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Finding generic filters by playing games

Heike Mildenberger

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Abstract We give some restrictions for the search for a model of the club principle with no Souslin trees. We show that $\diamondsuit(2^\omega, [\omega]^\omega$, is almost constant on) together with CH and "all Aronszajn trees are special" is consistent relative to ZFC. This implies the analogous result for a double weakening of the club principle.

Keywords Club principle · Aronszajn tree · Specialisation · Completeness system

Mathematics Subject Classification (2000) 03E05 · 03E15 · 03E35

1 Introduction

We show that "$\clubsuit_{w^2}$ (see Definition 1.4) and CH and all Aronszajn trees are special" is consistent relative to ZFC (see Theorem 2.3). To achieve this we work with the weak diamond for the reaping relation. The reaping relation is $\{(f, X) : f \in 2^\omega, X \in [\omega]^\omega, (\exists n \in \omega) f \restriction (X \setminus n)$is constant$\}$. The weak diamond for the reaping relation is an instance of Definition 1.1. In Theorem 2.1 we show that iterations where the NNR forcing is used to destroy all Souslin trees do not give a negative answer to Juhász' question as to whether the club principle (see Definition 1.2) implies the existence of a Souslin tree unless the bookkeeping is arranged particularly (and it is open whether this can be done).

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H. Mildenberger (✉)
Einstein Institute of Mathematics, The Hebrew University, Edmond Safra Campus Givat Ram, Jerusalem 91904, Israel
e-mail: heike@logic.univie.ac.at

The forcing used in the proof is the one from [23,2], where Aronszajn trees are specialised with forcings with side conditions that are $\mathbb{D}$-complete for a simple $\mathbb{D}$-completeness system. This is an application of Shelah's theory of iterating proper forcing without adding reals [23, Chap. 7]. Jensen [7] constructed the first model of CH where all Aronszajn trees are special.

In [18] Mildenberger and Shelah proved for two weaker parametrised diamonds that they are consistent together with CH and "all Aronszajn trees are special". For readers familiar with that work we point out the differences and the similarities: The main technical result, that for iterations of lengths γ of the specialisation forcings many (M, P)-generic filters can be computed in a Borel manner from arguments given by a game played in $\alpha = \mathrm{otp}(\gamma \cap M)$ rounds, is modified to give a different game: The second player imitates Miller forcing and the preservation of P-points to find infinite sets that serve as values for a function witnessing $\diamondsuit$(reaping). The first player finds a real coding the second order parameters in the completeness system such that all other reals not eventually below it (so for example the Miller reals) are equally suitable for coding these parameters. The transition from the game and from many guessed countable elementary substructures with little forcing scenarios to the weak diamond for the reaping relation in $\mathbf{V}^{P_{\omega_2}}$ is analogous to [18].

Now we review the definitions of the guessing principles that appear in this work. Let A and B be sets of reals and let $E \subset A \times B$. Here we work only with Borel sets A and B and absolute E, so that there are no difficulties in the interpretation of the notions in various ZFC models. The set A carries the topology inherited from the reals and 2^α carries the product topology.

Definition 1.1 (Definitions 4.3/4.4 of [20])

(1) A function $F\colon 2^{<\omega_1} \to A$ is called a Borel function if each part $F \restriction 2^\alpha, \alpha < \omega_1$, is a Borel function, possibly with a real parameter that depends on α. The complexity of the set $\{F \restriction 2^\alpha : \alpha < \omega_1\}$ can be high.

(2) Let $\diamondsuit(A, B, E)$ be the following statement: For every Borel map $F\colon 2^{<\omega_1} \to A$ the statement $\diamondsuit_F(A, B, E)$ holds, i.e., there is some $g\colon \omega_1 \to B$ such that for every $f\colon \omega_1 \to 2$ the set

$$\{\alpha \in \omega_1 : F(f \restriction \alpha)\, E\, g(\alpha)\}$$

is stationary. Commonly, if E is not the equality $\diamondsuit(A, B, E)$ is called a weak diamond.

The original diamond, $\diamondsuit_{\omega_1}$, is $\diamondsuit_F(A, B, E)$ with $A = B = 2^{<\omega_1}$ (so in this case A and B are not Borel subsets of the real line), E being equality, in the special case of F being the identity function. Jensen [14] showed that $\diamondsuit_{\omega_1}$ holds in L. Devlin and Shelah [8] showed that in the case $|B| = 2$ some diamond principles follow from $2^{\aleph_0} < 2^{\aleph_1}$.

Ostaszewski [21] introduced the club principle, $\clubsuit$, for a topological construction:

Definition 1.2 Let $\clubsuit$ be the following statement: There is some $\langle A_\alpha : \alpha < \omega_1, \alpha \text{ limit}\rangle$ such that for every α, A_α is cofinal in α and for every $X \subseteq \omega_1$ the set $\{\alpha \in \omega_1 : A_\alpha \subseteq X\}$ is stationary. Analogously, we define $\clubsuit_S$ for a stationary subset S of ω_1 by letting α range over S.

For sets A, X, we let $A \subseteq^* X$ denote that $A \setminus X$ is finite. Fuchino, Shelah and Soukup [10] and Džamonja and Shelah [9] also consider the following two weakenings of $\clubsuit$:

Definition 1.3 Let $\clubsuit_{\mathrm{w}}$ be the following statement: There is some $\langle A_\alpha : \alpha < \omega_1, \alpha \text{ limit}\rangle$ such that for every α, A_α is cofinal in α and for every $X \subseteq \omega_1$ the set $\{\alpha \in \omega_1 : A_\alpha \subseteq^* X\}$ is stationary.

Definition 1.4 Let $\clubsuit_{\mathrm{w}2}$ be the following statement: There is some $\langle A_\alpha : \alpha < \omega_1, \alpha \text{ limit}\rangle$ such that for every α, A_α is cofinal in α and for every $X \subseteq \omega_1$ the set $\{\alpha \in \omega_1 : A_\alpha \subseteq^* X \vee A_\alpha \subseteq^* \alpha \setminus X\}$ is stationary.

Finally, the following principle $\dagger$, also called the stick, is a weakening of both CH and $\clubsuit_{\mathrm{w}}$. The stick principle was introduced in [6].

Definition 1.5 The principle $\dagger$ says: There is a set $\{A_\alpha : \alpha < \omega_1\}$ such that for every uncountable $X \subseteq \omega_1$ there is some $A_\alpha \subseteq X$.

The stick is incomparable with $\clubsuit_{\mathrm{w}2}$. Fuchino, Shelah and Soukup [10, 4.1] show that $\clubsuit_{\mathrm{w}2}$ together with the negation of the stick principle is consistent. Moore, Hrušák and Džamonja [20, Theorem 8.3] give a model of CH, so in particular of stick, in which $\diamondsuit(\mathbb{R}^\omega, \mathbb{R}^\omega, \text{range}(f) \not\supseteq \text{range}(g))$ does not hold for some (quite concrete) Borel function F, so in particular also the weak diamond of the reaping relation and also the double weakening of the club fail.

Our notation on trees follows [24, Chap. 9]. Only in the forcing we stick to the older tradition that the stronger condition is the larger one.

2 Juhász' question and weaker club principles

Jensen [14] showed that $\diamondsuit_{\omega_1}$ implies the existence of a Souslin tree and Juhász asked whether the weakening $\clubsuit$ does so as well [19, Question 15.3]. Baumgartner [13, Section 4] and Shelah [23, Chap. 3, Theorem 7.4] showed that $\clubsuit$ is strictly weaker than $\diamondsuit$. Under CH, $\clubsuit$ implies $\diamondsuit$ by [22, Theorem 3. 7.3]. A modification of this proof that is carried out in the course of Theorem 2.1 shows that also $\clubsuit_{\mathrm{w}}$ together with the CH implies $\diamondsuit$. So our result about $\clubsuit_{\mathrm{w}2}$ is sharp for models of CH.

There is no Souslin tree iff every Aronszajn tree has an uncountable antichain; and an uncountable antichain exists iff there is an uncountable partial specialisation. Recall, a specialisation of an Aronszajn tree $(T, <_T)$ is a function $f : T \to \mathbb{Q}$ such that for any pair $x <_T y \in T$, $f(x) < f(y)$. An uncountable partial specialisation is a function $f : A \to \mathbb{Q}$ such that $A \subseteq T$ is uncountable and for any pair $x <_T y \in A$, $f(x) < f(y)$. An Aronszajn tree is called special if it has a (total) specialisation function. In order to destroy a Souslin tree without collapsing $\aleph_1$ one can add an uncountable partial specialisation or one can add a branch through the tree.

The following metatheorem shows that forcing "$\clubsuit_{\mathrm{w}}$ together with every Aronszajn tree is special" in a countable support iteration of proper iterands starting with a ground model of the CH and such that the iteration has the $\aleph_2$-c.c. and length ω_2 and that does not add reals together with the antichains destroying a Souslin tree in the intermediate models cannot be achieved at all. We write $\mathbf{V}_\beta$ for $\mathbf{V}^{P_\beta}$. A Souslin tree T is called *Souslin with respect to a diamond*, if there is a diamond sequence $\langle A_\delta : \delta \in S\rangle$, S stationary, such that for all $\delta \in S$ if A_δ is a maximal antichain in T_δ, then A_δ is maximal in the whole tree.

Theorem 2.1 *Metatheorem. Suppose that $\langle P_\beta, \underset{\sim}{Q}_\alpha : \alpha < \omega_2, \beta \leq \omega_2\rangle$ is a countable support iteration of proper iterands that has the $\aleph_2$-c.c. and that* CH *holds in $\mathbf{V}_\beta$ for all $\beta < \omega_2$. Suppose in addition that $\clubsuit_{\rm w}$ holds in $\mathbf{V}_{\omega_2}$. Then for every $\alpha < \omega_2$, if P_α adds an antichain to a Souslin tree that is Souslin with respect to a diamond in $\mathbf{V}_\gamma$ in all $\mathbf{V}_\gamma$, $\gamma < \alpha$, then a real is added by Q_γ if $\alpha = \gamma + 1$ or, if α is a limit, with Q_γ for cofinally many γ in α.*

Proof Suppose that $\langle A_\alpha : \alpha < \omega_1, \lim(\alpha)\rangle$ is a $\clubsuit_{\rm w}$ sequence. Then by [4, Theorem 4.2] there is some $\beta < \omega_2$ such that the sequence is in $\mathbf{V}_\beta$. Since $\mathbf{V}_\beta \models$ CH, by a strengthening of [23, Theorem I.7.3] from $\clubsuit$ to $\clubsuit_{\rm w}$, $\mathbf{V}^{P_\beta} \models \diamondsuit$.

For completeness' sake, we prove the strengthening: If CH and $\clubsuit_{{\rm w},S}$ holds then $\diamondsuit_S$ holds.

Suppose that $\langle A_\alpha : \alpha \in S\rangle$ is a witness that $\clubsuit_{{\rm w},S}$ holds. We replace each A_α by a cofinal subset of order type ω and we call the outcome A_α again and we still have $\diamondsuit_{{\rm w},S}$ sequence. Using CH, let $\langle B_i : i < \omega_1\rangle$ be a list in which every bounded subset of ω_1 appears $\aleph_1$ times and such that $\sup B_i \leq i$.

Now we define a $\diamondsuit^-_S$-sequence (see [15, Chap. 2, Theorem 7.14]) $\langle \mathfrak{D}_\alpha : \alpha \in S\rangle$ as follows: For $\alpha \in S$, we set $\mathfrak{D}_\alpha = \{D_{\alpha,n} : n \in \omega\}$, where $A_{\alpha,n}$ is A_α without the first n elements and $D_{\alpha,n} = \bigcup\{B_i : i \in A_{\alpha,n}\}$. We show that $\langle \mathfrak{D}_\alpha : \alpha \in S\rangle$ is a $\diamondsuit^-_S$-sequence. Let X be a subset of ω_1. If X is bounded, let X' be the set of i such that $B_i = X$. The set X' is unbounded in ω_1 and hence there are stationarily many points $\alpha \in S$ such that $A_\alpha \subseteq^* X'$. This implies that $X \in \mathfrak{D}_\alpha$. Now suppose that X is unbounded and we define by induction a function $j\colon \omega_1 \to \omega_1$ as follows: $j(\alpha)$ is the minimal $i \geq j(\beta), \beta < \alpha$, such that $B_i = X \cap \sup\{j(\beta) : \beta < \alpha\}$. Since j is strictly increasing for all α, $j(\alpha) \geq \alpha$. Let X' be the range of j. Let C be the club set of countable ordinals that are closed under j. By $\diamondsuit_{{\rm w},S}$ there is a $\delta \in S \cap C$ such that $A_\delta \subseteq^* X'$. We show that $X \cap \delta \in \mathfrak{D}_\delta$. Let $A_{\delta,n} \subseteq X'$. As for each $i \in A_{\delta,n}$, $B_i = X \cap \sup\{j(\beta) : \beta < \alpha\}$ for some $\alpha < \delta$, we have that $D_{\delta,n} = \bigcup\{B_i : i \in A_{\delta,n}\} \subseteq X \cap \delta$. Now we show that for every $\beta < \delta$ there are $\alpha \geq \beta$ and $i \in A_{\delta,n}$ such that $X \cap \alpha = B_i$. As $\delta \in C$, $j(\beta) < \delta$, and $A_{\delta,n}$ is cofinal in δ. So there is $\gamma \in A_{\delta,n}, \gamma \geq j(\beta)$. But $A_{\delta,n} \subseteq X'$ hence $\gamma = j(\alpha)$ for some $\alpha \geq \beta$. $B_{j(\alpha)} = X \cap \sup\{\beta' : \beta' < \alpha\}$ and the latter sup is $\geq \beta$. Since $j(\alpha) \in A_{\delta,n}$, we have $D_{\delta,n} \supseteq B_{j(\alpha)} \supseteq X \cap \beta$ for all $\beta < \delta$, and hence $D_{\delta,n} = X \cap \delta$. This finishes the proof of $\diamondsuit^-_S$.

