardinal characteristic $\mathfrak{u}$ is the minimal $\chi_\mathscr{U}$ when $\mathscr{U}$ ranges over the free ultrafilters on ω. The refining number is the smallest cardinality of a family $\mathscr{R} \subseteq [\omega]^\omega$ such that $\forall f \in 2^\omega\ \exists R \in \mathscr{R}\ f \restriction R$ is almost constant.

Balcar and Simon [1] showed that $\mathfrak{r} = \min\{\pi\chi_\mathscr{U} \mid \mathscr{U}$ free ultrafilter on $\omega\}$. Goldstern and Shelah [13] constructed a model of $\mathfrak{r} < \mathfrak{u}$.

A family $\mathscr{S} \subseteq [\omega]^\omega$ is called a splitting family if for every $X \in [\omega]^\omega$ there is some $S \in \mathscr{S}$ such that $S \cap X$ and $X \setminus S$ are both infinite (which is expressed as "S splits X"). The splitting number is the smallest size of a splitting family.

We define a preordering $\preceq$ on $\mathscr{P}(\omega^{\uparrow\omega})$ by $\mathscr{X} \preceq \mathscr{Y}$ iff $\exists r \in \omega^{\uparrow\omega}\ \forall f \in \mathscr{X}\ \exists g \in \mathscr{Y}\ f \leq^* g \circ r$. (This is $\preceq^2$ in [15].)

The preordering $\preceq$ on $\mathscr{P}(\omega^{\uparrow\omega})$ gives rise to the equivalence relation E on $\mathscr{P}(\omega^{\uparrow\omega})$, with $\mathscr{I} E \mathscr{J}$ if $\mathscr{I} \preceq \mathscr{J}$ and $\mathscr{J} \preceq \mathscr{I}$. We write E also for its restrictions. E-classes are commonly called $\preceq$-classes.

An ideal in $\omega^{\uparrow\omega}$ is a subset of $\omega^{\uparrow\omega}$ that is closed under pointwise maxima and under $\leq^*$-smaller functions. A growth type in $\omega^{\uparrow\omega}$ is an ideal of $\omega^{\uparrow\omega}$ that is closed under pointwise sums.

A function $f\colon \omega \to \omega$ is called finite-to-one if $\forall n \in \omega$ the f-preimage of the singleton $\{n\}$ is finite. A filter $\mathscr{F}$ on ω is called feeble if there is a finite-to-one function f such that $f(\mathscr{F})$ (i.e. the filter generated by $\{f[X] \mid X \in \mathscr{F}\}$) is the filter of all cofinite sets. A subset $\mathscr{X}$ of $\omega^{\uparrow\omega}$ is called unbounded iff $\forall f \in \omega^\omega\ \exists g \in \mathscr{X}\ g \not\leq^* f$. Note that $\omega^{\uparrow\omega}$ is dominating in ω^ω, and therefore unboundedness in $\omega^{\uparrow\omega}$ is the same as unboundedness in ω^ω. For $X \in [\omega]^\omega$, the "next" function is defined by $\mathrm{next}(X, n) = \min(X \cap [n, \infty))$. Following [6], for $\mathscr{X} \subseteq [\omega]^\omega$ we set $\mathscr{X}{\sim} = \{\omega \setminus X \mid X \in \mathscr{X}\}$ and ${\sim}\mathscr{X} = [\omega]^\omega \setminus \mathscr{X}$.

In order to increase the growth of a function we often use the $\tilde{\ }$-operation: For $f \in \omega^{\uparrow\omega}$ we define $\tilde{f}$ by

$$\begin{aligned}\tilde{f}(0) &= 0,\\ \tilde{f}(n+1) &= f(\tilde{f}(n)).\end{aligned}$$

Now we come to the chain of implications in $(*)$. First we recall the stated principles, starting with the strongest one.

First line. "Five classes in $\omega^{\uparrow\omega}$" is an abbreviation for: there are just five $\preceq$-classes of downward closed subsets of $\omega^{\uparrow\omega}$. "Trichotomy" stand for the following trichotomy principle: For every subset $\mathscr{Y}$ of $[\omega]^\omega$ that is closed under almost supersets, there is a finite-to-one f such that either $f[\mathscr{Y}] = [\omega]^\omega$ or $f[\mathscr{Y}]$ is an ultrafilter or $f[\mathscr{Y}]$ is the filter of cofinite sets.

Second line. The principle of 4 classes of ideals (4I) says: E restricted to the set of ideals has exactly four classes. We can name these: The first three classes are: the dominating ideals, the bounded ideals containing an unbounded function, and the ideals containing only bounded functions. Here, a function is called bounded iff its range is finite.

Under (4I), for any pair $\mathscr{I}$, $\mathscr{J}$ of unbounded, non-dominating, $=^*$-closed ideals in $\omega^{\uparrow\omega}$ we have $\mathscr{I} E \mathscr{J}$. Hence the set of ideals not belonging to any of these three classes is, under (4I) also one E-class, the fourth one.

Third Line. 4G says that there are four classes of growth types (4G) in the equivalence relation E. The classes have the same properties as the classes of ideals in (4I).

Fourth line. The filter dichotomy principle (FD) says: For any non-feeble filter $\mathscr{F}$ on ω there is a finite-to-one f such that $f(\mathscr{F})$ is an ultrafilter.

Fifth line. NCF says that any two ultrafilters are nearly coherent, i.e. there is a finite-to-one f such that their images under f coincide. NCF stands for near coherence of filters.

References to the implications. The first question is whether any of the stated principles is consistent. Initially, Blass and Shelah [9] have proved the relative consistency of NCF. Later Blass and Laflamme [7] have found that $\mathfrak{u} < \mathfrak{g}$ holds in the NCF-models from [9]. Somewhat simpler consistency proofs are presented in [4, 10].

Laflamme [15] has shown that $\mathfrak{u} < \mathfrak{g}$ implies that there are only five $\preceq$-classes and the trichotomy principle. Blass [5] has shown that each of these consequences of $\mathfrak{u} < \mathfrak{g}$ is actually equivalent to it.

The other four implications are shown in [7].

There are many equivalents to NCF in algebra and functional analysis, see [2, 3].

Cofinalities of reduced products. With a filter $\mathscr{F}$ on ω we get a partial ordering $\leq_{\mathscr{F}}$ on ω^ω which is the $\leq$ in the reduced product of $(\omega, \leq)^\omega$ modulo $\mathscr{F}$: $f \leq_{\mathscr{F}} g$ iff $\{n \in \omega \mid f(n) \leq g(n)\} \in \mathscr{F}$. If $\mathscr{U}$ is an ultrafilter, then $\leq_{\mathscr{U}}$ is a linear order, and we write $\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)$ for its cofinality. If $\mathscr{F}$ is just a filter, then the smallest size of an unbounded set and the smallest size of a dominating set in $(\omega, \leq)^\omega$ modulo $\mathscr{F}$ need not be the same. We write $\mathfrak{b}(\mathscr{F}{-}\mathrm{prod}\,\omega)$ for the former and $\mathrm{cf}(\mathscr{F}{-}\mathrm{prod}\,\omega)$ for the latter.

Some estimates for these $\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)$ are given in [8]. We write $\mathfrak{mcf}$ for the minimal $\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)$ when $\mathscr{U}$ ranges over the non-principal ultrafilters on ω.

Variants of $\mathfrak{g}$. Now we are going to define relatives $\mathfrak{g}_i$ to $\mathfrak{g}$ such that the right hand side in the i-th line of the scheme $(*)$ will be equivalent to $\mathfrak{u} < \mathfrak{g}_i$ (see Corollary 3.6) and to $\mathfrak{r} < \mathfrak{g}_i$ (see Theorems 4.1 and 4.2). A look at $\mathfrak{g}_2$ and at $\mathfrak{g}_4$ will yield the new fact that FD implies 4I. We shall show that $\mathfrak{r} < \mathfrak{g}_4$ implies $\mathfrak{r} = \mathfrak{u}$. From the weaker $\mathfrak{r} < \mathfrak{g}_5$ we derive $\mathfrak{u} < \mathfrak{g}_5$.

