Changing Cardinal Invariants of the Reals Without Changing Cardinals or the Reals

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Abstract

We show: The procedure mentioned in the title is often impossible. It requires at least an inner model with a measurable cardinal. The consistency strength of changing $\mathfrak{b}$ and $\mathfrak{d}$ from a regular κ to some regular $\delta < \kappa$ is a measurable of Mitchell order δ. There is an application to Cichoń's diagram.

1 Introduction

In order to show the consistency of one or more cardinal characteristics having prescribed values, e.g. $\mathfrak{b} = \aleph_1$, $\mathfrak{d} = \aleph_2$, $\mathfrak{c} = \aleph_3$ or $\mathfrak{u} < \mathfrak{g}$, the known technique is to add certain reals in a certain iteration manner. Obviously one can change some constellations merely by collapsing cardinals. But if we do not want to use either of these techniques, numerous questions arise:

> If $W \subseteq U$ are transitive models of ZFC with the same reals and the same cardinals, is there a cardinal invariant of the reals that is not the same in W and in U?

We use Vojtáš's framework [15] in which cardinal characteristics of the continuum can be regarded as norms of corresponding relations $\mathbf{A} = (A_-, A_+, A)$ with A_-,$A_+ \subseteq 2^\omega$, $A \subseteq A_- \times A_+$, and the norm

$$||\mathbf{A}|| = \min\{\text{card}(\mathcal{Z}) \,:\, \mathcal{Z} \subseteq A_+ \wedge \forall x \in A_- \,\exists z \in \mathcal{Z}\, A(x, z)\}.$$

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We concentrate on the case that A, A_- and A_+ are absolute relations, indeed, in our examples they will be Borel relations. We often write aAb instead of $A(a,b)$. $||\cdot||^W$ denotes the norm as computed in W.

Section 2 deals with situations in which some cardinal invariants cannot be changed without changing cardinals or the reals. Section 3 shows the consistency of changing cardinal invariants without changing cardinals or the reals relative to a measurable of high (the new cofinality) Mitchell order and the equiconsistency result. We show the following

Theorem 1.1 *If ZFC + "there is a measurable cardinal κ of Mitchell order $o(\kappa)=\delta$, $\omega_1 \leq \delta < \kappa$, δ regular" is consistent then the following is consistent:*

> *There are models $W \subset U$ of ZFC such that W and U have the same cardinals and the same reals, $W \models MA$ (and hence $\mathfrak{b}=\mathfrak{d}=\mathfrak{c}$), and $U \models$ "$\mathfrak{b}$ and $\mathfrak{d}$ are equal to δ less than $\mathfrak{c}$".*

Mitchell's work [12] gives the lower bound of the consistency strength of such a change:

Theorem 1.2 *If there is a model M of ZFC and an extension N, such that M and N have the same cardinals, and there is a cardinal κ regular in M that has uncountable cofinality $\delta < \kappa$ in N then there is an inner model with a measurable cardinal κ of Mitchell order $o(\kappa)=\delta$.*

Notation: Notation not defined here is taken from [7]. For the definition of the Mitchell order, see [11]. $\mathfrak{c}$ denotes the cardinality of the continuum. MA is Martin's Axiom for fewer than $\mathfrak{c}$ dense sets. For $f,g \in \omega^\omega$, we write $f \leq^* g$ iff $\exists n\, \forall k \geq n\; f(k) \leq g(k)$. The (un)bounding number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$ are defined as follows:

$$\begin{aligned} \mathfrak{b} &= \min\{\mathrm{card}(\mathcal{B}) : \forall f \in \omega^\omega\, \exists g \in \mathcal{B}\; g \not\leq^* f\}, \\ \mathfrak{d} &= \min\{\mathrm{card}(\mathcal{D}) : \forall f \in \omega^\omega\, \exists g \in \mathcal{D}\; f \leq^* g\}. \end{aligned}$$

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2 Characteristics may be preserved

Changing the norm of an absolute relation over the reals without changing the reals and the cardinals (hence: decreasing the norm) has strength of a measurable cardinal in an inner submodel of the lower model.