Hence by Kunen's result, for every $\beta \leq \delta < \omega_2$, there is a $\diamondsuit$-sequence $\bar{A}$ for $\mathbf{V}_\delta$ and a Souslin tree T_δ in $\mathbf{V}_\delta$ that is in $\mathbf{V}_\delta$ Souslin with respect to $\bar{A}$. We show that some T_δ is Souslin also in $\mathbf{V}_{\omega_2}$ unless the condition of the theorem is fulfilled.

Suppose for a contradiction that in $\mathbf{V}_{\omega_2}$ the tree T_δ (on ω_1) has an uncountable antichain and call it A', and suppose that $\delta < \alpha \leq \omega_2$ is minimal such that $A' \in \mathbf{V}_\alpha$. By the $\aleph_2$-c.c., $\alpha < \omega_2$.

We assume that there are no new reals in $\mathbf{V}_\alpha \setminus \mathbf{V}_\delta$. Then $\bar{A}$ is still a diamond sequence in $\mathbf{V}_\alpha$. In $\mathbf{V}_\delta$ and in $\mathbf{V}_\alpha$ the following holds: if A_ε is a maximal antichain in $T \cap \varepsilon$ then it is maximal in T, and for every maximal antichain A in T there are stationarily many ε that $A_\varepsilon = A \cap \varepsilon$ and that A_ε is a maximal antichain in $T_{<\varepsilon}$. Hence T is Souslin in $\mathbf{V}_\alpha$ as well. Contradiction. □

A related question is

Question 2.2 Is "$\clubsuit$ and every Aronszajn is special" consistent relative to ZFC?

It is known that this question differs from Juhász' question: There are models without Souslin trees where this is not established via the specialisation of (uncountable parts of) all Aronszajn trees; see e.g., Hirschorn's work [11] on RSH, the Souslin Hypothesis after adding random reals, and CH. Many models without Souslin trees where this is not established via the specialisation of all Aronszajn trees are given in [22, Chap. IX].

Now we prove a negative answer to a strengthening of Juhász' question:

Theorem 2.3 *"$\clubsuit_{\mathrm{w}2}$ and* CH *and every Aronszajn is special" is consistent relative to* ZFC.

We reduce this to a weak diamond and to showing that "$\diamondsuit$(reaping) and CH and every Aronszajn is special" is consistent relative to ZFC.

Lemma 2.4 $\diamondsuit(\mathit{Reaping})$ *implies* $\clubsuit_{\mathrm{w}2}$.

Proof For each limit ordinal $\alpha < \omega_1$ let $h_\alpha \colon \omega \to \alpha$ be a increasing injection such that range(h_α) is cofinal in α. We take the Borel function $F \colon 2^{<\omega_1} \to 2^\omega$ that is given by

$$F(f)(n) = f(h_\alpha(n)) \text{ for } f \in 2^\alpha.$$

We take $g \colon \omega_1 \to [\omega]^\omega$ witnessing $\diamondsuit_F(\mathit{reaping})$. Now we define $A_\alpha = h''_\alpha g(\alpha) \subseteq \alpha$. We show that $\langle A_\alpha : \alpha < \omega_1 \rangle$ witnesses $\clubsuit_{\mathrm{w}2}$.

Let $X \subseteq \omega_1$ be given. We apply $\diamondsuit(\mathit{reaping})$ to the characteristic function of X, $\chi_X \colon \omega_1 \to 2$. Then

$$\{\alpha : F(\chi_X \restriction \alpha) \text{ is almost constant on } g(\alpha)\}$$

is stationary in ω_1. Then we have

$$\begin{aligned}
& F(\chi_X \restriction \alpha) \text{ is almost constant on } g(\alpha) \\
\text{iff}\quad & g(\alpha) \subseteq^* F(\chi_X \restriction \alpha) \text{ or } g(\alpha) \subseteq^* \omega \setminus F(\chi_X \restriction \alpha) \\
\text{iff}\quad & g(\alpha) \subseteq^* \{n : \chi_X(h_\alpha(n)) = 1\} \text{ or } g(\alpha) \subseteq^* \{n : \chi_X(h_\alpha(n)) = 0\} \\
\text{iff}\quad & (\exists \ell \in \{0,1\})(A_\alpha = h''_\alpha g(\alpha) \subseteq^* h''_\alpha\{n : \chi_X(h_\alpha(n)) = \ell\} = \{k \in \alpha : \chi_X(k) = \ell\}), \\
\text{iff}\quad & A_\alpha \subseteq^* X \text{ or } A_\alpha \subseteq^* \alpha \setminus X.
\end{aligned}$$

□

$$\begin{tikzcd}
\diamondsuit(\text{reaping}) \arrow[d] \arrow[dddrr] & & & \\
\diamondsuit(\mathrm{Cov}(\mathcal{N})) \arrow[dd] & \boxed{\diamondsuit(\mathrm{Unif}(\mathcal{M}))} \arrow[l] \arrow[d] & \boxed{\diamondsuit(\mathrm{Cof}(\mathcal{M}))} \arrow[l] \arrow[d] & \boxed{\diamondsuit(\mathrm{Cof}(\mathcal{N}))} \arrow[l] \arrow[dd] \\
 & \diamondsuit(\omega^\omega, \omega^\omega, \not\geq^*) \arrow[d] & \diamondsuit(\omega^\omega, \omega^\omega, \leq^*) \arrow[l] \arrow[d] & \\
\diamondsuit(\mathrm{Add}(\mathcal{N})) & \diamondsuit(\mathrm{Add}(\mathcal{M})) \arrow[l] & \diamondsuit(\mathrm{Cov}(\mathcal{M})) \arrow[l] & \diamondsuit(\mathrm{Unif}(\mathcal{N})) \arrow[l]
\end{tikzcd}$$

Fig. 1 Only the framed weak diamonds imply the existence of a Souslin tree. The *arrows* indicate implications

3 The weak diamond for the reaping relation and CH

Jensen [14] showed that $\diamondsuit_{\omega_1}$ implies the existence of a Souslin tree. Hrušák [12] introduced $\diamondsuit_{\mathfrak{d}}$, a strengthening of $\mathfrak{d} = \aleph_1$, such that $\diamondsuit_{\mathfrak{d}}$ implies that $\mathfrak{a} = \aleph_1$. Moore, Hrušák and Džamonja [20] introduce and investigate numerous versions of weak diamonds that come from the relations in Cichoń's diagramme. Their weak diamond for the dominating relation is a slight strengthening of $\diamondsuit_{\mathfrak{d}}$. Let $\mathrm{Unif}(\mathcal{M})$ denote the relation (F_σ meager sets, ω^ω, $\not\ni$), and let $\mathrm{Unif}(\mathcal{N})$ denote the relation (G_δ null sets, ω^ω, $\not\ni$). They show that $\diamondsuit(\mathrm{Unif}(\mathcal{M}))$ implies the existence of a Souslin tree, and from work by Hirschorn [11] they derive that $\diamondsuit(\mathrm{Unif}(\mathcal{N}))$ does not imply the existence of a Souslin tree. Another model of $\diamondsuit(\mathrm{Unif}(\mathcal{N}))$, with larger continuum and no Souslin trees is given by Laver [16]. Since the Borel Galois-Tukey connections (see Vojtáš [25]) in the Cichoń diagramme can be translated into implications of the corresponding weak diamonds [20, Proposition 4.9], there is a Cichoń's diagramme of weak diamonds. So all its entries above $\diamondsuit(\mathrm{Unif}(\mathcal{M}))$ imply the existence of a Souslin tree, see Fig. 1. Also $\diamondsuit(\omega^\omega, \omega^\omega, \leq^*)$ together with "all Aronszajn trees are special" is consistent relative to ZFC according to [17]. In this model, the continuum is $\aleph_2$. In [18] it is shown that $\diamondsuit(\mathrm{Cov}(\mathcal{N}))$ together with CH and all Aronszajn trees are special is consistent.

Theorem 2.3 will follow from Lemma 2.4 and the following:

Theorem 3.1 *$\diamondsuit$(Reaping) together with* CH *and with "all Aronszajn trees are special" is consistent relative to* ZFC.

Remark As indicated in the diagramme, Theorem 3.1 extends the analogous results on $\diamondsuit(\mathrm{Cov}(\mathcal{M}))$ [20] and on $\diamondsuit(\mathrm{Cov}(\mathcal{N}))$ [18]. It does not extend, though, the intermediate results on a diamond for covering functions from ω to ω by small slaloms [18, Theorem 3.9].

Proof The proof of Theorem 3.1 takes the rest of the paper. One of the two main steps is Theorem 4.4, which is a strengthening of [18, Theorem 3.4]. We replace the one-iteration-step coding [18, Lemma 3.3] by some weaker codes. This will be Lemma 4.3. In order to explain the coding of (M, P, p) we need to recall a part of Shelah's theory of simple countable completeness systems for our particular notion of forcing. We do this in the rest of this section. In addition, later in the proof, in Lemma 5.1, the Laver game about the narrow slaloms [18, Lemma 3.10] will be replaced by a game with Miller reals. $\square$

We use the forcings "specialising an Aronszajn tree without adding reals" from [2] and [23, Chap. V, Section 6]. Let $\mathbf{T} = (\omega_1, <_\mathbf{T})$ be an Aronszajn tree. Now we consider partial specialisations whose domain is the union of countably many of its levels, so that the indices of the levels form a closed set C. We call such a pair (f, C) an *approximation*. For $\alpha < \omega_1$ let T_α denote the α-th level of $\mathbf{T}$. For $x \in T_\alpha$ and $\beta < \alpha$ we let $x\lceil\beta$ be the $y \in T_\beta$ such that $y <_\mathbf{T} x$. For making the notation easier, we consider only Aronszajn trees $\mathbf{T}$ whose α-th level $T_\alpha = [\omega\alpha, \omega(\alpha+1))$. This is no loss of generality since specialising all these Aronszajn trees suffices.

Definition 3.2 (A modification of [2, Definition 4.1]).

(1) An *approximation* is a pair (f, C) such that there is a countable ordinal α and $C \subseteq \alpha + 1$, C is closed and $\alpha \in C$, $f : \bigcup_{i \in C} T_i \to \mathbb{Q}$ is a partial specialisation function. The ordinal α is called $\mathrm{last}(f)$. We say "(f_2, C_2) extends (f_1, C_1)" and write $(f_1, C_1) \leq (f_2, C_2)$ iff $f_1 \subseteq f_2$ and C_2 is an end-extension of C_1, i.e., $C_1 \subseteq C_2$ and $(C_2 \setminus C_1) \cap (\bigcup C_1) = \emptyset$.
(2) We say H is a *requirement* of height $\gamma < \omega_1$ iff for some $n = n(H) < \omega$, H is a countable set of functions of the form $h : \mathrm{dom}(h) \to \mathbb{Q}$ with $\mathrm{dom}(h) \in [T_\gamma]^n$.
(3) We say that a finite function $h : T_\alpha \to \mathbb{Q}$ *bounds* an approximation f with $\mathrm{last}(f) = \alpha$ iff $\forall x \in \mathrm{dom}(h)$, $f(x) < h(x)$. More generally, if $\beta \geq \alpha = \mathrm{last}(f)$, then $h : T_\beta \to \mathbb{Q}$ bounds f iff $\forall x \in \mathrm{dom}(h)(f(x\lceil\alpha) < h(x))$.
(4) An approximation f with $\mathrm{last}(f) = \alpha$ is said to *fulfil* the requirement H of height $\beta \geq \alpha$ iff for every $t \in [T_\alpha]^{<\omega}$ there is some $h \in H$ which bounds f and such that $\{x\lceil\alpha : x \in \mathrm{dom}(h)\}$ is disjoint from t.