Definition 1.1. *a) Let $\mathscr{Y} \subseteq [\omega]^\omega$ be such that $\mathscr{Y}\sim$ is groupwise dense. Let $X \in [\omega]^\omega$.*

$$\begin{aligned}\mathscr{G}_1(X, \mathscr{Y}) = \{Z \mid \exists Y \in \mathscr{Y}\ (\forall a, b \in Z)\\ (a < b \to ([a, b) \cap Y \neq \emptyset \to [a, b) \cap X \neq \emptyset))\}.\end{aligned}$$

$$\mathfrak{g}_1(\mathscr{Y}) = \min\{|\mathscr{X}| \mid \mathscr{X} \subseteq [\omega]^\omega \ \wedge \bigcap_{X \in \mathscr{X}} \mathscr{G}_1(X, \mathscr{Y}) = \emptyset\}.$$
$$\mathfrak{g}_1 = \min\{\mathfrak{g}_1(\mathscr{Y}) \mid \mathscr{Y} \subseteq [\omega]^\omega \wedge \mathscr{Y}\sim \text{ is groupwise dense}\}.$$

b) Let $\mathscr{I} \subseteq \omega^{\uparrow\omega}$ be an unbounded non-dominating ideal of $\omega^{\uparrow\omega}$ that is closed under $=^*$.

$$\mathscr{G}_2(X, \mathscr{I}) = \{Z \mid \exists f \in \mathscr{I}\ \forall^\infty n \in \omega \ \mathrm{next}(X, n) \leq f(\mathrm{next}(Z, n))\}.$$
$$\mathfrak{g}_2(\mathscr{I}) = \min\{|\mathscr{X}| \mid \mathscr{X} \subseteq [\omega]^\omega \ \wedge \bigcap_{X \in \mathscr{X}} \mathscr{G}_2(X, \mathscr{I}) = \emptyset\}.$$
$$\mathfrak{g}_2 = \min\{\mathfrak{g}_2(\mathscr{I}) \mid \mathscr{I} \text{ unbounded non-dominating } =^*\text{-closed ideal in } \omega^{\uparrow\omega}\}.$$

c) We get $\mathscr{G}_3(X, \mathscr{G}T)$ and $\mathfrak{g}_3(\mathscr{G}T)$ by restricting the second component of the domain to growth types. We get $\mathfrak{g}_3$ by replacing every occurrence of an ideal $\mathscr{I}$ in b) by a growth type $\mathscr{G}T$ and otherwise taking the same definition.

d) Let $\mathscr{F}$ be a non-feeble filter on ω.

$$\mathfrak{g}_4(\mathscr{F}) = \min\{|\mathscr{X}| \mid \mathscr{X} \subseteq [\omega]^\omega \ \wedge \bigcap_{X \in \mathscr{X}} \mathscr{G}_1(X, \mathscr{F}) = \emptyset\}.$$
$$\mathfrak{g}_4 = \min\{\mathfrak{g}_4(\mathscr{F}) \mid \mathscr{F} \text{ non-feeble filter}\}.$$

e) Let $\mathscr{U}$ be a free ultrafilter on ω.

$$\mathfrak{g}_5(\mathscr{U}) = \min\{|\mathscr{X}| \mid \mathscr{X} \subseteq [\omega]^\omega \ \wedge \bigcap_{X \in \mathscr{X}} \mathscr{G}_1(X, \mathscr{U}) = \emptyset\}$$
$$\mathfrak{g}_5 = \min\{\mathfrak{g}_5(\mathscr{U}) \mid \mathscr{U} \text{ ultrafilter}\}.$$

Why are these variants of $\mathfrak{g}$? We shall first show that the families $\mathscr{G}_1(X, \mathscr{Y})$, $\mathscr{G}_2(X, \mathscr{I})$, $\mathscr{G}_2(X, \mathscr{G}T)$, $\mathscr{G}_1(X, \mathscr{F})$, $\mathscr{G}_1(X, \mathscr{U})$ are groupwise dense. So in all instances we are asking for the smallest size of a family of special groupwise dense families whose intersection is empty.

Lemma 1.2. *a) ([6, Proof of Thm 9.15]) Let $\mathscr{Y} \subseteq [\omega]^\omega$ be such that $\mathscr{Y}\sim$ is groupwise dense. Let $X \in [\omega]^\omega$. Then $\mathscr{G}_1(X, \mathscr{Y})$ is groupwise dense.*

b) ([7, Theorem 1, (0) ⇒ (1)]) Let $\mathscr{I} \subseteq \omega^{\uparrow\omega}$ be an unbounded non-dominating ideal of $\omega^{\uparrow\omega}$ that is closed under $=^*$. Then $\mathscr{G}_2(X, \mathscr{I})$ is groupwise dense.

c) ([6, Lemma 9.4]) Let $\mathscr{F}$ be a filter. Then $\mathscr{F}$ is non-feeble iff $\mathscr{F}\sim$ is groupwise dense.

d) ([6, Proposition 9.12]) Any ultrafilter is non-feeble.

Proof. For completeness' sake we write proofs.

a) First we show that $\mathscr{G}_1(X, \mathscr{Y})$ is open. Let $Z \in \mathscr{G}_1(X, \mathscr{Y})$ and $Z' \subseteq^* Z$, say $Z' \setminus n \subseteq Z$ and $n \in Z$. We set $n' = \text{next}(Z, n+1)$. We take some $Y \in \mathscr{Y}$ such that

$$\forall a < b \in Z\ ([a, b) \cap Y \neq \emptyset \to [a, b) \cap X \neq \emptyset).$$

Since $\mathscr{Y}\sim$ is closed under $=^*$, also $\mathscr{Y}$ is closed under $=^*$. Hence $Y' = Y \setminus n' \in \mathscr{Y}$. Now

$$\forall a < b \in Z'\ ([a, b) \cap Y' \neq \emptyset \to [a, b) \cap X \neq \emptyset)$$

is easily checked.

Now let $\Pi = \langle \pi_i \mid i \in \omega \rangle$ be given. We merge adjacent intervals in Π so that we may assume that

$$\forall i \in \omega\ X \cap [\pi_i, \pi_{i+1}) \neq \emptyset.$$

(For later use, note, that $\forall^\infty i \in \omega$ instead of $\forall i \in \omega$ is would suffice.)

Since $\mathscr{Y}\sim$ is groupwise dense, there is some infinite, coinfinite A such that

$$Z = \bigcup_{i \in A} [\pi_i, \pi_{i+1}) \in \mathscr{Y}\sim,$$
$$Y = \bigcup_{i \in \omega \setminus A} [\pi_i, \pi_{i+1}) \in \mathscr{Y}.$$

Now again

$$\forall a < b \in Z\ ([a, b) \cap Y \neq \emptyset \to [a, b) \cap X \neq \emptyset)$$

is easily checked.

b) First we show that $\mathscr{G}_2(X, \mathscr{I})$ is open. Let $Z \in \mathscr{G}_2(X, \mathscr{I})$ and $Z' \subseteq^* Z$. Note that we have $\forall^\infty n\ \text{next}(Z, n) \leq \text{next}(Z', n)$ and since f is increasing, $\forall^\infty n\ f(\text{next}(Z, n)) \leq f(\text{next}(Z', n))$. Hence if f is a witness for

$$\forall^\infty n \in \omega\ \text{next}(X, n) \leq f(\text{next}(Z, n)),$$

then the same f is also a witness for the formula with Z' instead of Z.

Now let $\Pi = \langle \pi_i \mid i \in \omega \rangle$ be given. We merge intervals of Π so that we have that there is some member of X in each interval. We write $\Pi = \langle \pi_i \mid i \in \omega \rangle$ for the new partition. Then we choose $g \colon \omega \to \omega$ as follows if $n \in [\pi_i, \pi_{i+1})$ then we set $g(n) = \pi_{i+3}$. Since $\mathscr{I}$ is not dominated by g, there is some $f \in \mathscr{I}$ such that for infinitely many n, we have that $f(n) > g(n)$. We take

$$Z = \bigcup \{[\pi_{i+1}, \pi_{i+2}) \mid \text{ for some } n \in [\pi_i, \pi_{i+1}) \text{ we have that } f(n) > g(n)\}.$$

Clearly Z is the union of infinitely many intervals from Π. We show that $\forall n \in \omega\ \text{next}(X, n) \leq f(\text{next}(Z, n))$. Suppose that $n \in [\pi_i, \pi_{i+1})$. Then by the choice of Π we have that $\text{next}(X, n) \leq \pi_{i+2}$. There is some $j \geq i$ such that $\text{next}(Z, n) \in$

$[\pi_j, \pi_{j+1})$ and such that there is some $m \in [\pi_{j-1}, \pi_j)$ such that $f(m) > g(m) = \pi_{j+2}$. So we have

$$f(\mathrm{next}(Z, n)) \geq f(\pi_j) \geq f(m) > g(m) \geq \pi_{j+2} \geq \pi_{i+2} \geq \mathrm{next}(X, n).$$

c) Let $\mathscr{F}$ be feeble and let $f \in \omega^\omega$ be finite-to-one such that $f(\mathscr{F})$ is the filter of cofinite sets. We set $\pi_i = \min\{f^{-1}\{i\}\}$ Then we have for any $F \in \mathscr{F}$ and for any $A \in [\omega]^\omega$ that $\bigcup_{i \in A}[\pi_i, \pi_{i+1}) \cap F$ is infinite, hence that $\bigcup_{i \in A}[\pi_i, \pi_{i+1}) \notin \mathscr{F} \sim$.