Proposition 2.1 *If $W \subset U$ have the same cardinals and there is a relation $\mathbf{A}^W = \mathbf{A}^U = \mathbf{A}$ and $||\mathbf{A}||^W > ||\mathbf{A}||^U \geq \aleph_1$, then in W there is an inner model with a measurable cardinal.*

Proof: If there is no inner model with a measurable cardinal W, then by [4] W is covered by K^W. As W and U have the same cardinals, we have $K^U = K^W$. This fact is a folklore result and in the hard case, when there in no inner model with a measurable cardinal in both of them, the proof involves a coiteration argument, see also [1] for the case of set generic extensions.

Hence $\forall Z' \in U \; \exists Z \in K^W \; (Z \supseteq Z'$ and $\mathrm{card}^W(Z) = \mathrm{card}^U(Z) \leq \mathrm{card}^U(Z') + \aleph_1)$. Any set of witnesses Z' for $||\mathbf{A}||^U$ can be covered by a set in W of the same cardinality. □

Since changing an invariant in the prescribed manner violates covering below the continuum, the hypothesis can also be changed and gives:

Proposition 2.2 *If $W \subset U$ have the same cardinals below $\mathfrak{c}$ and $\mathfrak{c}$ is a limit cardinal and there is a relation $\mathbf{A}^W = \mathbf{A}^U = \mathbf{A}$ and $||\mathbf{A}||^W > ||\mathbf{A}||^U \geq \aleph_1$, then in W there is an inner model with a measurable cardinal.*

Proof: Under these premises, the Dodd Jensen core models K^U and K^W agree on subsets of the reals of cardinality less than the continuum, hence on witnesses for $||\mathbf{A}||^U$, if this is less than the continuum. □

We fix the scenario: $W \subseteq U$ are transitive models of ZFC. $\mathbf{A} = (A_-, A_+, A)$ is a relation such that $\mathbf{A}$ is $\mathbf{\Sigma}^1_2$.

We require cardinals to be the same in W and in U in order to exclude trivial examples.

Proposition 2.3 (Blass) *If A is transitive, $A^W_- \supseteq A^U_+$ and $||\mathbf{A}^W||^W$ is regular in U, then in U the inequality $||\mathbf{A}^W||^W \leq ||\mathbf{A}^U||^U$ is true.*

Proof: Let $\mathcal{Z} = \{z_\alpha : \alpha < \mu\}$ witness $||\mathbf{A}^W||^W = \mu$, and $\mathcal{Z}' = \{z'_\alpha : \alpha < \mu'\}$ witness $||\mathbf{A}^U||^U = \mu'$. Since $A^U_+ \subseteq A^W_-$, in U there is a function $h\colon \mu' \to \mu$ such that for $\alpha < \mu$:

$$z'_\alpha A z_{h(\alpha)}.$$

If μ' were less than μ, then range(h) would be bounded in μ, say by a bound $\beta \in \mu$.

Then $\forall a \in A^W_- \, \exists \alpha \in \mu' \; aAz'_\alpha A z_{h(\alpha)}$. Hence $\{z_\alpha : \alpha \leq \beta\}$ is a witness for $||\mathbf{A}||^W \leq \text{card}(\beta) < \mu$. □

If we keep all the premises of the proposition except for the condition that $||\mathbf{A}^W||^W$ is regular in U, with the same proof we get in U the inequality $\text{cf}(||\mathbf{A}^W||^W) \leq ||\mathbf{A}^U||^U$.