Definition 3.3 $H \subseteq \mathbb{Q}^{[T_\gamma]^n}$ is called *dispersed* iff for each $t \in [T_\gamma]^{<\omega}$, there is some $h \in H$ such that $t \cap \mathrm{dom}(h) = \emptyset$.

Definition 3.4 (See [2, Definition 4.1 (4)].) Γ is a $\mathbf{T}$-promise iff $\mathrm{dom}(\Gamma)$ is club in ω_1 and $\Gamma = \langle \Gamma(\gamma) : \gamma \in \mathrm{dom}(\Gamma)\rangle$ has the following properties:

(a) For each $\gamma \in \mathrm{dom}(\Gamma)$, $\Gamma(\gamma)$ is a countable set of requirements of height γ.
(b) $(\forall \gamma \in \mathrm{dom}(\Gamma))(\forall H \in \Gamma(\gamma))H$ is dispersed.
(c) For $h : T_{\alpha_1} \to \mathbb{Q}$ and $\alpha_0 < \alpha_1$ we let $\mathrm{dom}(h\lceil\alpha_0) = \{y\lceil\alpha_0 : y \in \mathrm{dom}(h)\}$ and $h\lceil\alpha_0(x) = \min\{h(y) : y\lceil\alpha_0 = x, y \in \mathrm{dom}(h)\}$. We let $H\lceil\alpha_0 = \{h\lceil\alpha_0 : h \in H\}$. We require: $(\forall \alpha_0 < \alpha_1 \in \mathrm{dom}(\Gamma))(\Gamma(\alpha_0) \supseteq \{H\lceil\alpha_0 : H \in \Gamma(\alpha_1)\})$.

Condition (c) implies that for all α_0, $\{H\lceil\alpha_0 : (\exists \alpha_1 > \alpha_0)(H \in \Gamma(\alpha_1))\}$ is countable. For $\alpha < \gamma$, we write $\Gamma(\gamma)\lceil\alpha = \{H\lceil\alpha : H \in \Gamma(\gamma)\}$.

Definition 3.5 ([2, Definition 4.1 (5)]) We say that an approximation (f, C) *fulfils* the promise Γ iff $\mathrm{last}(f) \in \mathrm{dom}(\Gamma)$ and f fulfils each requirement H in $\Gamma(\mathrm{last}(f))$.

Definition 3.6 ([2, 4.2]) $Q_\mathbf{T}$ is the set of (f, C, Γ) such that (f, C) is an approximation, and Γ is a promise and (f, C) fulfils Γ. The partial order is defined as $(f_0, C_0, \Gamma_0) \leq (f_1, C_1, \Gamma_1)$ iff

(1) f_1 extends f_0,
(2) C_1 is an end-extension of C_0 and $C_1 \setminus C_0 \subseteq \mathrm{dom}(\Gamma_0)$, and
(3) $(\forall \gamma \in \mathrm{dom}(\Gamma_0 \setminus \mathrm{last}(f_1))(\gamma \in \mathrm{dom}(\Gamma_1) \mathrm{and} \Gamma_0(\gamma) \subseteq \Gamma_1(\gamma))$.

If $p = (f, C, \Gamma)$, we write $f = f^p$, $C = C^p$ and $\Gamma = \Gamma^p$, and we write $\mathrm{last}(p) = \mathrm{last}(f^p) = \max(C^p)$.

In the following lemma $\chi > 2^{\aleph_1}$ is sufficiently large. The following lemma provides an important step for the Borel computations in the next section.

Lemma 3.7 ([2, 4.3], [23, *Fact* 5 6.7]) *Let* $\mathbf{T}$ *be an Aronszajn tree. Let* $M \prec (H(\chi), \in)$ *be a countable elementary substructure with a sufficiently large regular* χ, $Q_{\mathbf{T}} \in M$, $p \in Q_{\mathbf{T}} \cap M$, $\mu = \omega_1 \cap M$ *and* $h \colon T_\mu \to \mathbb{Q}$ *be a finite function which bounds* f^p. *Let* $D \in M$, $D \subseteq Q_{\mathbf{T}}$ *be dense open. Then there is an* $q \geq p$, $q \in D \cap M$, *such that* h *bounds* q.

Definition 3.8 We take the iterands $Q_{\mathbf{T}}$ from Definition 3.6. Now we assume $\mathbf{V} \models \mathrm{CH} + \diamondsuit_{\omega_1} + 2^{\aleph_1} = \aleph_2$ and let $P_{\omega_2} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration with $\underset{\sim}{Q}_\alpha = Q_{\underset{\sim}{\mathbf{T}}_\alpha}$ being as above for some Aronszajn tree $\mathbf{T}_\alpha \in \mathbf{V}[G_\alpha]$, G_α P_α-generic over $\mathbf{V}$, such that $\Vdash_{P_\alpha}$ "$\underset{\sim}{\mathbf{T}}_\alpha$ is an Aronszajn tree and for $\gamma < \omega_1$ its γ-th level is $[\omega\gamma, \omega\gamma + \omega)$". The book-keeping shall be arranged so that every P_{ω_2}-name for an Aronszajn tree is used in some iterand.

Every Aronszajn tree in $\mathbf{V}^{P_{\omega_2}}$ has a P_α-name for some $\alpha < \omega_2$ since by [23, Chap. VIII, Section 2], each $Q_{\mathbf{T}}$ has the $\aleph_2$-p.i.c. (proper isomorphism condition), see [23, Chap. VIII, Definition 2.2], and hence by [23, Chap. VIII, Lemma 2.4], P_{ω_2} has the $\aleph_2$-c.c.

Since P_{ω_2} has the $\aleph_2$-c.c., by a lemma similar to the one of [4, 5.10], now for subsets of ω_1 instead of reals, every subset of ω_1 in a countable support iteration of proper forcings with the $\aleph_2$-c.c. at each initial segment has a name at some stage of cofinality ω_1. So we can carry out the desired book-keeping.

Definition 3.9 We call P *α-proper* if the following holds: Let M_i, $i < \alpha$, be countable elementary submodels of $(H(\chi), \in)$. Let $P \in M_0$ and let $\langle M_i : i < \alpha \rangle$ be an increasing sequence such that $\langle M_j : j \leq i \rangle \in M_{i+1}$ and for limit ordinals j, $M_j = \bigcup_{i<j} M_i$. Then for every $p \in P \cap M_0$ there is some $q \geq p$ that is (M_i, P)-generic for all $i < \alpha$. Such a sequence $\langle M_i : i < \alpha \rangle$ is called a *tower* of models and α is the height or the length of the tower.

Lemma 3.10 ([18, Lemma 2.29]) $Q_{\mathbf{T}}$ *is* α*-proper for all* $\alpha < \omega_1$.

Definition 3.11 ([23, V, 5.5])

(1) We call $\mathbb{D}$ a *completeness system* if for some χ, $\mathbb{D}$ is a function defined on the set of triples $\langle M, P, p \rangle$, $p \in M \cap P$, $P \in M$, $M \prec (H(\chi), \in)$, M countable, such that $\mathbb{D}(M, P, p)$ is a family of non-empty subsets of

$$\begin{aligned} \mathrm{Gen}(M, P, p) = \{ & G : G \subseteq M \cap P, G \text{ is directed and } p \in G \\ & \text{and } G \cap \mathcal{I} \neq \emptyset \\ & \text{for every dense subset } \mathcal{I} \text{ of } P \text{ which belongs to} M \}. \end{aligned}$$

(2) We call $\mathbb{D}$ a *λ-completeness system* if each family $\mathbb{D}(M, P, p)$ has the property that the intersection of any i elements is non-empty for $i < 1 + \lambda$ (so for $\lambda \geq \aleph_0$, $\mathbb{D}(M, P, p)$ generates a filter). $\aleph_1$-completeness systems are also called countably closed completeness systems.
(3) We say $\mathbb{D}$ is on χ if $M \prec (H(\chi), \in)$. We do not always distinguish strictly between $\mathbb{D}$ and its definition.

Definition 3.12 A condition p is *completely* (M, P)*-generic* if $G = \{q \in P \cap M : q \leq p\}$ is an (M, P)-generic filter. G is called bounded.

Definition 3.13 Suppose that $\mathbb{D}$ is a completeness system on χ. We say P is $\mathbb{D}$*-complete*, if for every countable $M \prec (H(\chi), \in)$ with $P \in M, \mathbb{D} \in M, p \in P \cap M$, the following set contains as a subset a member of $\mathbb{D}(M, P, p)$:

$$\mathrm{Gen}^+(M, P, p) = \{G \in \mathrm{Gen}(M, P, p) : \text{there is an upper bound for } G \text{ in} P\}.$$

Definition 3.14 ([23, V, 5.5])

(1) A completeness system $\mathbb{D}$ is called *simple* if there is a first order formula ψ such that

$$\mathbb{D}(M, P, p) = \{A_x : x \text{is a finitary relation on} M, \text{i.e.,} x \subseteq M^k \text{ for some } k \in \omega\},$$

where

$$A_x = \{G \in \mathrm{Gen}(M, P, p) : (M \cup \mathcal{P}(M), \in, p, M, P) \models \psi(x, G)\}. \quad (3.1)$$

(2) A completeness system $\mathbb{D}$ is called almost simple over $\mathbf{V}_0$ ($\mathbf{V}_0$ a class, usually a subuniverse) if there is a first order formula ψ such that

$$\mathbb{D}(M, P, p) = \{A_{x,z} : x \text{ is a finitary relation on } M, \text{i.e.,} \\ x \subseteq M^k \text{ for some } k \in \omega, z \in \mathbf{V}_0\},$$

where

$$A_{x,z} = \{G \in \mathrm{Gen}(M, P, p) : (\mathbf{V}_0 \cup M \cup \mathcal{P}(M), \\ \in^{\mathbf{V}_0}, \in^{M \cup P \cup \mathcal{P}(M)}, p, M, \mathbf{V}_0, P) \\ \models \psi(x, z, G)\},$$

where $\in^A = \{(y, y') \in A \times A : y \in y'\}$.
(3) If in (2) we omit z, we call $\mathbb{D}$ simple over $\mathbf{V}_0$.

We will use a completeness system $\mathbb{D}$ that is simple over $\mathbf{V}_0$. The technique of the following lemma comes from [2]. Actually a sketch of the elements of the $\aleph_1$-completeness system is also given in the end of the proof of [23, Chap. V, Theorem 6.1] on page 236. Let $P = Q_{\mathbf{T}}$. We conceive $x = (x_1, \bar{\beta})$ as one relation on

M. $x_1 \subseteq T_{<\mu} \times \omega$ describes the branches of $T_{<\mu}$ that have continuations in T_μ. We let $(y, m) \in x_1$ iff y is an elements of (the range of) the m-th such branch (in an enumeration in $\mathbf{V}_0$). Now we give a first order formula $\psi(x, G)$ that is inserted into A_x in Eq. (3.1) and that describes (M, P)-generic conditions above p in the structure $(M, P, <_P, p, x_1, \bar\beta)$. We let pr_i be the projection onto the i-th coordinate.

Lemma 3.15 ([18, Lemma 2.15]) *$Q_\mathbf{T}$ is $\mathbb{D}$-complete for the simple $\aleph_1$-completeness system $\mathbb{D}$ given by $\psi(x, G) = \psi_0(x) \wedge \psi_1(x, G)$, with*

$$\begin{aligned}\psi_0(x) \equiv\ & x = (x_1, \bar\beta) \wedge \bar\beta = \langle \beta_n : n \in \omega\rangle \textit{ increasing}\\ & \wedge M \cap \omega_1 = \bigcup \{\beta_n : n < \omega\}\end{aligned}$$

and

$$\begin{aligned}\psi_1(x, G) \equiv\ & (\forall \varepsilon > 0)(\exists m < \omega)(\forall n_1 < n_2 \in [m, \omega))(\forall y_1, y_2 \in \mathrm{pr}_1(x_1))\\ & \left((y_1 \in T_{\beta_{n_1}} \wedge y_2 \in T_{\beta_{n_2}} \wedge y_1 <_\mathbf{T} y_2 \to \underset{\sim}{f}[G](y_2) < \underset{\sim}{f}[G](y_1) + \frac{\varepsilon}{2^{n_2}}\right)\\ & \wedge \text{“}G \textit{ is a filter”}\\ & \wedge p \in G \wedge \forall D \in M((D \subseteq P \wedge D \textit{ dense in } P) \to D \cap G \neq \emptyset).\end{aligned}$$

Here M, P, x and G appear in the formulas as (names for) predicates and p is a constant.

Proof Our proof is a slight modification of the proof given in [18, Lemma 2.15]. □

First we proof the following claim:

Claim 3.16 *Let $\mu = M \cap \omega_1 = \sup\langle \beta_n : n < \omega\rangle$ and let the β_n be increasing. If*

$$(M \cup \mathcal{P}(M), \in^{M \cup \mathcal{P}(M)}, p, M, Q_\mathbf{T}) \models \psi_0(x),$$

then there is $G \subseteq Q_\mathbf{T}$, $G \in G(M, Q_\mathbf{T}, p) \cap A_x$.