Since $\mathscr{F}$ is a filter, we have that $\mathscr{F} \sim$ is open. Given a partition $\langle \pi_i \mid i \in \omega \rangle$ we stipulate $\pi_{-1} = 0$ and set $f(n) = i$ for $n \in [\pi_{i-1}, \pi_i)$. Since $\mathscr{F}$ is not feeble, we have that $f(\mathscr{F})$ is not the filter of cofinite sets, so there is some coinfinite $A \in f(\mathscr{F})$. Thus we have that $\bigcup_{i \in \omega \setminus A}[\pi_i, \pi_{i+1}) \in \mathscr{F} \sim$.

d) Suppose that $\mathscr{U}$ is an ultrafilter and that f is finite-to-one and $f(\mathscr{U})$ is the filter of the cofinite sets. We take some infinite, coinfinite $A \subseteq \omega$. Then $\omega \setminus A \notin f(\mathscr{F})$, and hence $\omega \setminus f^{-1}(A) = f^{-1}(\omega \setminus A) \notin \mathscr{U}$, so $f^{-1}(A) \in \mathscr{U}$. For $\omega \setminus A$ instead of A we get the same. But $f^{-1}(A)$ and $f^{-1}(\omega \setminus A)$ are disjoint. $\square_{1.2}$

Corollary 1.3. For $X, \mathscr{Y}, \mathscr{I}, \mathscr{G}T, \mathscr{F}, \mathscr{U}$ as in Definition 1.1, the families $\mathscr{G}_1(X, \mathscr{Y})$, $\mathscr{G}_2(X, \mathscr{I})$, $\mathscr{G}_3(X, \mathscr{G}T)$, $\mathscr{G}_1(X, \mathscr{F})$, $\mathscr{G}_1(X, \mathscr{U})$ are groupwise dense.

Hence for $i = 1, 2, 3, 4, 5$ we have that $\mathfrak{g}_i \geq \mathfrak{g}$.

2. Some comparisons

We know only the trivial inequalities between the $\mathfrak{g}_i$ that are listed in the first two parts of the next proposition. In situations of the type $\mathfrak{u} < \mathfrak{g}_i$ we get more, namely some implications of the form $\mathfrak{u} < \mathfrak{g}_i \Longrightarrow \mathfrak{u} < \mathfrak{g}_j$ even when we cannot prove $\mathfrak{g}_i \leq \mathfrak{g}_j$ in general. Though $\mathfrak{g} < \mathfrak{b}$ is consistent relative to ZFC (see [6]), all our relatives $\mathfrak{g}_i$ of $\mathfrak{g}$ will be greater than or equal $\mathfrak{b}$. (For the investigation of consequences of $\mathfrak{u} < \mathfrak{g}_i$ one could anyway replace $\mathfrak{g}_i$ by $\max(\mathfrak{g}_i, \mathfrak{u})$, but this does not make sense for our purpose here.)

Proposition 2.1. *a)* $\mathfrak{g} \leq \mathfrak{g}_1 \leq \mathfrak{g}_4 \leq \mathfrak{g}_5$,
b) $\mathfrak{g}_2 \leq \mathfrak{g}_3$,
c) $\mathfrak{g}_5 = \mathfrak{mcf}$,
d) $\mathfrak{b} \leq \mathfrak{g}_1$,
e) $\mathfrak{b} \leq \mathfrak{g}_2$.

Proof. The first inequality in a) follows from Lemma 1.2 a), that all the $\mathscr{G}_1(X, \mathscr{Y})$ in the definition of $\mathfrak{g}_1$ are groupwise dense. The other inequalities in a) follow from Lemma 1.2 c) that for a non-feeble $\mathscr{F}$ the set $\mathscr{F} \sim$ is groupwise dense, and from 1.2 d) that ultrafilters are not feeble.

b) From the definitions follows that growth types are ideals.

c) First we show that for every ultrafilter $\mathscr{U}$ we have $\mathfrak{g}_5(\mathscr{U}) \geq \mathrm{cf}(\mathscr{U}\text{−prod}\,\omega)$: We take an $\mathscr{X}$ of cardinality $\mathfrak{g}_5(\mathscr{U})$ such that $\bigcap\{\mathscr{G}_1(X, \mathscr{U}) \mid X \in \mathscr{X}\} = \emptyset$. Then we have

$$\forall Z\, \exists X \in \mathscr{X}\, \forall U \in \mathscr{U}\, \exists a < b \in Z\ ([a, b) \cap U \neq \emptyset \wedge [a, b) \cap X = \emptyset).$$

From this we get

$$\forall Z\, \exists X \in \mathscr{X}\ \{u \mid \mathrm{next}(X, u) < \mathrm{next}(Z, u)\} \notin \mathscr{U}.$$

Since $\mathscr{U}$ is an ultrafilter, we get

$$\forall Z\, \exists X \in \mathscr{X}\ \{u \mid \mathrm{next}(X, u) \geq \mathrm{next}(Z, u)\} \in \mathscr{U}.$$

Since $\mathscr{U}$ is closed under finite intersections, we have

$$\begin{aligned}
&\forall Z_0, Z_1, Z_2 \exists X_0, X_1, X_2 \in \mathscr{X} \\
&\{u \mid \max(\mathrm{next}(X_0, u), \mathrm{next}(X_1, u), \mathrm{next}(X_2, u)) \geq \\
&\max(\mathrm{next}(Z_0, u), \mathrm{next}(Z_1, u), \mathrm{next}(Z_2, u))\} \\
&\in \mathscr{U}.
\end{aligned}$$

For every $f \in \omega^{\uparrow\omega}$ there are Z_0, Z_1, Z_2 such that

$$\forall n \in \omega\ f(n) \leq \max(\mathrm{next}(Z_0, n), \mathrm{next}(Z_1, n), \mathrm{next}(Z_2, n)).$$

Just choose by induction π_i such that $\forall n \leq \pi_i\ \ f(n) \leq \pi_{i+1}$ and set $Z_j = \bigcup\{[\pi_{3i+j}, \pi_{3i+j+1}) \mid i \in \omega\}$ for $j = 0, 1, 2$. Therefore

$$\{\max(\mathrm{next}(X_0, n), \mathrm{next}(X_1, n), \mathrm{next}(X_2, n)) \mid X_0, X_1, X_2 \in \mathscr{X}\}$$

is dominating in $\leq_{\mathscr{U}}$.

Now we show that for every ultrafilter $\mathscr{U}$ we have $\mathfrak{g}_5(\mathscr{U}) \leq \mathrm{cf}(\mathscr{U}\text{−prod}\,\omega)$: Let $\{f_\alpha \mid \alpha < \mathrm{cf}(\mathscr{U}\text{−prod}\,\omega)\}$ be cofinal in $\leq_{\mathscr{U}}$. Take X_α such that $f_\alpha \leq_{\mathscr{U}} \mathrm{next}(X_\alpha, \cdot)$. Such an X_α exists because we first can take three X_α's such that the maximum over their next functions dominates f_α everywhere. Then one of the three dominates f_α on a set in $\mathscr{U}$. Then we have

$$\bigcap\{\{f \mid f \geq_{\mathscr{U}} \mathrm{next}(X_\alpha, \cdot)\} \mid \alpha \in \mathrm{cf}(\mathscr{U}\text{−prod}\,\omega)\} = \emptyset.$$

We claim that also

$$\bigcap\{\mathscr{G}_1(X_\alpha, \mathscr{U}) \mid \alpha \in \mathrm{cf}(\mathscr{U}\text{−prod}\,\omega)\} = \emptyset.$$

Suppose that the claim is false, i.e. that there is some Z such that $\forall \alpha\ \exists U \in \mathscr{U}\ \forall a < b \in Z\ ([a, b) \cap U \neq \emptyset \to [a, b) \cap X_\alpha \neq \emptyset)$. We let $\langle z_n \mid n \in \omega\rangle$ be the increasing enumeration of Z and define f by setting $f(i) = z_{n+2}$ for $i \in [z_n, z_{n+1})$. Then we have $\forall\alpha\ \exists U \in \mathscr{U}\ \forall u \in U\ \exists n\ (u \in [z_n, z_{n+1}) \wedge f(u) = z_{n+2} \wedge \mathrm{next}(X_\alpha, u) \leq z_{n+1})$. Hence f would be in the above intersection. Contradiction.