We extract a scheme from the proof of proposition 2.3 that describes the situation of not necessarily transitive relations:

Proposition 2.4 *Let $\mathcal{Z} = \{z_\alpha : \alpha < \mu\}$ witness $||\mathbf{A}^W||^W = \mu$, and $\mathcal{Z}' = \{z'_\alpha : \alpha < \mu'\}$ witness $||\mathbf{A}^U||^U = \mu'$. If in U there is a function $h\colon \mu' \to \mu$ such that for $\alpha < \mu$:*

$$\{a \in A^U_- : aAz'_\alpha\} \subseteq \{a \in A^W_- : \exists \beta \in h(\alpha) \; aAz_\beta\}$$

and $||\mathbf{A}||^W$ is regular in U, then in U the inequality $||\mathbf{A}^W||^W \leq ||\mathbf{A}^U||^U$ is true.

The proof is the same as that of 2.3. □

If $A^W_- = A^U_-$, then $||\mathbf{A}^W||^W \geq ||\mathbf{A}^U||^U$, and hence under the premises of the propositions, they will be equal.

We require from now on that additionally W and U have the same reals. Then $A^W_- = A^U_-$ is true (or can be arranged by choosing suitable cofinal subsets of the ideals) for the relations corresponding to the Cichoń diagram and many others from [14]. We consider some well-known examples from [14]. Let I be an ideal of subsets of the real line $\mathbb{R}$. The additivity, covering number, uniformity, and cofinality of the ideal are defined by:

$$\text{add}(I) = \min\{\mathcal{Z} : \mathcal{Z} \subseteq I \text{ and } \bigcup \mathcal{Z} \notin I\},$$

$$\begin{aligned}
\mathrm{cov}(I) &= \min\{\mathcal{Z} : \mathcal{Z} \subseteq I \text{ and } \bigcup \mathcal{Z} = \mathbb{R}\},\\
\mathrm{unif}(I) &= \min\{\mathcal{Z} : \mathcal{Z} \subseteq \mathbb{R} \text{ and } \mathcal{Z} \notin I\},\\
\mathrm{cof}(I) &= \min\{\mathcal{Z} : \mathcal{Z} \subseteq I \text{ and } \forall B \in I\, \exists Z \in \mathcal{Z}\ B \subseteq Z\}.
\end{aligned}$$

Superscripts U, W denote in which model the corresponding invariant is computed. In any fixed model of ZFC we have: If I' is generated by I, i.e. $\forall x \in I'\, \exists y \in I\ x \subseteq y$, then $\mathrm{add}(I') = \mathrm{add}(I)$ and so on. For I being the meager or the Lebesgue null ideal, we have I^U is generated by I^W, if $\mathbb{R}^W = \mathbb{R}^U$, as the ideals are generated by the set of meager F_σ-sets and by the set of G_δ-nullsets respectively. Also for the ideal K_σ of countable unions of compact sets there are the same generating sets in W and in U if W and U have the same reals. By abuse of notation, we often write I. It shall be clear from the context which interpretation is meant.

Proposition 2.5 *Suppose I^U is generated by I^W, $\mathbb{R}^W = \mathbb{R}^U = \mathbb{R}$.*
a) If in W, $\mathrm{cov}(I) = \mathrm{cof}(I)$ and this is regular in U, then in U, $\mathrm{cov}(I) = \mathrm{cof}(I) = \mathrm{cov}^W(I)$.
b) If in W, $\mathrm{add}(I) = \mathrm{cof}(I)$ and this is regular in U, then in U, $\mathrm{add}(I) = \mathrm{cof}(I) = \mathrm{add}^W(I)$. For $I = K_\sigma$, the ideal of countable unions of compact sets, this reads: If in W, $\mathfrak{d} = \mathfrak{b}$ and these remain regular in U, then in U, $\mathfrak{d}^U = \mathfrak{b}^U = \mathfrak{d}^W$.
c) If in W, $\mathrm{add}(I) = \mathrm{cov}(I)$ and this is regular in U, then in U, $\mathrm{unif}(I) \geq \mathrm{cov}^W(I)$.