Proof Let $\{I_n : n \in \omega\}$ be an enumeration of all open dense subsets of $Q_\mathbf{T}$ that are in M. Let $\{t_k : k \in \omega\}$ enumerate T_μ: Now we choose by induction on $n < \omega$, p_n such that

(1) $p_0 = p$,
(2) $p_{n+1} \geq p_n \in M$,
(3) $\mathrm{last}(p_{n+1}) \geq \beta_{n+1}$,
(4) $p_{n+1} \in I_n$,
(5) $(\forall t \in \{t_k : k \leq n\})(\forall y <_\mathbf{T} t)\big(y \in T_{\beta_{n+1}} \to f^{p_{n+1}}(y) < f^{p_n}(y \lceil \beta_n) + \frac{1}{2^{n+1+n}}\big)$.

Then $G = \{r : (\exists n \in \omega)(r \leq p_n)\} \in \mathrm{Gen}(M, Q_\mathbf{T}, p) \cap A_x$.

Why is this choice possible? For Properties (4) and (5) we use Lemma 3.7 for h with

$$\begin{aligned}\mathrm{dom}(h) &= \{t_k \lceil \beta_{n+1} : k \leq n\},\\ h(y) &= f^{p_n}(y \lceil \beta_n) + \frac{1}{2^{n+1+n}},\end{aligned}$$

which is a finite function that bounds p_n and we find some p_{n+1} of length β_{n+1}. $\square$

Claim 3.17 *If* $(M \cup \mathcal{P}(M), \in, p, M, Q_{\mathbf{T}}) \models \psi(x, G)$ *for some* x*, then* G *has an upper bound in* $Q_{\mathbf{T}}$.

Proof Again let $\{I_n : n \in \omega\}$ be an enumeration of all open dense subsets of $Q_{\mathbf{T}}$ that are in M. Let x be as in $\psi(x, G)$. Let $G \supseteq \{q_n : n \in \omega\}$, $q_n \in M \cap I_n$, $\mathrm{last}(q_n) = \beta_n$ such that the β_n and the q_n are increasing. We set $\mu = M \cap \omega_1 = \bigcup \beta_n$. We let $f^q \restriction T_{<\mu} = \bigcup_{n\in\omega} f^{q_n}$, and let $f^q \restriction T_\mu$ a slightly larger rational variant of $\bigcup f^{q_n} \cup \{(z, \sup\{f^{q_n}(z\lceil\beta_n) : n \in \omega\}) : z \in T_\mu\}$. We set $C^q = \bigcup_{n\in\omega} C^{q_n} \cup \{\mu\}$, which is closed since for each n, $C^{q_{n+1}}$ is an end extension of C^{q_n}, $\mathrm{dom}(\Gamma^q) = (\bigcup_{n\in\omega} \mathrm{dom}\, \Gamma^{q_n} \cap [\mu, \omega_1)) \cup \{\mu\}$, and for $\mu' > \mu$, $\Gamma^q(\mu') = \bigcup_{n\in\omega} \Gamma^{q_n}(\mu')$ and $\Gamma^q(\mu) = \bigcup_{\mu'\geq\mu} \bigcup_{n\in\omega} \Gamma^{q_n}(\mu')\lceil\mu$.

We claim that q is an upper bound of G: First we check that $q \in Q_{\mathbf{T}}$. Note that if ν dominates all $h_{\bar{\beta},z}$, $z \in T_\mu$, then for every $z \in T_\mu$ the limit $f^q(z)$ exists, because if $h_{z,\bar{\beta}} \leq^* \nu$, then for almost all n, $z\lceil\beta_n = \omega\beta_n + h_{z,\bar{\beta}}(n)$ and $h_{z,\bar{\beta}}(n) \leq \nu(n)$. So we have that (f^q, C^q) is an approximation. Now let $H \in \Gamma^q(\mu)$ be a $\mathbf{T}$-promise. For some $\mu' \geq \mu$, $k \in \omega$, $H \in \Gamma^{q_k}(\mu')\lceil\mu$. Then, since q_k fulfils the promise, also q fulfils the promise. $\square$

Proof of Lemma 3.15 continued:

We showed that $A_x \subseteq G^+(N, Q_{\mathbf{T}}, p)$. So we have that $Q_{\mathbf{T}}$ is $\mathbb{D}$-complete. It remains to show that $\mathbb{D}$ is countably closed, i.e., that given x^ℓ with $\psi(x^\ell, G)$, $\ell < \omega$, the intersection $\bigcap_{\ell\in\omega} A_{x^\ell}$ is not empty. But this is now easy: Let $x^\ell = (x_1^\ell, \bar{\beta}^\ell)$. x_1^ℓ, coding the cofinal branches in $T_{<\mu}$, are defined from $\mathbf{T}$ and p and do depend on ℓ.

There is only some little twist because the $\bar{\beta}^\ell = \langle \beta_u^\ell : u < \omega\rangle$ are not the same. We choose $\beta = \langle \beta_m : m < \omega\rangle$ such that $\beta_0 = 0$, $(\forall \ell \leq m)(\exists u < \omega)(\beta_u^\ell \in [\beta_m, \beta_{m+1}))$. Then we let $x_1 = x_1^0$ and $x = (x_1, \bar{\beta})$. Then $A_x \subseteq A_{x^\ell}$, $\ell < \omega$. $\square$

Now we can cite Theorem V.7.1 (2) of [23] for $\aleph_1$-complete systems. A very clear proof, even in a more general context when "almost simple over $\mathbf{V}_0$" is replaced by "in $\mathbf{V}_0$", is given in [1, Theorem 5.17].

Theorem 3.18 *Let* $P_\gamma = \langle P_j, \underset{\sim}{Q}_i : j \leq \gamma, i < \gamma\rangle$ *be a countable support iteration. If each* $\underset{\sim}{Q}_i$ *is* β*-proper for every* $\beta < \omega_1$ *and* $\mathbb{D}_i$*-complete for some almost simple* $\aleph_1$*-completeness system* $\mathbb{D}_i$ *over* $\mathbf{V}_0$ *(not over the current stage of the iteration), then* P_γ *does not add reals.*

So we know that P_{ω_2} from Definition 3.10 specialises all Aronszajn trees and does not add reals. The remaining task is to obtain the weak diamond $\diamondsuit$(reaping) in $\mathbf{V}^{P_{\omega_2}}$.

4 Coding by unbounded functions

Now, we will be given only $(M, P, <_P, p, \bar{\beta})$ and partial information about x_1. We want that this partial information nevertheless "codes" enough of the structure such that from this partial information (then called η and ν, appearing in innings of a game) (M, P)-generic filters can be computed.

The trick is to find a real ν coding x_1 (after a transitive collapse) and code in such a way that every $\eta \not\leq^* \nu$ codes as well. Coding means we want to redo Lemma 3.15 now with ν taking the role of x_1. The parameter $\bar{\beta}$ can stand as it is, since it depends only on the transitive collapse of M and not on $T_{\mathrm{otp}(M\cap\omega_1)}$.

We translate the task of x_1.

Definition 4.1 Let $\mathbf{T}$ be an Aronszajn tree with levels T_α. Let μ be a limit ordinal in ω_1. Given $\bar{\beta}$ converging to μ, we can write cofinally many nodes of a branch b of $T_{<\mu}$ into a function $h_{b,\bar{\beta}} \colon \omega \to \omega$, such that for all n,

$$b \cap T_{\beta_n} = \{\omega\beta_n + h_{b,\bar{\beta}}(n)\}$$

and we can describe each node $t = \omega\mu + k \in T_\mu$, by $h_{t,\bar{\beta}} \colon \omega \to \omega$, such that for all n,

$$t \restriction \beta_n = \omega\beta_n + h_{t,\bar{\beta}}(n).$$

We recall that $T_{<\mu} \subseteq M$ for $\mu = \mathrm{otp}(\omega_1 \cap M)$ and $T_\mu \cap M = \emptyset$. The point is: for finding a generic condition, the partial specialisations given in the first component of the approximating conditions in the generic must not diverge along any branch of $T_{<\mu}$ that has a continuation in T_μ. In [18], an (M, P)-generic filter and a condition that is stronger than all filter elements were computed from a function $\eta \colon \omega \to \omega$ that dominates all the functions in M, without the knowledge of T_μ.

Now we use that η must dominate just the countably many (codes of) branches of $\mathbf{T}$ that have continuations in T_μ. Later we replace the dominating function η by an unbounded function. We let $<^*_\chi$ be a well-ordering of $H(\chi)$, and we we let $e \colon \omega \times \omega \to \omega$ be a bijective recursive function.

Lemma 4.2 *Let $p \in Q_{\mathbf{T}} \cap M$. Let $\mu = M \cap \omega_1 = \sup\langle \beta_n : n < \omega\rangle$, $\beta_{n+1} > \beta_n$. Let $c \colon \omega \to M$ be a bijection with $c(0) = Q_{\mathbf{T}}$, $c(1) = p$, $c(2n+2) = \beta_n$, and let*

$$\begin{aligned} U &= U(M, Q_{\mathbf{T}}, p) \\ &= \{2e(n_1, n_2) : c(n_1) \in c(n_2)\} \cup \{2e(n_1, n_2) + 1 : c(n_1) <^*_\chi c(n_2)\}. \end{aligned}$$

We let η, ν stand for functions from ω to ω, and we let the functions $h_{t,\bar{\beta}}$ for $t \in T_\mu$ be defined as in Definition 4.1.

There is a Borel function $\mathbf{B}_1 \colon \omega^\omega \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, such that if

$$(\forall t \in T_\mu)(h_{t,\bar{\beta}} \leq^* \eta), \tag{4.1}$$

for

$$G = \{c(n) : n \in \mathbf{B}_1(\eta, U)\}$$

the following holds: G is $(M, Q_{\mathbf{T}})$-generic and $p \in G$ and there is an upper bound r of G.

Remark r is an upper bound of G iff we have over $\mathbf{V}$ for every $Q_{\mathbf{T}}$-generic filter $G^{\mathbf{V}}$ over $\mathbf{V}$ with $r \in G^{\mathbf{V}}$ and name $\underset{\sim}{G}^{\mathbf{V}}$ that

$$r \Vdash_{Q_{\mathbf{T}}} \underset{\sim}{G}^{\mathbf{V}} \cap M = \{c(n) : n \in \mathbf{B}_1(\eta, U)\}.$$

Proof We verify that each step in the proof of Lemma 3.15 is Borel-computable from (η, U). Let $M \prec (H(\chi), \in, <^*_\chi)$ be countable. Then we take an $<^*_\chi$-increasing enumeration $\langle I_n : n \in \omega\rangle$ of all dense subsets of $Q_{\mathbf{T}}$ that are in M.

Now, we compute from η and U by induction on $n < \omega$, p_n such that

(1) $p_0 = p$, $\mathrm{last}(p) = \beta_0$
(2) p_{n+1} is the $<^*_\chi$-least element of M such that
  (2a) $p_{n+1} \geq p_n$,
  (2b) $\mathrm{last}(p_{n+1}) \geq \beta_{n+1}$,
  (2c) $p_{n+1} \in I_n$,
  (2d) $(\forall t \in T_{\beta_{n+1}})\big(h_{t,\bar\beta}(n+1) \leq \eta(n+1) \to f^{p_{n+1}}(t) < f^{p_n}(t \lceil \beta_n) + \frac{1}{2^{n+1+n}}\big)$.

Note that this is like in Lemma 3.15 now with η instead of x_1 and going along the well-order $<^*_\chi$ instead of choosing arbitrary conditions. The existence of p_{n+1} is guaranteed by Lemma 3.7 applied to the initial segment $y \restriction (\beta_{n+1}+1)$ with $y(\beta_{n+1}) \leq \eta(n+1)$ and with the following bound h:

$$\mathrm{dom}(h) = \{x \in T_{\beta_{n+1}} : h_{x,\bar\beta}(n+1) \leq \eta(n+1)\},$$
$$h(x) = f^{p_n}(x \lceil \beta_n) + \frac{1}{2^{n+1+n}}.$$

If Eq. (4.1) holds, then η is sufficiently large to take care of all branches of $T_{<\mu}$ that lead to points $x \in T_\mu$. Set $\mathbf{B}_1(\eta, U) = \{c^{-1}(q) \in N \cap Q_{\mathbf{T}} : (\exists n) q \leq p_n\}$. Then $c''\mathbf{B}_1(\eta, U) \in \mathrm{Gen}^+(M, Q_{\mathbf{T}}, p) \cap A_x$ and there is an upper bound of $c''\mathbf{B}_1(\eta, U)$ as in Lemma 3.15.