So $\{X_\alpha \mid \alpha \in \mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)\}$ witnesses that $\mathfrak{g}_5(\mathscr{U}) \leq \mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)$.

d) Let $\mathscr{Y}\sim$ be groupwise dense. Let $\mu < \mathfrak{b}$ and let X_α, $\alpha \in \mu$, be infinite sets. We show that

$$\bigcap\{\mathscr{G}_1(X_\alpha, \mathscr{Y}) \mid \alpha \in \mu\}$$

is groupwise dense. (All we need is that this intersection is nonempty.)

Since each of the $\mathscr{G}_1(X_\alpha, \mathscr{Y})$ is open, also their intersection is open. Now suppose that a partition Π is given. We take f_α as the increasing enumeration of X_α. Then we take $g \in \omega^{\uparrow\omega}$ such that for $\alpha < \mu$ we have that $\tilde{f}_\alpha \leq^* g$.

Then we take $i(0) = 0$, and $i(n+1) = \min\{i > i(n) \mid g(\pi_{i(n)}) < \pi_{i(n+1)}\}$. Now we show that

$$\forall\alpha \in \mu\ \forall^\infty n\ [\pi_{i(n)}, \pi_{i(n+1)}) \cap X_\alpha \neq \emptyset.$$

By the proof of Lemma 1.2 a) we are then finished. Suppose α is given. Let k be such that $\tilde{f}_\alpha(n) \leq g(n)$ for all $n \geq k$. Suppose that $\tilde{f}_\alpha(k) \in [\pi_{i(n)}, \pi_{i(n+1)})$. Then

$$\tilde{f}_\alpha(k+1) = f_\alpha(\tilde{f}_\alpha(k)) \leq g(\tilde{f}_\alpha(k)) < g(\pi_{n+1}) < \pi_{i(n+2)}.$$

Hence, starting from some interval $[\pi_{i(n)}, \pi_{i(n+1)})$ all later intervals contain an element of $\mathrm{range}(\tilde{f}_\alpha)$.

e) Let $\mathscr{I} \subseteq \omega^{\uparrow\omega}$ be an unbounded non-dominating ideal of $\omega^{\uparrow\omega}$ that is closed under $=^*$. Let X_α, $\alpha < \kappa$, $\kappa < \mathfrak{b}$ be given. We show that $\bigcap_{\alpha\in\kappa} \mathscr{G}_2(X_\alpha, \mathscr{I}) \neq \emptyset$. Let g dominate all $\mathrm{next}(X_\alpha, \cdot)$, $\alpha \in \kappa$. As in c), we take three infinite sets $\tilde{X}_j$, $j = 0, 1, 2$, such that $g \leq^* \max\{\mathrm{next}(\tilde{X}_j, \cdot) \mid j = 0, 1, 2\}$. In Lemma 1.2(b) it is shown that the $\mathscr{G}_2(\tilde{X}_j, \mathscr{I})$ are groupwise dense. As groupwise dense sets are dense, we can choose $Z_0 \supseteq^* Z_1 \supseteq^* Z_2$ such that $Z_j \in \mathscr{G}_2(\tilde{X}_j, \mathscr{I})$ with witnessing function $f_j \in \mathscr{I}$. Since $\mathscr{I}$ is an ideal we have that $\max\{f_0, f_1, f_2\} \in \mathscr{I}$. As all functions are increasing and as $\mathrm{next}(Z_2, \cdot) \geq^* \mathrm{next}(Z_1, \cdot) \geq^* \mathrm{next}(Z_0, \cdot)$ we get

$$\begin{aligned}\forall\alpha \in \kappa\ \forall^\infty n \in \omega\ \mathrm{next}(X_\alpha, n) \leq g(n) &\leq \max\{\mathrm{next}(\tilde{X}_j, n) \mid j = 0, 1, 2\} \\ &\leq (\max\{f_0, f_1, f_2\})(\mathrm{next}(Z_2, n)).\end{aligned}$$

Thus $Z_2 \in \bigcap_{\alpha\in\kappa} \mathscr{G}_2(X_\alpha, \mathscr{I})$. □

Remark. Blass [6, 4.8] gives a model where $\mathfrak{c} = \aleph_{\aleph_1}$ and $\mathfrak{g} = \aleph_1$ and $\mathfrak{b} = \aleph_2$. Hence $\mathfrak{g} < \mathfrak{g}_1$ is consistent relative to ZFC.

In [8] $\min\{\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega) \mid \mathscr{U}$ ultrafilter on $\omega\} \geq \mathfrak{g}$, which follows also from a) and c), is proved. Canjar [11] has shown that $\min\{\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega) \mid \mathscr{U}$ ultrafilter on $\omega\} \leq \mathrm{cf}(\mathfrak{d})$ and this gave the new estimate in [8] that $\mathfrak{g} \leq \mathrm{cf}(\mathfrak{d})$. Here we give a direct proof for this.

Theorem 2.2. $\mathfrak{g} \leq \mathrm{cf}(\mathfrak{d})$.

Proof. Let α_i, $i < \mathrm{cf}(\mathfrak{d})$ be cardinals less than $\mathfrak{d}$ such that $\lim_{i<\mathrm{cf}(\mathfrak{d})} \alpha_i = \mathfrak{d}$ and let $\{f_\beta \mid \beta < \mathfrak{d}\}$ be a dominating family. We assume that the enumeration is chosen in a way such that for every $i < \mathrm{cf}(\mathfrak{d})$, the set $\mathscr{C}_i := \{f_\beta \mid \beta < \alpha_i\}$ is closed under pointwise maxima. Then we set

$$\mathscr{G}_i := \{Z \in [\omega]^\omega \mid \forall \beta < \alpha_i \ \mathrm{next}(Z, \cdot) \not\leq^* f_\beta\}.$$

Since $\{f_\beta \mid \beta < \mathfrak{d}\}$ is a dominating family, we have that $\bigcap_{i \in \mathrm{cf}(\mathfrak{d})} \mathscr{G}_i = \emptyset$.

We show that each $\mathscr{G}_i$ is groupwise dense. Obviously, it is closed under $\subseteq^*$. Let Π be a partition of ω into finite intervals. Since $\mathscr{C}_i$ is not dominating, there is some g such that $\forall \beta < \alpha_i \ g \not\leq^* f_\beta$. We fix such a g. From the closure property of $\mathscr{C}_i$ we get that

$$\mathscr{F} = \{\{n \mid f_\beta(n) < g(n)\} \mid \beta < \alpha_i\}$$

generates a filter. After merging intervals, we may assume that $\Pi = \{[\pi_i, \pi_{i+1}) \mid i \in \omega\}$ is such that

$$\forall i \in \omega \, \forall n \leq \pi_i \ g(n) \leq \pi_{i+1}.$$

We set for $j = 0, 1, 2$,

$$Z_j = \bigcup \{[\pi_{3i+j}, \pi_{3i+j+1}) \mid i \in \omega\},$$

and we take j such that $\mathscr{F} \cup \{Z_j\}$ has the strong finite intersection property. Then $Z_{j+2 \bmod 3} \in \mathscr{G}_i$, because for each $\beta < \alpha_i$ and for each n in the infinite set $Z_j \cap \{n \mid f_\beta(n) < g(n)\}$, say for n additionally $\in [\pi_{3i+j}, \pi_{3i+j+1})$, we have

$$\mathrm{next}(Z_{j+2 \bmod 3}, n) \geq \pi_{3i+j+2} \geq g(n) > f_\beta(n).$$

□

Remark. Let $\mathscr{F}$ any non-principal filter on ω. The above proof can be modified for reduced products $(\omega, <)^\omega/\mathscr{F}$ and the partial order $\leq_{\mathscr{F}}$. If we add more sets to $\mathscr{F}$ then $\mathrm{cf}(\mathscr{F}\text{-prod}\omega)$ can decrease, and thus $\mathrm{cf}(\mathscr{F}\text{-prod}\omega) \geq \mathrm{cf}(\mathscr{U}\text{-prod}\omega)$ for any ultrafilter $\mathscr{U}$ above $\mathscr{F}$.

Corollary 2.3. For any non-principal filter $\mathscr{F}$ we have that $\mathfrak{g} \leq \mathrm{cf}(\mathscr{F}\text{-prod}\,\omega)$.

Discussion. For $\mathscr{F}$ non-feeble, corollary 2.3 is superseded by Theorem 16 of [8], which says that $\mathfrak{g} \leq \mathfrak{b}(\mathscr{F}\text{-prod}\,\omega)$. Besides the fact that we did not require non-feebleness, another reason for the weakness of 2.3 might be that in the proof in [8] yet another type of groupwise dense families is considered. For feeble $\mathscr{F}$, $\mathrm{cf}(\mathscr{F}\text{-prod}\omega) = \mathfrak{d}$ and $\mathfrak{b}(\mathscr{F}\text{-prod}\omega) = \mathfrak{b}$. In particular, the Theorem 16 of [8] cannot apply to feeble filters.