Proof: a) Let $\{z'_\alpha : \alpha < \mu'\}$ be in U a covering of $\mathbb{R}$ with elements from I. Let $\{z_\alpha : \alpha < \mu\}$ be in W a cofinal subfamily of I. $\{a \in \mathbb{R} : a \in z'_\alpha\} \subseteq \{a \in \mathbb{R} : a \in z_{\tilde{h}(\alpha)}\}$ for some $\tilde{h}(\alpha)$ such that $z'_\alpha \subseteq z_{\tilde{h}(\alpha)}$. Hence $h(\alpha) = \tilde{h}(\alpha) + 1$ is as required in the previous proposition.

b) Let $\{z'_\alpha : \alpha < \mu'\} \subseteq I$ be in U with $\bigcup\{z'_\alpha : \alpha < \mu'\} \notin I$. Again, let $\{z_\alpha : \alpha < \mu\}$ be in W a cofinal subfamily of I. $\{a \in I : a \subseteq z'_\alpha\} \subseteq \{a \in I : a \subseteq z_{\tilde{h}(\alpha)}\}$ for some $\tilde{h}(\alpha)$ such that $z'_\alpha \subseteq z_{\tilde{h}(\alpha)}$. For the additivity this yields: $\bigcup\{z'_\alpha : \alpha < \mu'\} \notin I$ implies $\bigcup\{z_{h(\alpha)} : \alpha < \mu'\} \notin I$.

c) Let $\{z'_\alpha : \alpha < \mu'\} \subseteq \mathbb{R}$ in U. Let $\{z_\alpha : \alpha < \mu\}$ be in W a covering subfamily of I and let us assume $\mu' < \mu$ and μ is (still) regular in U. $\{a \in I : z'_\alpha \notin a\} \subseteq \{a \in I : z_{\tilde{h}(\alpha)} \not\subseteq a\}$ for some $\tilde{h}(\alpha)$ such that $z'_\alpha \in z_{\tilde{h}(\alpha)}$.

Set $s = \sup\{\tilde{h}(\alpha) : \alpha \in \mu'\}$. Since μ is regular in U, $s < \mu$. Since $s < \mathrm{add}(I)$, we have $\{z'_\alpha : \alpha < \mu'\} \subseteq \bigcup_{\alpha<\mu'} z_{\tilde{h}(\alpha)} \subseteq \bigcup_{\alpha\in s} z_\alpha \in I(\cap V)$. □

We do not have any use for the full extent of proposition 2.4, as we only need singletons as values of h. The regularity in U is a necessary condition in 2.3, 2.4, and 2.5, as we will see in the next sections.

3 Changing Scales

In this section we prove theorem 1.1 and give for completeness' sake some hints on the proof of theorem 1.2. We start from the premise that there is a measurable κ of Mitchell order δ, $\omega_1 \leq \delta < \kappa$, δ a regular cardinal. The main ingredient of the proof is taken from [6]. We use the following

Fact 3.1 *Let M, N be inner models of ZFC, $M \subseteq N$, $N \models {}^{\mu}M \subseteq M$. Let $P \in M$ be a forcing notion, such that $N \models P$ is μ^+-c.c., and let G be P-generic over N. Then $N[G] \models {}^{\mu}(M[G]) \subseteq M[G]$.*

Proof: See [7], §37 or, for a more explicit statement, [2].

Lemma 3.2 *Suppose V is a model of $\forall\alpha < \kappa\ \alpha^{\omega_1} < \kappa$ and in V there is an ω-distributive forcing P_1 that preserves cardinals and changes the cofinality of κ into δ without adding a bounded subset of κ. Let P in V be a c.c.c. forcing that forces $MA + \mathfrak{c} = \kappa$, G_1 be P_1-generic over V and G be P-generic over $V[G_1]$. Then $V[G]$ and $V[G_1][G]$ are as stated in theorem 1.1, i.e.*

1) *$V[G] \subset V[G_1][G]$ are models of ZFC,*

2) *they have the same reals, indeed the same ω-sequences with ranges in $V[G]$,*

3) *$V[G]$ is a model of $MA + 2^\omega = \kappa$,*

4) *$V[G]$ and $V[G_1][G]$ have the same cardinals,*

5) *in $V[G_1][G]$, $\mathfrak{d} = \mathfrak{b} = \delta$.*

Proof: For 2, we apply the fact 3.1 and that P has c.c.c. in $V[G_1]$, which is proved below under 4.