So we have to guarantee only that for every $x \in T_\mu$, $\lim f^{p_n}(x \lceil \beta_n) \in \mathbb{R}$ and this is done by Eq. (4.1) together with condition (2d). $\square$

Now we show an important technical fact: If ν fulfils Eq. (4.1) in the place of η, then also every $\eta \not\leq^* \nu$ codes so much information that an (M, P)-generic filter can be computed from η. Note that this replacing "dominating" by "being unbounded" uses the knowledge that there are only countably many branches that are continued on the next level of the Aronszajn tree. Since T_μ has only countably many nodes t, there is ν dominating all the $h_{t,\bar\beta}$, $t \in T_\mu$.

Lemma 4.3 *Let* $p \in Q_{\mathbf{T}} \cap M$. *Let* $\mu = \mathrm{otp}(M \cap \omega_1)$, $\langle \beta_n : n < \omega\rangle$ *be strictly increasing and cofinal in* $M \cap \omega_1$. *Let* $c\colon \omega \to M$ *be a bijection with* $c(0) = Q_{\mathbf{T}}$, $c(1) = p$, $c(2n+2) = \beta_n$, *and let*

$$\begin{aligned} U &= U(M, Q_{\mathbf{T}}, p) \\ &= \{2e(n_1, n_2) : c(n_1) \in c(n_2)\} \cup \{2e(n_1, n_2) + 1 : c(n_1) <^*_\chi c(n_2)\}. \end{aligned}$$

We let η, ν stand for functions from ω to ω and we let the functions $h_{y,\bar{\beta}}$ be defined as in Definition 4.1.

There is a Borel function $\mathbf{B}_1 : \omega^\omega \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, such that if

$$(\forall t \in T_\mu)(h_{t,\bar{\beta}} \leq^* \nu), \tag{4.2}$$

and

$$\eta \not\leq^* \nu \tag{4.3}$$

then for

$$G = \{c(n) \,:\, n \in \mathbf{B}_1(\eta, U)\}$$

the following holds: G is $(M, Q_{\mathbf{T}})$-generic and $p \in G$ and there is an upper bound r of G.

Proof The idea is: on each level T_{β_n} there is all information about all earlier levels. If η sticks out infinitely often over the breadth of T_{β_n} in the sense of Eq. (4.2), then we can redo the inductive construction of the second part of the proof of Lemma 4.2.

Again we verify that G is Borel-computable from (η, U). This time there will be some vain trials and some bootstrapping through all the sequences. By Lemma 4.2 we know that the given function ν is large enough for computing a generic. Now $\eta \not\leq^* \nu$ can be used as an input for a modified computation. The reasoning is as follows: We always can assume that the $h_{x,\bar{\beta}}$ coding $x \in T_\mu$ are increasing. Assume $\eta \not\leq^* \nu$ and both in $\omega^{\uparrow\omega}$. Then let $\{b_n \,:\, n \in \omega\}$ enumerate the arguments so that $\eta(b_n) > \nu(b_n)$. Then η' with $\eta'(k) = \eta(\mathrm{next}(k, \{b_n \,:\, n \in \omega\}))$ eventually dominates ν. We use this fact to strengthen a guess of $h_{x,\bar{\beta}}$ as in (2d) retroactively.

Let $M \prec (H(\chi), \in, <^*_\chi)$ be countable. Then we take an enumeration $\langle I_n \,:\, n \in \omega\rangle$ of all dense subsets of $Q_{\mathbf{T}}$ that are in M, ordered according to $<^*_\chi$.

Now, we compute from $\eta \not\leq^* \nu$ and U by induction on $n < \omega$, p_n such that

(1) $p_0 = p$, $\mathrm{last}(p) = \beta_0$
(2) p_{n+1} is the $<^*_\chi$-least element of M such that
  (2a) $p_{n+1} \geq p_n$,
  (2b) $\mathrm{last}(p_{n+1}) \geq \beta_{n+1}$,
  (2c) $p_{n+1} \in I_n$,
  (2d') $(\forall x \in T_{\beta_{n+1}})(h_{x,\bar{\beta}}(n+1) \leq \eta(n+1) \to (\forall i \leq n+1)(f^{p_{i+1}}(x) < f^{p_i}(x\lceil\beta_i) + \frac{1}{2^{i+1+i}}))$.

If we read (2d') only for $i = n$ then we are back to the former lemma, and the existence of p_{n+1} follows from Lemma 3.7 for the initial segment $y \restriction (\beta_{n+1} + 1)$ with $y(\beta_{n+1}) \leq \eta(n+1)$ and with the following bound h:

$$\mathrm{dom}(h) = \{x \in T_{\beta_{n+1}} \,:\, h_{x,\bar{\beta}}(n+1) \leq \eta(n+1)\},$$
$$h(x) = f^{p_n}(x\lceil\beta_n) + \frac{1}{2^{n+1+n}}.$$

However, now the choice of p_{n+1} might fail in the other requirements in item (2d'). Then we revise p_n and search for (p_n, p_{n+1}) simultaneously. If this fails, then revise (p_{n-1}, p_n, p_{n+1}), and so on. If $\eta \not\leq^* \nu$ with witness $\{b_n : n \in \omega\}$, and $\nu \geq^* h_{x,\bar{\beta}}$ for all $x \in T_\mu$, then by Lemma 4.2 we know that there is some sequence $(p_n : n \in \omega)$ as desired: Namely a sequence chosen in a recursion with $\eta' = (k \mapsto (\eta(\mathrm{next}(k, \{b_n : n \in \omega\}))))$ in (2d') instead of η.

If Equations (4.2) and (4.3) hold, then η is sufficiently large to take care of all branches of $T_{<\mu}$ that lead to points $x \in T_\mu$. Each p_i needs to be revised only finitely many times: p_n is revised at most b_n times. Set $\mathbf{B}_1(\eta, U) = \{c^{-1}(q) \in M \cap Q_{\mathbf{T}} : (\exists n) q \leq p_n\}$. Then $c'' \mathbf{B}_1(\eta, U) \in \mathrm{Gen}^+(M, Q_{\mathbf{T}}, p) \cap A_x$, and there is an upper bound of $c'' \mathbf{B}_1(\eta, U)$, namely just the union of the p_n. □

We take the lemma for the iteration step and we define a modified game for which we have a modification of [18, Theorem 3.4]. The following theorem works for arbitrary iteration length.

Theorem 4.4 *Let $P_{\omega_2} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration of iterands of the form $Q_{\mathbf{T}}$. Suppose that χ is sufficiently large and regular, that $M \prec (H(\chi), \in, <^*_\chi)$ is a countable elementary model, and that*

(a) $P_\gamma \in M$,
(b) $p \in P_\gamma \cap M, \gamma \leq \omega_2$,
(c) $\alpha = \mathrm{otp}(M \cap \gamma)$.

Let $\bar{\beta}$ be cofinal in $M \cap \omega_1$. Let $c \colon \omega \to M$ be a bijection with $c(0) = P_\gamma, c(1) = p, c(2n+2) = \beta_n$, and let

$$U(M, P_\gamma, p) = \{2e(n_1, n_2) : c(n_1) \in c(n_2)\} \cup \{2e(n_1, n_2) + 1 : c(n_1) <^*_\chi c(n_2)\}.$$

Then there is a Borel function $\mathbf{B} = \mathbf{B}_\alpha \colon (\omega^\omega)^\alpha \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, such that in the following game $\partial_{(M, P_\gamma, p)}$ the generic player has a winning strategy σ, which depends only on the isomorphism type of $(M, \in, <^*_\chi, P_\gamma, p, \bar{\beta})$:

(α) *a play lasts α moves,*
(β) *in the ε-th move the generic player chooses some real ν_ε and the anti-generic player chooses some $\eta_\varepsilon \not\leq^* \nu_\varepsilon, \eta_\varepsilon \in \omega^\omega$,*
(γ) *in the end the generic player wins iff the following is true*:

$$\begin{aligned} G_\gamma = 7\{c(n) : n \in \mathbf{B}_\alpha(\langle \eta_\varepsilon : \varepsilon < \alpha \rangle, U(M, P_\gamma, p))\} &\text{ is } (M, P_\gamma) - \text{generic and} \\ &p \in G_\gamma \text{ and} \\ &(\exists q \in P_\gamma)(p \leq q \text{ and } q \text{ bounds } G_\gamma). \end{aligned}$$

Proof The proof of how to perform the iteration is literally the proof of [18, Theorem 3.4]. However, for the single iterands we now we use the new Lemma 4.3. For completeness' sake, we repeat the proof of how to organise the iteration. We follow Abraham's exposition in [1, Theorem 5.17]. This theorem works inductively: For Q_α in $\mathbf{V}^{P_\alpha}$ to be $\mathbb{D}$–complete with respect to a system that lies in $\mathbf{V}$ we need that P_α does not add new countable sets of ordinals. So every countable transitive set in $\mathbf{V}^{P_\alpha}$ is in $\mathbf{V}$.

To prove the theorem we shall first define for every countable $M \prec (H(\chi), \in, <^*_\chi)$ with $P_\gamma \in M$, $p \in P_\gamma \cap M$, with $\alpha = \mathrm{otp}(M \cap \gamma)$, an (M, P_γ)-generic filter $G_\gamma = c'' \mathbf{B}_\alpha(\langle \eta_i : i < \alpha \rangle, U)$; and then we shall prove that G_γ is bounded in P_γ by a completely (M, P_γ)-generic condition. The bounding condition is not computed in a Borel manner. Its existence is sufficient, and its existence is proved along the iteration. $\square$

Remark The bounding condition also appears in an argument about the truth in forcing extensions at the very end of Lemma 5.2.

The definition of G_γ is by induction and we shall define for every $\gamma_0 < \gamma$ and G_{γ_0} that is (M, P_{γ_0})-generic and every $p \in P_\gamma \cap M$ with $p \restriction \gamma_0 \in G_0$ a filter G_γ that extends G_{γ_0} and contains p. Once the induction is performed, we shall set $\gamma_0 = 0$, $G_0 = \{0_{P_0}\}$. There will be two main cases in this definition: γ successor and γ limit, and likewise there will be two cases in the proofs that G_γ is bounded. We start with the preparations for the successor case. When looking at complexity, we regard G_0 as a parameter.

Two step iteration

Let P be a poset and let $\underset{\sim}{Q} \in \mathbf{V}^P$ be a name forced by 0_P to be a poset. Let χ be sufficiently large and regular (as said, $\chi = (2^{\aleph_2})^+$ is always sufficiently large) and $M_0 \prec (H(\chi), \in, <^*_\chi)$ be a countable elementary submodel such that $P, \underset{\sim}{Q} \in M_0$. Henceforth we write just $H(\chi)$ instead of $(H(\chi), \in, <^*_\chi)$. We want to find a criterion for when a condition $(q_0, q_1) \in P * \underset{\sim}{Q}$ is completely $(M_0, P * \underset{\sim}{Q})$-generic. Let $\pi \colon M_0 \to N_0$ be a transitive collapsing map. Suppose that $q_0 \in \tilde{P}$ is completely generic over (M_0, P) and let $G_0 \subseteq P \cap M_0$ be the (M_0, P)-generic filter induced by q_0. Then $\mathcal{G}_0 = \pi'' G_0$ is an $(N_0, \pi(P))$-generic filter and we can form the transitive extension $N_0^* = N_0[\mathcal{G}_0]$. $\pi(\underset{\sim}{Q})$ is a name in N_0, and its interpretation $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$ is a poset in N_0^*.

Let $\underset{\sim}{G} \in \mathbf{V}^P$ be the canonical name of the P-generic filter over $\mathbf{V}$. If F is a $(\mathbf{V}, P)$ generic filter containing q_0 then $M_0[F] \prec H(\chi)[F]$ can be formed and the collapsing map π on M_0 can be extended to collapse $M_0[F]$ onto N_0^*. Let $\underset{\sim}{\pi}$ be the name of the extended collapse. Then $q_0 \Vdash_P \underset{\sim}{\pi} \colon M_0[\underset{\sim}{G}] \to N_0^*$. We phrase now the desired criterion and we shall use the direction from right to left later.

Lemma 4.5 *Using the above notation,* (q_0, q_1) *is completely generic over* $(M_0, P * \underset{\sim}{Q})$*, iff*

(1) q_0 *is completely* (M_0, P)*-generic, and*
(2) *for some* $\mathcal{G}_1 \subseteq Q_0^*$ *that is* (N_0^*, Q_0^*)*-generic* $q_0 \Vdash$ *“*$\underset{\sim}{\pi}^{-1}{}'' \mathcal{G}_1$ *is bounded by* q_1*”.*

In this case the filter induced by (q_0, q_1) *over* $M_0 \cap P * \underset{\sim}{Q}$ *is* $\pi^{-1}{}'' \mathcal{G}_0 * \mathcal{G}_1$.

Given a countable $M_0 \prec H(\chi)$ such that the two step iteration $P * \underset{\sim}{Q}$ is in M_0, our aim is to extent each (M_0, P)-generic filter G_0 to an $(M_0, P * \underset{\sim}{Q})$-generic filter. This definition depends not only on M_0 but also on another countable elementary submodel $M_1 \prec H(\chi)$ such that $M_0 \in M_1$ and $G_0 \in M_1$. In addition we fix a $p_0 \in P * \underset{\sim}{Q}$ which we want to include in the extended filter. All of this leads us to a five place function $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ that we define now.