3. The correspondences and coincidences

In this section we add left hand sides to the scheme (∗) and show that the middle three lines are equivalent. We cite many known facts and complement them by proving their converses.

First there is a lemma that we shall be using several times and therefore put separately.

Lemma 3.1. *(see [16, Theorem 1 and 3] or [2, Theorem 16]). For every ultrafilter $\mathscr{V}$ we have the inequality $\pi\chi_{\mathscr{V}} \cdot \mathrm{cf}(\mathscr{V}{-}\mathrm{prod}\,\omega) \geq \mathfrak{d}$.*

Proof. Let $\{X_\alpha \mid \alpha \in \pi\chi_{\mathscr{V}}\}$ be a π-base of $\mathscr{V}$. Let $\{f_\beta \mid \beta < \mathrm{cf}(\mathscr{V}{-}\mathrm{prod}\,\omega)\}$ be a dominating family in $\leq_{\mathscr{V}}$, such that the f_β are increasing functions. Then $\{f_\beta \circ \mathrm{next}(X_\alpha, \cdot) \mid \beta \in \mathrm{cf}(\mathscr{V}{-}\mathrm{prod}\,\omega), \alpha \in \pi\chi_{\mathscr{V}}\}$ is $\leq^*$-dominating. □

Proposition 3.2. *a)* $\mathfrak{u} < \mathfrak{g}_1 \Leftrightarrow \mathfrak{u} < \mathfrak{g}$,
b) $\mathfrak{u} < \mathfrak{g}_2 \Rightarrow$ 4I,
c) $\mathfrak{u} < \mathfrak{g}_3 \Rightarrow$ 4G,
d) $\mathfrak{u} < \mathfrak{g}_4 \Rightarrow$ FD,
e) $\mathfrak{u} < \mathfrak{g}_5 \Leftrightarrow$ NCF.

Proof. a) In [6, Lemma 9.15 and Theorem 9.22] Laflamme's trichotomy is derived from $\mathfrak{u} < \mathfrak{g}_1$. The trichotomy is equivalent to $\mathfrak{u} < \mathfrak{g}$ by [15] and [5]. The backward direction is obvious, as $\mathfrak{g} \leq \mathfrak{g}_1$.
b) This is the proof of (0) ⇒ (1) in [7, Theorem 1]. Only groupwise dense families of the type $\mathscr{G}_2(X, \mathscr{I})$ are used there.
c) Rewrite the proof described in b) for growth types instead of ideals.
d) This is 9.15 and 9.16 of [6].
e) "⇐" According to [2], under NCF we have $\mathfrak{mcf} = \mathfrak{d}$ and $\mathfrak{u} < \mathfrak{d}$, and by Proposition 2.1c) we have that $\mathfrak{mcf} = \mathfrak{g}_5$.
"⇒" Theorem 12 of [8] says that NCF follows even from the apparently (see Section 4) weaker $\mathfrak{r} < \mathfrak{g}_5$. An alternative proof is to use $\min\{\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)\} = \mathfrak{g}_5$ and the reworking 9.15 of [6] with $\mathscr{G}_1(X, \mathscr{U})$. □

What about the reverse direction in b), c), and d)? The following theorems complete the picture and answer open questions about the equivalence of FD and 4G and of 4G and 4I: Not only do we have all five equivalences but also that the three strict inequalities and the three principles in b), c), and d) are all equivalent.

Theorem 3.3. *The filter dichotomy principle implies* $\mathfrak{u} < \mathfrak{g}_4$.

Proof. FD implies NCF (see [7]), and NCF in turn implies $\mathfrak{u} < \mathfrak{d}$ (see [2]). Therefore, it suffices to show that FD implies $\mathfrak{g}_4 \geq \mathfrak{d}$. In order to show the latter we modify the beginning of the proof of Theorem 8 in [5]: We fix a non-feeble filter $\mathscr{F}$ and first show that $\mathfrak{g}_4(\mathscr{F}) \geq \mathfrak{d}$. Suppose we are given $\mu < \mathfrak{d}$ and $\{X_\alpha \mid \alpha \in \mu\} \subset [\omega]^\omega$. We show that $\bigcap\{\mathscr{G}_1(X_\alpha, \mathscr{F}) \mid \alpha \in \mu\}$ is not empty.

We set

$$\mathscr{G}_*(X_\alpha, \mathscr{F}) := \{Z \mid \exists Y \in \mathscr{F}\ \forall a \in Z \ \mathrm{next}(Y, a) \geq \mathrm{next}(X_\alpha, a)\}.$$

Since $\forall a \in Z\ \forall b \in Z\ (\mathrm{next}(Y, a) < b \rightarrow \mathrm{next}(X_\alpha, a) < b)$ is implied by $\forall a \in Z\ \mathrm{next}(X_\alpha, a) \leq \mathrm{next}(Y, a)$, we have

$$\mathscr{G}_1(X_\alpha, \mathscr{F}) \supseteq \mathscr{G}_*(X_\alpha, \mathscr{F}).$$

Now we shall show that $\bigcap\{\mathscr{G}_*(X_\alpha, \mathscr{F}) \mid \alpha \in \mu\}$ is not empty.

We have that $\mathscr{G}_*(X_\alpha, \mathscr{F})$ is groupwise dense: It is closed under almost subsets, because the $\forall z \in Z$ may be replaced in its definition by $\forall^\infty z \in Z$ as $\mathscr{F}$ is closed under finite modifications. Given a partition Π, first we merge intervals so that each of the new intervals contains at least one element of X_α. Call this new partition again Π and write it as $\Pi = \{[\pi_i, \pi_{i+1}) \mid i \in \omega\}$.

Since $\mathscr{F}$ is not feeble, by 1.2 c) we have that $\mathscr{F} \sim$ is groupwise dense; and hence there in an infinite A such that $\bigcup\{[\pi_{2i}, \pi_{2(i+1)}) \mid i \in A\} \in \mathscr{F} \sim$ and so $\bigcup\{[\pi_{2i}, \pi_{2(i+1)}) \mid i \notin A\} \in \mathscr{F}$. Then $Z = \bigcup\{[\pi_{2i}, \pi_{2i+1}) \mid i \in A\} \in \mathscr{G}_*(X_\alpha, \mathscr{F})$ with witness $Y \in \mathscr{F}$, because for all $z \in Z$, say for $z \in [\pi_{2i}, \pi_{2i+1})$, we have

$$\mathrm{next}(Y, z) = \pi_{2i+2} > \mathrm{next}(X_\alpha, z).$$

$\mathscr{G}_*(X_\alpha, \mathscr{F})$ is an ideal, because $\mathscr{F}$ is a filter. Hence

$$\mathscr{A}_\alpha := \{Z \mid \omega \setminus Z \in \mathscr{G}_*(X_\alpha, \mathscr{F})\}$$

is a filter and again by 1.2 c) it is not feeble. Now we apply the filter dichotomy principle and then we could literally take the end of Blass' proof of Theorem 8 in [5].

For completeness' sake and because we need it in the next theorem as well, we present a somewhat simplified version of Blass' proof here:

Lemma 3.4. *([5, Part of the proof of Theorem 8]) FD implies that fewer than $\mathfrak{d}$ groupwise dense ideals have a common element.*

Proof. Let $\mathscr{G}_\alpha$, $\alpha < \mu < \mathfrak{d}$, be groupwise dense ideals, and let $\mathscr{A}_\alpha = \{\omega \setminus Z \mid Z \in \mathscr{G}_\alpha\}$ be the dual filters, which are non-feeble, because the $\mathscr{G}_\alpha$ are groupwise dense.

By FD, for each α there is some finite-to-one function f_α such that $f_\alpha(\mathscr{A}_\alpha)$ is an ultrafilter. These fewer than $\mathfrak{d}$ ultrafilters have according to [2, Theorem 19] a common finite-to-one image $\mathscr{U}$. So, by composing f_α with an appropriate finite-to-one map and renaming the result f_α, we may assume that $f_\alpha(\mathscr{A}_\alpha) = \mathscr{U}$ for all α.

We can also arrange that all the f_α are increasing.

Since $\omega \in \mathscr{A}_\alpha$ we have that $f_\alpha(\omega) = \text{range}(f_\alpha) \in \mathscr{U}$.

This means that the following functions are defined on $\mathscr{U}$-almost all arguments:

$$f_\alpha^{-1,upper}(n) = \max\{x \mid f_\alpha(x) = n\},$$
$$f_\alpha^{-1,lower}(n) = \min\{x \mid f_\alpha(x) = n\}.$$

Since we have NCF, we have that $\text{cf}(\mathscr{U}\text{−prod}\,\omega) = \mathfrak{d}$ and hence there is some $h \in \omega^{\uparrow\omega}$ that $\leq_{\mathscr{U}}$-dominates all the $f_\alpha^{-1,upper}$, $\alpha \in \mu$.