Ad 4: P_1 preserves cardinals, so V and $V[G_1]$ have the same cardinals. We show that P has c.c.c. in $V[G_1]$. We suppose the contrary: P_1 adds a new uncountable antichain A to P. In $V[G_1]$, P is still a iteration of forcings of cardinality less than κ of iteration-length κ with finite supports. Hence the Δ-lemma (Ch. II, theorem 1.6 in [8]) gives a finite root r for the supports of all the conditions in an uncountable subset A' of A. The forcings whose preimage in the iteration is a subset of $\max(r)+1$ are (after a suitable injection) a subset of an ordinal below κ, because $(\max(r)+1)^{\omega_1} < \kappa$. Since Gitik's forcing P_1 does not add any bounded subset of κ, there is no new uncountable antichain in the forcings attached to a subset of $\max(r)+1$. As every old antichain is countable, among $\{p \restriction (\max(r)+1) \, : \, p \in A'\}$ there are two compatible or same ones belonging to different p's. These yield two compatible elements of A.

Ad 5: In $V[G]$, $MA + \mathfrak{c} = \kappa$ holds and hence there is an increasing cofinal sequence $\langle f_\beta \, : \, \beta \in \kappa \rangle$ in $(\omega^\omega, \leq^*)$. In $V[G_1][G]$ there are also κ reals, but now κ has cofinality δ and we can choose a subsequence of $\langle f_\beta \, : \, \beta \in \kappa \rangle$ in $(\omega^\omega, \leq^*)$ whose indices are cofinal in κ. Since there are no additional reals, this subsequence is cofinal in $(\omega^\omega, \leq^*)$. $\square$

In order to get a model V of $\forall \alpha \in \kappa \; \alpha^{\omega_1} < \kappa$ where a forcing P_1 with the above nice yet strong properties exists, we rely on [6]:

Fact 3.3 (Gitik) *If there is a measurable cardinal κ of Mitchell order δ, $\omega_1 \leq \delta < \kappa$, then the following is consistent with ZFC: GCH, κ is inaccessible and there is a κ^+-c.c. forcing notion that does not add bounded subsets to κ and does force* $\mathrm{cf}(\kappa) = \delta$.

Such a forcing notion does not destroy cardinals and does not add a sequence of length $< \delta$: It is $(<\delta, \kappa)$-distributive, and therefore $<\delta$-distributive because of the κ^+-c.c. 3.2 and 3.3 together prove theorem 1.1: We take $W = V[G]$, $U = V[G_1][G]$. $\square$

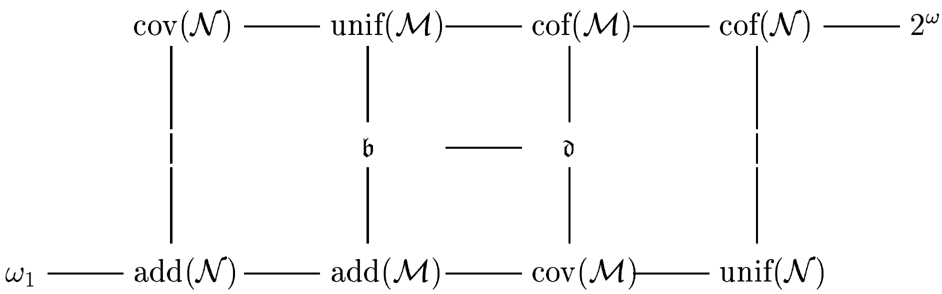
Now we sketch a proof of theorem 1.2: We use the core model $\mathbf{K} = K(\vec{U}_{max})$ of [13]. In [12] there is the following theorem:

Theorem 3.4 (Mitchell) *Suppose κ is a cardinal in V, κ is regular in* $\mathbf{K}$, *and* $\mathrm{cf}(\kappa) = \delta < \kappa$ *in V. Then $o(\kappa) \geq 1$ in* $\mathbf{K}$, *and if $\delta > \omega$, then $o(\kappa) \geq \delta$ in* $\mathbf{K}$.