Definition 4.6 Let P be a poset that adds no new countable sets of ordinals and suppose that $\underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in \mathbf{V}^P$ are such that

$$\Vdash_P \quad \underset{\sim}{\mathbb{D}} \in \mathbf{V}\text{is an } \aleph_1\text{-completeness system and}$$
$$\underset{\sim}{Q}\text{is } \mathbb{D}\text{-complete with respect to } \underset{\sim}{\mathbb{D}}.$$

Let χ be sufficiently large and $M_0 \prec M_1 \prec (H(\chi), \in, <^*_\chi)$ be countable elementary submodels with $M_0 \in M_1$ and $P, \underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in M_0$. Let $G_0 \subseteq M_0 \cap P$ be (M_0, P)-generic and suppose that $G_0 \in M_1$. Let $p_0 \in P * \underset{\sim}{Q} \cap M_0$ be given $p_0 = (a, \underset{\sim}{b})$ with $a \in G_0$. Then we define

$$G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0),$$

an $(M_0, P * \underset{\sim}{Q})$-generic filter containing p_0 (dominating G_0) by the following procedure:

Let $\pi : M_1 \to N_1$ with $\pi(M_0) = N_0$ be the transitive collapse and $\mathcal{G}_0 = \pi'' G_0$. Form $N_0^* = N_0[\mathcal{G}_0]$. Observe that $N_0^* \in N_1$. Let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and let $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. Then $\mathbb{D}_0 \in N_0$, because it is forced to be in the ground model. So $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably closed completeness system. Thus $\mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is defined in N_1, where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is countable,

$$\text{there is some } \mathcal{G}_1 \in \bigcap(N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)). \tag{4.4}$$

$\mathcal{G}_1$ is (N_0^*, Q_0^*)-generic and $b^* \in \mathcal{G}_1$. Form $\mathcal{G}_0 * \mathcal{G}_1 = \mathcal{G}$, an $(N_0, \pi(P * \underset{\sim}{Q}))$-generic filter. Then $\pi(p_0) \in \mathcal{G}$. Finally we define

$$G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0) = \pi^{-1\prime\prime}\mathcal{G}. \tag{4.5}$$

Now observe that if for some ν with Eq. (4.2) for (N_0^*, Q_0^*, b^*) instead of $(M, Q_{\mathbf{T}}, p)$, the real η fulfils Eq. (4.3), then the existence of Eq. (4.4) is given by

$$\pi^{-1\prime\prime}\mathcal{G}_1 = c^{\prime\prime}\mathbf{B}_1(\eta, U(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0]))$$

and hence is Borel computable from η and the code U of the intermediate model (N_0^*, Q_0^*, b^*).

In fact, we want to define a formula ψ so that

$$H(\chi) \models \psi(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$$

iff Eq. (4.5) holds. That is, we want to define $\mathbb{E}$ in $H(\chi)$. We cannot take the above definition verbally, because it relies on the assumption that M_0 and M_1 are elementary substructures of $H(\chi)$, something which is not expressible in $H(\chi)$. Whenever the definition above relies on some fact that happens not to hold we let $\mathcal{G}$ have an arbitrary value. For example if N_0^* is not in N_1 or if $N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is empty, then we let

$\mathcal{G}$ be some arbitrary fixed N_0-generic filter. The Borel computation does not invoke N_1, since $\pi^{-1\prime\prime}\mathcal{G}_1 = c^{\prime\prime}\mathbf{B}_1(\eta, U(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0]))$. Here, G_0 is a parameter and will be set $\{0_{P_0}\}$ later, so that in the end (that means in Lemma 3.11) only the possible isomorphism types of $(M_0, \in\restriction M_0, <^*_\chi\restriction M_0, P_\gamma, p, \bar{\beta})$ need to be guessed stationarily often alongside with names for the F and f from the statement of the weak diamond.

The following lemma shows the second part of the argument: We want to show the G given in Eq. (4.5) is bounded. The lemma analyses the iteration of two posets when the second is $\mathbb{D}$-complete.

Lemma 4.7 *The One Step Extension Lemma. Let P be poset and suppose that $\underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in \mathbf{V}^P$ are such that*

$$\Vdash_P \underset{\sim}{\mathbb{D}} \in \mathbf{V} \textit{is an } \aleph_1\textit{-completeness system and}$$
$$\underset{\sim}{Q} \textit{ is } \mathbb{D}\textit{-complete with respect to } \underset{\sim}{\mathbb{D}}.$$

Let χ be sufficiently large and $M_0 \prec M_1 \prec H_\chi$ be countable elementary submodels with $M_0 \in M_1$ and $P, \underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in M_0$. Suppose that $q_0 \in P$ is (M_1, P)-generic as well as completely (M_0, P)-generic, and let $G_0 \subseteq M_0 \cap P$ be the M_0 filter over $M_0 \cap P$ induced by q_0. Let $p_0 \in P * \underset{\sim}{Q}$, $p_0 \in M_0$ be given, so that $p_0 = (a, \underset{\sim}{b})$ and $a \in G_0$. Then there is $q_1 \in \mathbf{V}^P$ such that (q_0, q_1) is completely generic over $(M_0, P * \underset{\sim}{Q})$ and $p_0 \leq (q_0, q_1)$, in fact (q_0, q_1) bounds $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0) = G_0 * c^{\prime\prime}\mathbf{B}_1(\eta, U(N_0^*, Q_0^*, \pi(\underset{\sim}{b})))$.

Proof This is literally [1, The Gambit Lemma]. For completeness' sake we repeat Abraham's proof here. Notice that $G_0 \in M_1$ by the following argument: Let R be the collection of all conditions $r \in P$ that are completely generic over M_0. Then $R \in M_1$ and $q_0 \in R \cap M_1$. Since q_0 is (M_1, P)-generic, it follows that it is compatible with some $r \in R \cap M_1$. But any two compatible conditions in R induce the same filter, and hence G_0 is the filter induced by r.

Let $\pi : M_1 \to N_1, \pi(M_0) = N_0$, be the transitive collapse and $\mathcal{G}_0 = \pi''G_0$. We recall the definition of $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$. Form $N_0^* = N_0[\mathcal{G}_0]$ and let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and let $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. Then $\mathbb{D}_0 \in N_0$ because it is forced to be in the ground model. So $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably closed completeness system. Thus $\mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is defined in N_1, where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is countable, there is some $\mathcal{G}_1 \in \bigcap(N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*))$. $\mathcal{G}_1$ is (N_0^*, Q_0^*)-generic and $b^* \in \mathcal{G}_1$. Form $\mathcal{G}_0 * \mathcal{G}_1 = \mathcal{G}$, an $(N_0, \pi(P * \underset{\sim}{Q}))$-generic filter. Then $\pi(p_0) \in \mathcal{G}$. We defined $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ as $\pi^{-1\prime\prime}\mathcal{G}$.

Let $\underset{\sim}{G} \in \mathbf{V}^P$ be the canonical name of the generic filter over P. Then q_0 forces that π can be extended to a collapse $\underset{\sim}{\pi}$ which is onto N_0^*, that is

$$q_0 \Vdash_P \underset{\sim}{\pi} : M_0[\underset{\sim}{G}] \to N_0^*.$$

The conclusion of our lemma follows if we show that

$$q_0 \Vdash_P \underset{\sim}{\pi}^{-1\prime\prime}\mathcal{G}_1 \text{is bounded in} \underset{\sim}{Q}. \tag{4.6}$$

In this case, if we define $q_1 \in \mathbf{V}^P$ so that $q_0 \Vdash_P q_1$ bounds$\underset{\sim}{\pi}^{-1\prime\prime}\mathcal{G}_1$, then the previous lemma implies that the $(M_0, P * \underset{\sim}{Q})$-generic filter induced by (q_0, q_1) is $\pi^{-1\prime\prime}\mathcal{G}_0 * \mathcal{G}_1$.

So let F be $(\mathbf{V}, P)$-generic with $q_0 \in F$. $\underset{\sim}{\pi}[F]$ collapses $M_0[F]$ onto N_0^* and there is a set $X \in \mathbb{D}_0(N_0^*, Q_0^*, b^*)$, so that if $\mathcal{H} \in X$ is any filter then $\pi^{-1\prime\prime}\mathcal{H}$ is bounded in $\underset{\sim}{Q}[F]$. As $N_1[F] \prec H_\chi[F]$, we can have $X \in N_1[F]$. But since $\mathbb{D}_0$ is in the ground model, $X \in N_1$. Thus $\mathcal{G}_1 \in X$, where $\mathcal{G}_1$ is the filter defined above. This proves Eq. (4.6). □

The iteration theorem

Let P_γ be a countable support iteration of length γ obtained by choosing iterands $Q_\alpha \in \mathbf{V}^{P_\alpha}$ as in the theorem. That is, each Q_α is $\mathbb{D}$-complete in $\mathbf{V}^{P_\alpha}$ for some $\aleph_1$-completeness system taken from $\mathbf{V}$. Let χ be a sufficiently large regular cardinal. To prove the theorem we first describe a machinery for obtaining generic filters over countable submodels of $H(\chi)$. We define a function $\mathbb{E}$ that takes five arguments, $\mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0)$ of the following types.

(1) $M_0 \prec H_\chi$ is countable, $P_\gamma \in M_0$, so $\gamma \in M_0$. Moreover, $p_0 \in M_0 \cap P_\gamma$.
(2) For some $\gamma_0 \in M_0 \cap \gamma$, G_0 is an (M_0, P_{γ_0})-generic filter and such that $p_0 \restriction \gamma_0 \in G_0$. We assume that $G_0 \in M_1$.
(3) The order type of $M_0 \cap [\gamma_0, \gamma)$ is α.
(4) $\bar{M} = \langle M_\xi : 0 \le \xi \le \alpha \rangle$ is an $\alpha + 1$-tower of countable elementary submodels of $H(\chi)$ and $M_0 = M$. Note that only $M_0 = M$ appears in the statement of the theorem. The rest $\langle M_\xi : 1 \le \xi \le \alpha \rangle$ of the tower is a technical means for the proof.

The value returned, $G_\gamma = \mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0)$ is an (M_0, P_γ)-generic filter that extends G_0 and contains p_0. Formally, in saying that G_γ extends G_0, we mean that the restriction projection takes G_γ onto G_0. The definition of $\mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0)$ is by induction on $\alpha < \omega_1$.

Assume that $\alpha = \alpha' + 1$ is a successor ordinal. Then $\gamma = \gamma' + 1$ is also a successor. Assume first that $\gamma_0 = \gamma'$. Then $\alpha = 1$ and we have only two structures: M_0 and M_1. Since P_γ is isomorphic to $P_{\gamma_0} * Q_{\gamma_0}$ we can define G_γ by Eq. (4.5). So, if for some ν with Eq. (4.2) η fulfils Eq. (4.3) for $(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0])$ in the role of of $(M, Q_{\mathbf{T}}, p)$, then

$$G_\gamma = \mathbb{E}(M_0, M_1, P_{\gamma_0} * Q_{\gamma_0}, G_0, p_0) = G_0 * c''\mathbf{B}_1(\eta_0, U(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0])).$$

Assume next that $\gamma_0 < \gamma'$. Then by induction hypothesis, if all $\eta_i, i < \alpha'$, are sufficiently large, then

$$\begin{aligned} G_{\gamma'} &= \mathbb{E}(M_0, \langle M_\xi : 1 \le \xi \le \alpha' \rangle, P_{\gamma'}, G_0, p_0 \restriction \gamma') \\ &= G_0 * c''\mathbf{B}_{\alpha'}(\langle \eta_i : 0 \le i < \alpha' \rangle, U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma')}[G_0], \\ &\quad p_0 \restriction [\underset{\sim}{\gamma_0}, \gamma')[G_0]]) \end{aligned} \tag{4.7}$$

is defined and is an $(M_0, P_{\gamma'})$-generic filter that extends G_0 and contains $p_0 \restriction \gamma'$. Moreover by elementarity, $G_{\gamma'} \in M_\alpha$. When we finish this definition it will be evident that it continues for every $\alpha < \omega_1$ since $M_\alpha \prec H(\chi)$ and the parameters are all in M_α. This brings us to the previous case and we choose $\eta_{\alpha'}$ such that it fulfils Eq. (4.3) for some ν with (4.2) in which $(M, Q_{\mathbf{T}}, p)$ is replaced by

$$(M_0[G_{\gamma'}], \underset{\sim}{Q}_\gamma[G_{\gamma'}], p_0(\underset{\sim}{\gamma}')[G_{\gamma'}]).$$

Now from Eq. (4.7) we define temporarily

$$U' = U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma')}[G_0], p_0 \restriction [\underset{\sim}{\gamma_0}, \gamma')[G_0)]). \tag{4.8}$$

Then

$$\begin{aligned} G_\gamma &= \mathbb{E}(M_0, M_\alpha, P_{\gamma'} * Q_{\gamma'}, G_{\gamma'}, p_0) \\ &= G_0 * c'' \mathbf{B}_1(\eta_{\alpha'}, U(M_0[G_0 * c'' \mathbf{B}_{\alpha'}(\langle \eta_i : i < \alpha' \rangle, U')], \\ &\qquad\qquad \underset{\sim}{Q}_\gamma[G_0 * c'' \mathbf{B}_{\alpha'}(\langle \eta_i : i < \alpha' \rangle, U')], \\ &\qquad\qquad p_0(\underset{\sim}{\gamma}')[G_0 * c'' \mathbf{B}_{\alpha'}(\langle \eta_i : i < \alpha' \rangle, U')])) \\ &=: G_0 * c'' \mathbf{B}_\alpha(\langle \eta_i : i < \alpha \rangle, U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma)}[G_0], \underset{\sim}{p_0}[G_0])) \end{aligned} \tag{4.9}$$

and the middle U' is defined above in Eq. (4.8). This justifies that the Borel functions given by induction hypothesis can be composed to one Borel function of the required arguments.