There is some $\ell' \in \omega^{\uparrow\omega}$ that is unbounded and $\leq^*$-dominates all the $f_\alpha, \alpha \in \mu$, and hence $f_\alpha(\omega)$ witnesses $\ell \leq_{\mathscr{U}} f_\alpha^{-1,lower}$, where $\ell(n) = \max\{x \mid \ell'(x) < n\}$, for $\alpha < \mu$.

By induction we define a sequence $a_0 < a_1 < \ldots$ by setting $a_0 = 0$ and choosing a_{n+1} so large that $\ell(a_{n+1}) > h(a_n)$. Let $X = \{h(a_n) \mid n \in \omega\}$.

We set

$$D_\alpha = \{n \in \omega \mid \ell(n) \leq f_\alpha^{-1,lower}(n) \wedge f_\alpha^{-1,upper}(n) \leq h(n)\} \in \mathscr{U}.$$

Then we have that $f_\alpha \restriction (f_\alpha^{-1}(D_\alpha) \cap X)$ is injective. Proof: Suppose that $x < y \in f_\alpha^{-1}(D_\alpha) \cap X$. Then there are $k < l$ such that $x = h(a_k)$ and $y = h(a_l)$. We have that $f_\alpha(x), f_\alpha(y) \in D_\alpha$. Hence we have that

$$\begin{aligned}&\ell(f_\alpha(x)) \leq f_\alpha^{-1,lower}(f_\alpha(x)) \leq x = h(a_k) < h(a_l)\\ &= y \leq f_\alpha^{-1,upper}(f_\alpha(y)) \leq h(f_\alpha(y)).\end{aligned}$$

Hence $f_\alpha(x) < a_{k+1}$ and $a_{k+1} \leq a_l \leq f_\alpha(y)$.

Let $\mathscr{Z}$ be a family of 2^ω almost disjoint subsets of X. By the injectivity property just proved we have that for each α the sets $f_\alpha(Z) \cap D_\alpha$, $Z \in \mathscr{Z}$, are almost disjoint, so at most one of them is in $\mathscr{U}$. Since there are $\mathfrak{c}$ $Z's$ and fewer than $\mathfrak{d}$ α's there is $Z \in \mathscr{Z}$ such that for all α

$$f_\alpha(Z) \cap D_\alpha \notin \mathscr{U}.$$

We fix such a Z.

Since $D_\alpha \in \mathscr{U}$ we have that $f_\alpha(Z) \notin \mathscr{U}$ and therefore $\omega \setminus f_\alpha(Z) \in \mathscr{U}$. Since $\mathscr{U} \subseteq f_\alpha(\mathscr{A}_\alpha)$ we get that $\omega \setminus f_\alpha^{-1}(f_\alpha(Z)) = f_\alpha^{-1}(\omega \setminus f_\alpha(Z)) \in \mathscr{A}_\alpha$, and therefore, by definition of $\mathscr{A}_\alpha$, that $Z \subseteq f_\alpha^{-1}(f_\alpha(Z)) \in \mathscr{G}_\alpha$.

Thus we have that Z belongs to all the given $\mathscr{G}_\alpha$'s. $\square_{3.4}$

Now we finish the proof of 3.3: The lemma gives some $Y \in \bigcap\{\mathscr{G}_*(X_\alpha, \mathscr{F}) \mid \alpha \in \mu\}$. $\square_{3.3}$

Theorem 3.5. *The filter dichotomy principle implies* $\mathfrak{u} < \mathfrak{g}_2$.

Proof. We show that $\mathfrak{g}_2 = \mathfrak{d}$.

This proof is easier than the previous one, because we already have ideals.

Fix an unbounded non-dominating ideal $\mathscr{I}$ in $\omega^{\uparrow\omega}$ that is closed under $=^*$. Let X_α, $\alpha \in \mu < \mathfrak{d}$ be given infinite sets. Note that $\mathscr{G}_2(X_\alpha, \mathscr{I})$ is a groupwise dense ideal, because for any $Z_0, Z_1 \in \mathscr{G}_2(X_\alpha, \mathscr{I})$, witnessed by f_0, f_1, we have

$$\max(f_0, f_1)(\mathrm{next}((Z_0 \cup Z_1), \cdot)) \geq \\ \min(f_0(\mathrm{next}(Z_0, \cdot)), f_1(\mathrm{next}(Z_1, \cdot))) \geq^* \mathrm{next}(X, \cdot).$$

Hence $\mathscr{A}_\alpha = \{\omega \setminus Z \mid Z \in \mathscr{G}_2(X_\alpha, \mathscr{I})\}$ is a non-feeble filter.

Hence we are again in a position to apply Lemma 3.4. □

Corollary 3.6. a) $\mathfrak{u} < \mathfrak{g}_1 \Leftrightarrow \mathfrak{u} < \mathfrak{g}$,
b) $\mathfrak{u} < \mathfrak{g}_2 \Leftrightarrow$ 4I $\Leftrightarrow \mathfrak{u} < \mathfrak{g}_3 \Leftrightarrow$ 4G $\Leftrightarrow \mathfrak{u} < \mathfrak{g}_4 \Leftrightarrow$ filter dichotomy,
c) $\mathfrak{u} < \mathfrak{g}_5 \Leftrightarrow$ near coherence of filters.

4. Pseudobases

In this section we show that in each of the inequalities $\mathfrak{u} < \mathfrak{g}_i$ the ultrafilter character $\mathfrak{u}$ can equivalently be replaced by $\mathfrak{r}$. For $i \leq 4$ this follows from the stronger result that $\mathfrak{r} < \mathfrak{g}_4$ implies $\mathfrak{r} = \mathfrak{u}$; for $i = 5$ we just have that $\mathfrak{r} < \mathfrak{g}_5$ implies that $\mathfrak{u} < \mathfrak{g}_5$. Note that Goldstern and Shelah [13] proved that $\mathfrak{r} < \mathfrak{u}$ is consistent relative to ZFC.

Theorem 4.1. $\mathfrak{r} < \mathfrak{g}_5$ *implies* $\mathfrak{u} < \mathfrak{g}_5$ *and* $\mathfrak{g}_5 = \mathfrak{d}$.

Proof. We check that the proof of "$\mathfrak{u} < \mathfrak{g}_5$ implies NCF" (see e.g. Proposition 3.2 e)) can be done with the apparently weaker premise $\mathfrak{r} < \mathfrak{g}_5$. We take into account that the X in the $\mathscr{G}_1(X, \mathscr{U})$ can just as well range only over a pseudo base. From NCF we get by 3.2 $\mathfrak{u} < \mathfrak{g}_5$.

The second equality in the conclusion follows from $\mathfrak{g}_5 = \mathfrak{mcf}$ and NCF $\rightarrow \mathfrak{mcf} = \mathfrak{d} = \mathrm{cf}(\mathfrak{d})$ [2]. □

Theorem 4.2. $\mathfrak{r} < \mathfrak{g}_4$ *implies* $\mathfrak{r} = \mathfrak{u}$ *and* $\mathfrak{g}_4 = \mathfrak{d} = \mathfrak{c}$ *and* $\mathfrak{u} = \mathfrak{b}$.

Proof. I am indebted to Andreas Blass for his hints how to shorten my original, too complicated proof. We re-do and modify Theorem 5 of [5]. First we prove as there, only replacing base by pseudo base:

Claim: If $\mathscr{F}$ is a non-feeble filter and $\mathscr{H}$ is a pseudobase (of some ultrafilter $\mathscr{U}$) of cardinality $\kappa < \mathfrak{g}_4(\mathscr{F})$ then there is a finite-to-one function f such that

$$f(\mathscr{H}) \subseteq \text{ some } \kappa\text{-generated filter } \subseteq f(\mathscr{F}).$$

For completeness' sake we give a proof: We may assume that $\mathscr{H}$ is closed under those finite intersections that are infinite sets. For each $H \in \mathscr{H}$ consider the groupwise dense set $\mathscr{G}_1(H, \mathscr{F})$. Since $|\mathscr{H}| = \kappa < \mathfrak{g}_4(\mathscr{F})$, there is some X common to all $\mathscr{G}_1(H, \mathscr{F})$. Let X be strictly increasingly eumerated by $\langle x_i \mid i \in \omega \rangle$ and define $f\colon \omega \to \omega$ by letting $f(n) = i$ if $n \in [x_{i-1}, x_i)$ (with $x_{-1} = 0$). Then we have that $\{f[H] \mid H \in \mathscr{H}\}$ is a base for a filter that is contained in $f(\mathscr{F})$ $\square_{\text{claim}}$

Then we take for $\mathscr{H}$ a pseudobase of cardinality $\kappa = \mathfrak{r}$ for some ultrafilter $\mathscr{U}$ and get that $\{f[H] \mid H \in \mathscr{H}\}$ is a base for $f(\mathscr{U}) = f(\mathscr{F})$ of cardinality $\mathfrak{r}$.