We relativize (in the sense of model theory) this fact: Assume we have a model M of ZFC and an extension N with the same cardinals, and that κ is a cardinal in N, κ is regular in M, whereas $\mathrm{cf}(\kappa) = \delta < \kappa$, $\delta > \omega$, in N. Then κ is regular in $\mathbf{K}^M$, as this is a submodel of M. Since $\mathbf{K}^M = \mathbf{K}^N$ (folklore as in Proposition 2.1), κ in regular in $\mathbf{K}^N$. Hence theorem 3.4 applied in N yields $o(\kappa) \geq \delta$ in $\mathbf{K}^M$. $\square$

4 Application to Cichoń's diagram

Let $\mathcal{N}$ be the ideal of Lebesgue null subsets of the real line, and let $\mathcal{M}$ be the ideal of meager subsets. The following partial order is called Cichoń's diagram:



The invariants further up or right from an entry are greater or equal than that entry; proofs can be found in [5]. Under MA, all these invariants except ω_1 are equal to $\mathfrak{c}$, cf. [10]. Moreover, as $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ and $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$, there are $\subseteq$-increasing sequences of length κ that are cofinal in $\mathcal{M}$ or $\mathcal{N}$, respectively. In $V[G_1][G]$ of lemma 3.2, all invariants except ω_1 are equal to δ. Hence we have the

Theorem 4.1 *If ZFC + "there is a measurable cardinal κ of Mitchell order $o(\kappa) = \delta$, $\omega_1 \leq \delta < \kappa$" is consistent then the following is consistent:*

There are models $W \subset U$ of ZFC such that W and U have the same cardinals and the same reals, in W the cardinals in the Cichoń diagram are equal to $\mathfrak{c} > \omega_1$, and in U these cardinals are equal to $\delta < \mathfrak{c}$.

5 An open question

We briefly discuss the necessary ingredients for the changing procedure in question. Suppose $W \subseteq U$, W and U have the same cardinals and the same reals, there is some relation $\mathbf{A}^W = \mathbf{A}^U = \mathbf{A}$ whose norm $||\mathbf{A}||$ has value $||\mathbf{A}|| = \kappa \geq \aleph_2$ in W and value $\aleph_1 \leq ||\mathbf{A}|| = \lambda < \kappa$ in U. Then a set of ordinals in U of cardinality λ cannot be covered by a set of ordinals in W of cardinality $\leq \lambda$. Hence one of the premises of the following theorem of [9] is not fulfilled:

Theorem 5.1 (Magidor) *If $W \subseteq U$ are two models of ZFC, $W \models GCH$, W, U agree on cofinalities, every countable set in U of ordinals can be covered in W by a set of cardinality $\leq \lambda$, then every set x in U can be covered by a set in W of cardinality $\leq \max(\text{card}(x), \lambda)$.*

Now, regarding the models from section 3, $W = V[G]$ and $U = V[G_1][G]$ have the same ω-seqences of ordinals, so necessarily $V[G] \not\models GCH$ or a cofinality is changed. Both are true. In order to change a cardinal characteristic of the reals, the smaller model does not fulfill CH, otherwise all characteristics are already $\aleph_1$ and cannot be lowered any more. So there is the question: Is there a changing procedure that does not change cofinalities? Magidor's theorem shows: Using an ω-distributive component and the exchange of the order in the product of the two forcings does now exclude starting from any W that is gotten from a model V of GCH by a c.c.c. forcing extension. Proposition 2.3 excludes starting with a regular $||\mathbf{A}||^W$.

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