Now it is also clear how to define the *strategy* $\sigma(\langle \nu_i, \eta_i : i < \alpha' \rangle)$: The generic player plays $\nu_{\alpha'}$ so that it fulfils Eq. (4.2), where $(M, Q_{\mathbf{T}}, p)$ is replaced by $(M_0[G_{\gamma'}], \underset{\sim}{Q}_\gamma[G_{\gamma'}], p_0(\underset{\sim}{\gamma}')[G_{\gamma'}])$ with $G_{\gamma'}$ as in Eq. (4.7).

Now assume that α is a limit ordinal and let $\langle \alpha_n : n \in \omega \rangle$ be an increasing cofinal sequence with $\alpha_0 = 0$. Let $\gamma_n \in M_0$ be such that $\alpha_n = \mathrm{otp}(M_0 \cap [\gamma_0, \gamma_n))$. Let $\langle I_n : n \in \omega \rangle$ be an enumeration of all dense subsets of P_γ that are in M_0 in such a way that I_n is the $<^*_\chi$-least dense subset of P_γ that is not among $\{I_m : m < n\}$.

We define

$$\begin{aligned} G_\gamma &= \mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0) \\ &= G_0 * c'' \mathbf{B}_\alpha(\langle \eta_i : i < \alpha \rangle, U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma)}[G_0], p_0 \restriction [\underset{\sim}{\gamma_0}, \gamma)[G_0])) \end{aligned}$$

as follows. We define by induction on $n \in \omega$ a condition $p_n \in P_\gamma \cap M_0$ and an (M_0, P_{γ_n})-generic filter $G_n \in M_{\alpha_{n+1}}$ such that

(1) G_0 and p_0 are given. $p_n \restriction \gamma_n \in G_n$.
(2) $p_n \leq p_{n+1}$ and $p_{n+1} \in I_n$.

Suppose that G_n and p_n are defined. First we can find $p_{n+1} \in I_n \cap M_0$ such that $p_{n+1} \restriction \gamma_n \in G_n$ (for an existence proof see [1, Lemma 1.2]) and we take the $<^*_\chi$-least in M_0 so that it is Borel computed. Now define

$$\begin{aligned} G_{n+1} &= \mathbb{E}(M_0, \langle M_\xi : \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle, P_{\gamma_{n+1}}, G_n, p_{n+1} \restriction \gamma_{n+1}) \\ &= G_0 * c'' \mathbf{B}_{\alpha_{n+1} - \alpha_n}(\langle \eta_i : i \in [\alpha_n, \alpha_{n+1}) \rangle, U^*) \end{aligned}$$

Here we have

$$
\begin{aligned}
U^* = {} & U(M_0[G_0 * c''\mathbf{B}_{\alpha_n}(\langle \eta_i : i < \alpha_n\rangle, U'')],\\
& P_{[\underset{\sim}{\gamma_n}, \gamma_{n+1})}[G_0 * c''\mathbf{B}_{\alpha_n}(\langle \eta_i : i < \alpha_n\rangle, U'')],\\
& p_{n+1} \restriction [\underset{\sim}{\gamma_n}, \gamma_{n+1})[G_0 * c''\mathbf{B}_{\alpha_n}(\langle \eta_i : i < \alpha_n\rangle, U'')]) \text{ and}\\
U'' = {} & U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma_n)}[G_0], p_{n+1} \restriction \underset{\sim}{[}\gamma_0, \gamma_n)[G_0]).
\end{aligned}
$$

Finally let

$$G_\gamma = \text{the generic filter generated in } M_0 \text{ by } \{p_n : n \in \omega\}.$$

From the above induction on $n < \omega$ and from the induction hypothesis it is clear that there is a Borel function $\mathbf{B}_\alpha$ such that

$$G_\gamma = G_0 * c''\mathbf{B}_\alpha(\langle \eta_i : i < \alpha\rangle, U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma)}[G_0], p_0 \restriction [\underset{\sim}{\gamma_0}, \gamma)[G_0])) \quad (4.10)$$

This ends the definition of $\mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0)$ and of $\mathbf{B}_\alpha$.

The *strategy* σ for the generic player is defined by the prescription, that in the limit game of length α he plays according to the strategies for the initial segments of the game. (This justifies that σ_α is just named σ, for all lengths α.) This is a winning strategy, as the Borel function was just derived. It gives a generic filter. We still have to show that the given generic filter is bounded.

Now the missing part is to show that "all the generic filters are bounded" is preserved in the limit steps of the iteration. Again there is nothing new to our work and we repeat Abraham's proof to [1, The Extension Lemma].

Lemma 4.8 *Let $\langle P_\alpha, \underset{\sim}{Q}_\beta : \beta < \gamma, \alpha \leq \gamma\rangle$ be a countable support iteration of forcing posets such that each iterand Q_α satisfies the following in $\mathbf{V}^{P_\alpha}$:*

(1) *Q_α is δ-proper for every countable δ.*
(2) *Q_α is $\mathbb{D}$-complete with respect to some countably closed completeness system in the ground model that has the property that all $\eta \geq^* \nu$ serve as parameters.*

Suppose that $M_0 \prec H(\chi)$ is countable, $P_\gamma \in M_0$ and $p_0 \in P_\gamma \cap M_0$. For any $\gamma_0 \in \gamma \cap M_0$ with $\alpha = \mathrm{otp}(M_0 \cap [\gamma_0, \gamma))$ and $\bar{M} = \langle M_\xi : \xi \leq \alpha\rangle$ is a tower of countable elementary substructures starting with the given M_0, then the following holds:

For every $q_0 \in P_{\gamma_0}$ that is completely (M_0, P_{γ_0})-generic as well as $(\bar{M}, P_{\gamma_0})$-generic, if $p_0 \restriction \gamma_0 < q_0$, then there is some $q \in P_\gamma$ such that $q_0 = q \restriction \gamma_0$ and $p_0 < q$ and q is completely (M_0, P_γ)-generic. In fact, the filter induced by q is $\mathbb{E}(M_0, \langle M_\xi : 1 \leq \xi \leq \alpha\rangle, P_\gamma, G_0, p_0)$ where $G_0 \subseteq P_{\gamma_0} \cap M_0$ is the filter induced by q_0.

Proof Let $G_0 \subseteq P_{\gamma_0} \cap M_0$ be the M_0-generic filter induced by q_0. Observe that $G_0 \in M_1$ follows from the assumption that q_0 is also M_1-generic. We shall prove by induction on $\alpha = \mathrm{otp}(M_0 \cap [\gamma_0, \gamma))$ that q can be found that bounds $G_\gamma = \mathbb{E}(M_0, \langle M_\xi : 1 \le \xi \le \alpha\rangle, P_\gamma, G_0, p_0)$.

Suppose first that $\alpha = \alpha' + 1$ and consequently $\gamma = \gamma' + 1$ are successor ordinals. Define in M_α, $X \subseteq P_{\gamma_0}$ as maximal antichain of conditions r so that

(1) r bounds G_0,
(2) r in $\langle M_\xi : 1 \le \xi \le \alpha'\rangle$-generic.

Then $X \in M_\alpha$ is predense above q_0. By our inductive assumption, every $r_0 \in X$ has a prolongation $r_1 \in P_{\gamma'}$ that bounds $G_{\gamma'} = \mathbb{E}(M_0, \langle M_\xi : 1 \le \xi \le \alpha'\rangle, G_0, p_0 \restriction \gamma')$. Since all the parameters are in M_α, we get that $G_{\gamma'} \in M_\alpha$. Since $M_\alpha \prec H(\chi)$ we can choose $r_1 \in M_\alpha$ whenever $r_0 \in X \cap M_\alpha$. This defines a name $\underset{\sim}{r}_1 \in \mathbf{V}^{P_{\gamma_0}}$, forced by q_0 to be in $M_\alpha \cap P_{\gamma'}$. Namely, if G is any $(\mathbf{V}, P_{\gamma_0})$-generic filter containing q_0, then $X \cap G$ contain a unique condition r_0, and we let $\underset{\sim}{r}_1[G] = r_1$. By the Properness Extension Lemma [1, Lemma 2.8] we can find $q_1 \in P_{\gamma'}$, $q_1 \restriction \gamma_0 = q_0$, q_1 is $(M_\alpha, P_{\gamma'})$-generic, and $q_1 \Vdash_{P_{\gamma'}}$ "$\underset{\sim}{r}_1$ is in the generic filter $\underset{\sim}{G}_{\gamma'}$". It follows that q_1 bounds $G_{\gamma'}$. We find $q_2 \in P_\gamma$, such that $q_2 \restriction \gamma' = q_1$ and q_2 bounds G_γ. In order to define $q_2(\gamma)$ we use the Two Step Lemma and Eq. (4.6).

Now assume that α is a limit ordinal. We follow the definition of G_γ see Eq. (4.10). Recall that we had an ω-sequence $\langle \alpha_n : n \in \omega\rangle$ cofinal in α and we defined γ_n cofinal in γ as the resulting sequence $\alpha_n = \mathrm{otp}(M_0 \cap [\gamma_0, \gamma_n))$. We defined by induction $p_n \in P_\gamma \cap M_0$ and filters $G_n \subseteq P_{\gamma_n}$, $G_n \in M_{\alpha_n+1}$ and defined G_γ as the filter generated by the p_n's. We shall define now $q_n \in P_{\gamma_n}$ by induction on n so that the following hold

(1) q_n bounds G_n,
(2) $p_n \restriction \gamma_n \le q_n$,
(3) $q_n = q_{n+1} \restriction \gamma_n$,
(4) q_n is $\langle M_\xi : \alpha_n + 1 \le \xi \le \alpha\rangle$-generic over P_{γ_n}.

Thus q_n gains in length and looses in status as an M_ξ-generic condition for $0 < \xi \le \alpha_n$. But q_n is completely (M_0, P_{γ_n})-generic for all n. Finally $q = \bigcup q_n$ is not M_ξ-generic for any $\xi > 0$. However, q is completely (M_0, P_γ)-generic.

Suppose that q_n is defined. Let X be in $M_{\alpha_{n+1}+1}$ be a maximal antichain in P_{γ_n} of conditions r that induce G_n and are $\langle M_\xi : \alpha_n + 1 \le \xi \le \alpha_{n+1}\rangle$-generic over P_{γ_n}. Observer that X is predense above q_n. For each $r_0 \in X$, define by the induction assumption $r_1 \in P_{\gamma_{n+1}}$ such that r_1 bounds G_{n+1}, $p_{n+1} \restriction \gamma_{n+1} < r_1$ and $r_1 \restriction \gamma_n = r_0$. If $r_0 \in X \cap M_{\alpha_{n+1}+1}$, then r_1 is taken from $M_{\alpha_{n+1}+1}$. Now view $\{r_1 : r_0 \in X\}$ as a name $\underset{\sim}{r}$ for a condition forced by q_n to lie in $M_{\alpha_{n+1}+1}$. By the α-Extension Lemma [1, Lemma 5.6], define q_{n+1} that satisfies items 2 to 4 from the above list and such that $q_{n+1} \Vdash_{P_{\gamma_{n+1}}} \underset{\sim}{r} \in \underset{\sim}{G}_{n+1}$. Then q_{n+1} bounds G_{n+1} and is a required. $\square$

End of proof of Theorem 4.4: Now that the induction is performed,we set $\gamma_0 = 0$, $G_0 = \{0_{P_0}\}$, $p_0 = p \in P_\gamma$ from the statement of Theorem 4.4. Then $N_0^* = N_0 = \pi(M_0)$, $\pi(P_{[\gamma_0,\gamma)})[\underset{\sim}{\mathcal{G}}_0] = \pi(P_\gamma)$ and $\pi(p_0)_{[\gamma_0,\gamma)})[\underset{\sim}{\mathcal{G}}_0] = \pi(p)$ and the $\mathbf{B}_\alpha$'s second argument is just the isomorphism type of $(M_0, \in, <^*_\chi, P_\gamma, p, \bar{\beta})$ $\square$.