The second and the third equalities in the conclusion follow the fact that FD implies $\mathfrak{u} = \mathfrak{b} \leq \mathfrak{d} = \mathfrak{c}$ ([5]) and the

Claim: FD implies $\mathfrak{g}_4 = \mathfrak{d}$.

Proof of the claim. The first proof is by the proof of Theorem 3.3. However, with the help of Theorem 4.1, we give here a shorter proof: Let $\langle \pi_i \mid i \in \omega \rangle$ be a strictly increasing sequence of natural numbers. We set $\pi_{-1} = 0$ and take $f\colon \omega \to \omega$ such that $f(n) = i$ for $n \in [\pi_{i-1}, \pi_i)$. Now we claim for any filter $\mathscr{F}$,

$$\mathfrak{g}_4(\mathscr{F}) \geq \mathfrak{g}_4(f(\mathscr{F})).$$

In order to show this inequality, fix some $\mu < \mathfrak{g}_4(f(\mathscr{F}))$. Let X_α, $\alpha < \mu$, be infinite subsets of ω. We show that $\bigcap_{\alpha<\mu} \mathscr{G}_1(X_\alpha, \mathscr{F}) \neq \emptyset$. Since $\mu < \mathfrak{g}_4(f(\mathscr{F}))$, we have that $\bigcap_{\alpha<\mu} \mathscr{G}_1(f[X_\alpha], f(\mathscr{F})) \neq \emptyset$, and we take a witness Z_0 for this. Now $\pi[Z_0] \in \bigcap_{\alpha<\mu} \mathscr{G}_1(X_\alpha, \mathscr{F})$ follows from:

Subclaim. For all infinite sets Z and X, if $Z \in \mathscr{G}_1(f[X], f(\mathscr{F}))$ then $\pi[Z] \in \mathscr{G}_1(X, \mathscr{F})$.

Let $Z \in \mathscr{G}_1(f[X], f(\mathscr{F}))$ be witnessed by $Y \in f(\mathscr{F})$. Hence we have for all $a < b$ in Z, if $[a, b) \cap Y \neq \emptyset$, then $[a, b) \cap f[X] \neq \emptyset$. We take two elements $c = \pi(a)$ and $d = \pi(b)$ in $\pi[Z]$ with $c < d$. Then $[c, d) \cap f^{-1}[Y] \neq \emptyset$ implies $[a, b) \cap Y \neq \emptyset$ and hence $[a, b) \cap f[X] \neq \emptyset$ and $[c, d) \cap X \neq \emptyset$.

Hence $f^{-1}[Y]$ witnesses that $\pi[Z] \in \mathscr{G}_1(X, \mathscr{F})$ and the subclaim and the displayed inequality are proved.

In order to finish the proof of the claim we use FD and apply the subclaim with some finite-to-one f (of the above orderly interval form) such that $f(\mathscr{F})$ is an ultrafilter. Such an f exists by [2]. Then we use that FD implies NCF. Hence Theorem 4.1 yields the last equality in the following chain: $\mathfrak{g}_4(\mathscr{F}) \geq \mathfrak{g}_4(f(\mathscr{F})) = \mathfrak{g}_5(f(\mathscr{F})) \geq \mathfrak{g}_5 = \mathfrak{d}$. $\square_{4.2}$

We close this section with another theorem about interrelations:

Theorem 4.3. *FD implies* $\mathfrak{s} \leq \mathfrak{g}$.

Proof. Let fewer than $\mathfrak{s}$ groupwise dense sets $\mathscr{G}_\alpha$, $\alpha \in \mu < \mathfrak{s}$, be given. We show that they have a common element.

Since $\mathfrak{s} \leq \mathfrak{d}$ we have that $\mu < \mathfrak{d}$, and by 3.4 we can find an infinite X that is in $I(\mathscr{G}_\alpha)$ for every $\alpha \in \mu$ where $I(\mathscr{G})$ denotes the ideal generated by $\mathscr{G}$.

So for every α, X is the union of finitely many members of $\mathscr{G}_\alpha$, that is $X = \bigcup_{i<n_\alpha} X(i,\alpha)$, $X(i,\alpha) \in \mathscr{G}_\alpha$. The $X(i,\alpha)$, $i < n_\alpha$, $\alpha < \mu$, do not form a splitting family on X, because $\mu < \mathfrak{s}$. Hence there is some infinite $Z \subseteq X$ such that for every α there is some $i < n_\alpha$ such that $Z \subseteq^* X(i,\alpha)$. Since the $\mathscr{G}_\alpha$ are open we have that $Z \in \mathscr{G}_\alpha$. □

Corollary 4.4. FD and $\mathfrak{s} > \mathfrak{u}$ implies $\mathfrak{u} < \mathfrak{g}$.

Proof. By the previous theorem FD implies that $\mathfrak{s} \leq \mathfrak{g}$. Now our second hypothesis allows us to put things together. □

Since FD implies $\mathfrak{u} < \mathfrak{d}$ we also have

Corollary 4.5. FD and $\mathfrak{s} = \mathfrak{d}$ implies $\mathfrak{u} < \mathfrak{g}$.

5. A better upper bound for $\mathfrak{s}$

We prove $\mathfrak{s} \leq \mathrm{cf}(\mathfrak{d})$, thus improving the well-known inequality $\mathfrak{s} \leq \mathfrak{d}$. Thus we have the following diagram:

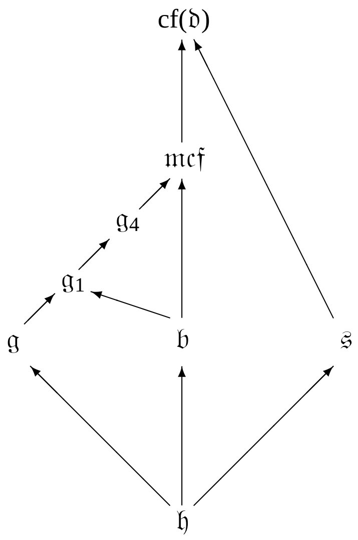

Fig. 1.

The lower three inequalities and $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$ follow immediately for the definitions or are very easy to see (see [18]). The inequality $\mathfrak{g} \leq \mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)$ was proved in [8], and Canjar [11] proved $\mathfrak{mcf} \leq \mathrm{cf}(\mathfrak{d})$. From work in [17,9,14,10] or from more explicit unpublished work of Eisworth one can put together examples showing that each of the eight assignments of $\aleph_1$ or $\aleph_2$ to each of $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{s}$ is realized in some model. The relations to the other entries of Cichoń's Diagram are $\mathfrak{s} \leq \mathrm{unif}(\mathscr{M})$ and $\mathfrak{s} \leq \mathrm{unif}(\mathscr{N})$ and all that follows from transitivity. In [8] we have shown that the splitting number may be larger than $\mathfrak{mcf}$.

Theorem 5.1. $\mathfrak{s} \leq \mathrm{cf}(\mathfrak{d})$

Proof. We assume that $\mathfrak{s} > \mathrm{cf}(\mathfrak{d})$ and work towards a contradiction. First, we take one of Canjar's ultrafilters (see [11]) such that $\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega) = \mathrm{cf}(\mathfrak{d}) < \mathfrak{s}$. We note that for such an ultrafilter we have

$$\begin{array}{l} \forall \text{ finite-to-one } f \\ f(\mathscr{U}) \text{ is not (pseudo) generated by } < \mathfrak{d} \text{ sets.} \end{array} \tag{$*$}$$

Proof of $(*)$: It is easy to see that for any finite-to-one f we have that $\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega) = \mathrm{cf}(f(\mathscr{U}){-}\mathrm{prod}\,\omega) = \mathrm{cf}(\mathfrak{d}) < \mathfrak{d}$. Together with the well-known inequality $\mathrm{cf}(f(\mathscr{U}){-}\mathrm{prod}\,\omega) \cdot \pi\chi_{f(\mathscr{U})} \geq \mathfrak{d}$ (see 3.1) we get $\pi\chi_{f(\mathscr{U})} \geq \mathfrak{d}$.

We fix a dominating family $\mathscr{D}$ of size $\mathfrak{d}$. By Theorem 15 of [2] there is a set $\mathscr{D}^+ \cup \mathscr{D}^-$ of size $\mathfrak{d}$, such that for any two filters, if they are nearly coherent then this is witnessed by an $f \in \mathscr{D}^+ \cup \mathscr{D}^-$. We also fix such a $\mathscr{D}^+ \cup \mathscr{D}^-$.

Now we construct, combining the proofs of Theorem 15 of [2] and of the theorem of [11], a second ultrafilter $\mathscr{V}$ such that

$$\begin{array}{l} \mathrm{cf}(\mathscr{V}{-}\mathrm{prod}\,\omega) = \mathrm{cf}(\mathfrak{d}) < \mathfrak{s}, \text{ and} \\ \forall f \in \mathscr{D}^+ \cup \mathscr{D}^- \quad f(\mathscr{U}) \neq f(\mathscr{V}). \end{array}$$

By the choice of $\mathscr{D}^+ \cup \mathscr{D}^-$, the second condition implies that for every finite-to-one f we have that $f(\mathscr{U}) \neq f(\mathscr{V})$, i.e. that $\mathscr{U}$ and $\mathscr{V}$ are not nearly coherent. Thus we have a contradiction to Theorem 8 of [8]: If $\mathrm{cf}(\mathscr{U}{-}\mathrm{prod}\,\omega)$ and $\mathrm{cf}(\mathscr{V}{-}\mathrm{prod}\,\omega)$ are smaller than $\mathfrak{s}$, then $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent.

Construction of $\mathscr{V}$. We let $\mathscr{D}^+ \cup \mathscr{D}^-$ be enumerated as $\{f_\alpha \mid \alpha < \mathfrak{d}\}$, and we let $\mathscr{D}$ be enumerated as $\{h_\alpha \mid \alpha < \mathfrak{d}\}$. By induction on α we define $\mathscr{V}_\alpha$, A_α, and H_α such that

1. $\mathscr{V}_0$ is the set of cofinite sets.
2. For limit λ, $\mathscr{V}_\lambda = \bigcup\{\mathscr{V}_\alpha \mid \alpha < \lambda\}$.
3. $\mathscr{V}_{\alpha+1}$ is generated by $\mathscr{V}_\alpha \cup \{A_\alpha\} \cup (|\alpha| + \omega)$ many suitable sets.
4. $f_\alpha[A_\alpha] \notin f_\alpha(\mathscr{U})$.
5. $\{n \mid h_\alpha(n) \leq H_\alpha(n)\} \in \mathscr{V}_{\alpha+1}$.
6. For all $\beta < \alpha$, $\{n \mid H_\beta(n) \leq H_\alpha(n)\} \in \mathscr{V}_{\alpha+1}$.

7. $\mathscr{V} \supseteq \bigcup\{\mathscr{V}_\alpha \mid \alpha < \mathfrak{d}\}$. (So, if $V_\mathfrak{d}$ is not an ultrafilter, then we add more sets to it until it is an ultrafilter, and call this ultrafilter $\mathscr{V}$.)

If this construction is accomplished, then for any $\langle \alpha_i \mid i < \mathrm{cf}(\mathfrak{d})\rangle$ cofinal in $\mathfrak{d}$ we have that $\langle H_{\alpha_i} \mid i < \mathrm{cf}(\mathfrak{d})\rangle$ is cofinal in $\omega^\omega/\mathscr{V}$, and that $\mathscr{U}$ and $\mathscr{V}$ are not nearly coherent.

Only the successor step requires some work: Let $\mathscr{V}_\alpha$ and $\langle H_\beta \mid \beta < \alpha\rangle$ be already chosen. We first select A_α such that $\mathscr{V}_\alpha \cup \{A_\alpha\}$ has the strong finite intersection property and that $f_\alpha[A_\alpha] \notin f_\alpha(\mathscr{U})$.

With the help of $(*)$, this is done exactly as in Blass' proof of Theorem 15 in [2]. For completeness' sake, we insert his argument here:

It suffices to find a $B \in f_\alpha[\mathscr{U}]$ such that $f_\alpha^{-1}[B] \notin \mathscr{V}_\alpha$, for we can then set $A_\alpha = \omega \setminus f_\alpha^{-1}[B]$.

Indeed, since $\mathscr{V}_\alpha$ is a filter not containing $f_\alpha^{-1}[B]$ it contains no subsets of $f_\alpha^{-1}[B]$, i.e. no sets disjoint from A_α. Furthermore $f_\alpha[A_\alpha]$ is disjoint from B, hence not in $f_\alpha(\mathscr{U})$.

To complete the proof, we suppose that no B of the desired sort exists and derive a contradiction. The supposition means that each $B \in f_\alpha(\mathscr{U})$ also belongs to $f_\alpha(\mathscr{V}_\alpha)$. But by our inductive hypotheses $f_\alpha(\mathscr{V}_\alpha)$ is generated by fewer than $\mathfrak{d}$ sets. This contradicts $(*)$ that no finite-to-one image (such as $f_\alpha(\mathscr{U}) = f_\alpha(\mathscr{V}_\alpha)$) is generated by fewer than $\mathfrak{d}$ sets.

Then we add $|\alpha| + \omega$ suitable elements to $\mathscr{V}_\alpha \cup \{A_\alpha\}$ such that the resulting union generates a filter $\mathscr{V}_{\alpha+1}$ and such that there is some H_α such that $\{n \mid h_\alpha(n) \leq H_\alpha(n)\} \in \mathscr{V}_{\alpha+1}$ and for all $\beta < \alpha$, $\{n \mid H_\beta(n) \leq H_\alpha(n)\} \in \mathscr{V}_{\alpha+1}$. This is done as in [11]. Again, for completeness' sake we add a proof: Let $\{B_\alpha \mid \alpha < \lambda\}$ be a generating set for $\mathscr{V}_\alpha$ which is closed under finite intersections and which has cardinality $\lambda < \mathfrak{d}$. We may assume that $\{H_\beta \mid \beta \in \alpha\}$ is closed under finite maxima and contains only increasing functions. Since $|\alpha \cdot \lambda| < \mathfrak{d}$ there is an $H_\alpha \in \omega^\omega$ such that

$$\forall \beta \in \alpha : \forall \tau \in \lambda \; H_\beta(\mathrm{next}(B_\tau, \cdot)) \not\geq^* H_\alpha.$$

We show that for such an H_α, the set

$$\mathscr{V}_\alpha \cup \{\{n \mid H_\beta(n) \leq H_\alpha(n)\} \mid \beta \in \alpha\}$$

has the strong finite intersection property. By the closure properties imposed on $\{B_\alpha \mid \alpha < \lambda\}$ and on $\{H_\beta \mid \beta \in \alpha\}$ it suffices to show that for any $\beta \in \alpha$ and $\tau \in \lambda$ the intersection $B_\tau \cap \{n \mid H_\beta(n) \leq H_\alpha(n)\}$ is infinite. This is true because $B_\tau \cap \{n \mid H_\beta(n) \leq H_\alpha(n)\} \supseteq \{n \mid H_\beta(\mathrm{next}(B_\tau, n)) \leq H_\alpha(n)\} \cap B_\tau$ and the latter set is infinite because, by our choice of H_α, we have that $X = \{n \mid H_\beta(\mathrm{next}(B_\tau, n)) \leq H_\alpha(n)\}$ is infinite and for each $n \in X$ we have $\mathrm{next}(B_\tau, n) \in \{n \mid H_\beta(n) \leq H_\alpha(n)\} \cap B_\tau$. $\square$

6. Concluding remarks

There are other families of groupwise dense sets of a special form that are useful in the study of filters and cardinal characteristics. For a non-feeble filter $\mathscr{F}$ and $f \in \omega^\omega$,

$$\mathscr{G}_f = \{X \in [\omega]^\omega \mid \{n \mid f(n) < \mathrm{next}(X, n)\} \in \mathscr{F}\}$$

from [8, Theorem 16] is groupwise dense and $\{\mathscr{G}_f \mid f \in \omega^\omega\}$ is useful in connection with the unboundedness number of the partial orders $<_{\mathscr{F}}$. Another type of groupwise dense set is: For $\mathscr{Y} \subseteq [\omega]^\omega$, $|\mathscr{Y}| < \mathfrak{c}$,

$$\mathscr{G}_{\mathscr{Y}} = \{X \in [\omega]^\omega \mid \forall Y \in \mathscr{Y}\ Y \not\subseteq^* X\},$$

which is used in [6, 8.6] to show that $\mathfrak{g} \leq \mathrm{cf}(\mathfrak{c})$. These types of groupwise dense families are not (yet) put into relation with the ones from Definition 1.1. So far, there seem to be propitious special forms for each purpose, but not one set of "smallest" or best groupwise dense families, easily definable and witnessing the computation of $\mathfrak{g}$.

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