5 Working with miller reals

In this section we show that the antigeneric player in the game $\partial(M, P_\gamma, p)$ can influence the outcome of another game. This second game aims at providing a function g witnessing $\diamondsuit$(reaping). In the end $g(\delta)$ will be $C_{\mathbf{B}'}$ from the next lemma for a suitably chosen Borel function $\mathbf{B}'$.

In [18, Lemma 2.11] we worked with the Laver property and coverings by small slaloms. Now we work with Miller reals for analysing the second game. We recall Miller forcing: The conditions are of the form $p = (t, T)$ with $t \in \omega^{<\omega}$ (called the trunk of p, $\mathrm{tr}(p)$) and $T \subseteq \omega^{<\omega}$ being a superperfect tree, i.e., a non-empty subset T of $\omega^{<\omega}$ that is closed under initial segments and that contains for every $s \in T$ a $t \in T$ such that $t \restriction |s| = s$ whose set of immediate successors

$$\mathrm{succ}(t, p) := \{n \in \omega \,:\, t\hat{}\langle n\rangle \in T\}$$

is infinite. The latter is called "t is infinitely splitting".

Lemma 5.1 *Suppose that*

(α) $\gamma < \omega_1$, and
(β) $\mathbf{B}'$ is a Borel function from $(\omega^\omega)^\gamma$ to 2^ω.

Then we can find some $C = C_{\mathbf{B}'} \in [\omega]^\omega$ such that in the following game $\partial_{(\gamma,\mathbf{B}')}$ between two players, IN and OUT, the player IN has a winning strategy, the play lasts γ moves and in the ε-th move OUT chooses $\nu_\varepsilon \in \omega^\omega$ and then IN chooses $\eta_\varepsilon \not\leq^* \nu_\varepsilon$. In the end IN wins iff $\mathbf{B}'(\langle \eta_\varepsilon \,:\, \varepsilon < \gamma\rangle)$ is almost constant on C.

Proof Assume that $P^*_\gamma = \langle P^*_\xi, \underset{\sim}{Q}^*_\zeta \,:\, \xi \leq \gamma, \zeta < \gamma\rangle$ is a c.s. iteration of Miller forcing and assume that $p \in P^*_\gamma$ and $\langle \underset{\sim}{\rho}_\xi \,:\, \xi < \gamma\rangle$ is a sequence of names for the P^*_ξ-generics. By CH, there is a P-point U in the ground model and by [5], Miller forcing preserves P-points. Moreover, the P-point preservation property is preserved in countable support iterations [4, Sect.4] or [3]. Hence we have

$$\Vdash_{P^*_\gamma} \mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon \,:\, \varepsilon < \gamma\rangle) \text{ is almost constant on a set in U.}$$

Given $M^* \prec H(\chi)$ such that $\mathbf{B}' \in M^*$ and a P-point $U \in M^*$ with pseudo-intersection $C_{\mathbf{B}'} \subseteq^* x$ for $x \in U \cap M^*$,

$$\Vdash_{P^*_\gamma} \mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon \,:\, \varepsilon < \gamma\rangle) \text{ is almost constant on } C_{\mathbf{B}'}.$$

Now we show that player IN can play for $C_{\mathbf{B}'}$ in a way that he imitates the Miller-generic reals over the countable elementary submodel M^*, so that actually everything is in the ground model.

Let $M^* \prec (H(\chi), \in)$ be countable such $\mathbf{B}', P^*_\gamma, p^* \in M^*$. (So M^* is not the M from the next proof, but rather contains a non-trivial part of the power-set of that latter M.) Now we prove by induction on $j \leq \gamma$ for all $i < j$

$\otimes_{i,j}$ Assume that $P^*_j \in M^*$ and $G_i \subseteq P^*_i \cap M^*$ generic over M^*, and p^* is so that $p^* \in P^*_j \cap M^*$ and $p^* \restriction i \in G_i$. Then in the following game $\eth^*_{(i,j,G_i,p^*)}$ player II has a winning strategy $\sigma_{(i,j,G_i,p^*)}$. There are $j-i$ moves indexed by $\varepsilon \in [i,j)$, and in the ε-th move $(p_\varepsilon, \nu_\varepsilon, \eta_\varepsilon)$ are chosen such that player I chooses $p_\varepsilon \in P_\varepsilon/G_i$, $p_\varepsilon \geq p^* \restriction \varepsilon$, and $\nu_\varepsilon \in \omega^\omega$ and player II chooses $\eta_\varepsilon \not\leq^* \nu_\varepsilon$.
First case: there is a (P^*_ε, M^*)-generic $G_\varepsilon \subseteq P^*_\varepsilon \cap M^*$, such that $p^*(\varepsilon) \in G_\varepsilon$ and $G_\varepsilon \supset G_i$ and $(\forall \xi \in [i,\varepsilon)\underset{\sim}{\rho}_\xi[G_\varepsilon] = \eta_\xi$ and $M^*[G_\varepsilon \cap P^*_\xi] \models p_\xi \geq p^*(\xi)$. In this case player I chooses $p_\varepsilon \in G_\varepsilon$ forcing this and so that $M^*[G_\varepsilon] \models p^*(\varepsilon) \leq_{P^*_\varepsilon} p_\varepsilon$. Then player I chooses ν_ε dominating $M^*[G_\varepsilon]$ and the second player chooses $\eta_\varepsilon \not\leq^* \nu_\varepsilon$.
Second case: There is no such G_ε. Then player I won the play.

We prove by induction on j that player II wins the game $\eth^*_{(i,j,G_i,p^*)}$: Case 1: $j=0$. Nothing to do. Case 2: $j = j^*+1$. For $\varepsilon \in [i,j)$ we use the strategy for $\eth^*_{(i,j,G_i,p^*)}$, and for $\varepsilon = j$ we make the following move: just say that we should find a generic G^{j^*} of ${Q^*}_{j^*}^{M^*[G_{j^*}]}$ to which $p^*(j^*)$ belongs and such that $\underset{\sim}{\rho}_{j^*}[G^{j^*}] \not\leq^* \nu_{j^*}$.

First take $q \geq p^*(j^*)$ such that q is $(M^*[G_{j^*}], {Q^*}_{j^*}^{M^*[G_{j^*}]})$-generic. $q \in \mathbf{V}$ is a Miller condition. Now we take a stronger condition q' by letting $\mathrm{tr}(q) = \mathrm{tr}(q')$ and for every splitting node $s \in q'$ of length n, $\mathrm{succ}(q',s) = \{k \in \mathrm{succ}(q,s) : k \geq \nu_{j^*}(n)\}$. Now let

$$G^{j^*} = \{r \in M^*[G_{j^*}] : q' \geq r\}.$$

Since q' is a $(M^*[G_{j^*}], {Q^*}_{j^*}^{M^*[G_{j^*}]})$-generic condition, G^{j^*} is a $(M^*[G_{j^*}], {Q^*}_{j^*}^{M^*[G_{j^*}]})$-generic filter. The generic real is $\underset{\sim}{\rho}_{j^*}[G^{j^*}] = \bigcup\{s : (s,T) \in G^{j^*}\}$. Then $q' \Vdash \underset{\sim}{\rho}_{j^*} \not\leq^* \nu_{j^*}$.

Now player II takes $\eta_{j^*} = \underset{\sim}{\rho}_{j^*}[G^{j^*}]$. We set $G_j = G_{j^*} * G^{j^*}$. Case 3: j is a limit. From the proof of the preservation of properness (see, e.g., [23, Chap. II, Theorem 3.2, Chap. II., Section 3.3, or Chap. XII, Theorem 1.8]) we get that existence of p_ε, so player I can never win the game on the ground of the second case.

The winning condition for player II is preserved in the limit steps, since it is a requirement on all formerly chosen η_ε.

Now we show that $\otimes_{i,j}$ suffices. We use $i = 0, j = \gamma, \mathbf{B}' \in M_*$. Take $P^*_\gamma \in M^*$, $p^* \in P^*_\gamma \cap M^*$. Let $\sigma(0,\gamma,\{\emptyset\},p^*)$ be a winning strategy for player II in the game $\eth^*_{(0,\gamma,\{\emptyset\},p^*)}$. During the play of $\eth_{(\gamma,\mathbf{B}')}$ let ν_ε be chosen in stage $\varepsilon < \gamma$. The player IN simulates on the side a play of $\eth^*_{(0,\gamma,\{\emptyset\},p^*)}$: As a move of I he assumes the ν_ε chosen by OUT in the play of $\eth_{(\gamma,\mathbf{B}')}$ and p_ε, $p_\varepsilon \restriction \delta = p_\delta$ for $\delta < \varepsilon$, the p_δ gotten from earlier simulations. Then player IN uses $\sigma(0,\gamma,\{\emptyset\},p^*)$ for player II, applied to $(p_\varepsilon, \nu_\varepsilon)$, to compute an η_ε, which he presents in this move in $\eth_{(\gamma,\mathbf{B}')}$. So p_ε forces that there is a Miller generic $\underset{\sim}{\rho}_\varepsilon[G^\varepsilon] =: \eta_\varepsilon$ over $M^*[G_\varepsilon]$ and that $\eta_\varepsilon \not\leq^* \nu_\varepsilon$. The requirement $\eta_\varepsilon \not\leq^* \nu_\varepsilon$ is fulfilled.

Suppose that a run of the game has been played. Let $C_{\mathbf{B}'}$ be a pseudo-intersection of $U \cap M^*$ for an ultrafilter $U \in M^*$. So we have $\langle \nu_\varepsilon, \eta_\varepsilon : \varepsilon < \gamma\rangle$ and there is $p = \bigcup_{\varepsilon<\gamma} p_\varepsilon \geq p^*$, and for $\varepsilon < \gamma$ there is the name for the Q^*_ε-generic real, namely $\underset{\sim}{\rho}_\varepsilon \in M_*$, such that for all $\varepsilon < \gamma$, $p \Vdash_{P^*_\gamma} \underset{\sim}{\rho}_\varepsilon = \check{\eta}_\varepsilon$. So as $p \Vdash_{P^*_\gamma}$

"$\mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon : \varepsilon < \gamma\rangle)$is almost constant on $C_{\mathbf{B}'}$", we have

$$\mathbf{B}'(\langle \eta_\varepsilon : \varepsilon < \gamma\rangle)\text{is almost constant on } C_{\mathbf{B}'}. \qquad \square$$

The following final lemma is analogous to [18, Lemma 3.11], just for another Borel relation. We include it for completeness' sake.

Lemma 5.2 *Assume that* $\mathbf{V} \models \diamondsuit_S$ *for some stationary set* S*. Then*

$$\Vdash_{P_{\omega_2}} \diamondsuit_S(\textit{reaping}).$$

Proof Let G be P_ω-generic over $\mathbf{V}$. We use the $\diamondsuit_S$-sequence $\langle A_\delta : \delta \in S\rangle$ in the following manner: By easy coding we have $\langle (N^\delta, \bar{\beta}^\delta, \underset{\sim}{f}^\delta, \underset{\sim}{F}^\delta, \underset{\sim}{C}^\delta, P^\delta_{\omega_2}, p^\delta, <^\delta) : \delta \in S\rangle$ such that

(a) N^δ is a transitive collapse of a countable $M \prec H(\chi, \in, <^*_\chi)$, $<^\delta$ is a well-ordering of N^δ, U^δ codes the isomorphism type of $(N^\delta, P^\delta_{\omega_2}, p^\delta, \bar{\beta}^\delta)$.
(b) $N^\delta \models P^\delta_{\omega_2} = \langle P^\delta_\alpha, \underset{\sim}{Q}^\delta_\beta : \alpha \le \omega_2^{N^\delta}, \beta < \omega_2^{N^\delta}\rangle$ is as in Definition 3.5.
(c) $N^\delta \models (p^\delta \in P^\delta_{\omega_2}, \underset{\sim}{f}^\delta$ is a $P^\delta_{\omega_2}$-name of a member of ${}^{\omega_1}2\,\underset{\sim}{F}^\delta : 2^{<\omega_1} \to 2^\omega$ is Borel).
(d) If $p \in P_{\omega_2}$,

$$p \Vdash_{P_{\omega_2}} \underset{\sim}{f} \in 2^{\omega_1} \wedge \underset{\sim}{F} : 2^{<\omega_1} \to 2^\omega \text{is Borel, } \underset{\sim}{C} \subseteq \omega_1 \text{is club,}$$

and $p, P_{\omega_2}, \underset{\sim}{F}, \underset{\sim}{f}, \underset{\sim}{C} \in H(\chi)$, then

$$\begin{aligned} S(p, \underset{\sim}{F}, \underset{\sim}{f}) := \{\delta \in S : \ & \text{there is a countable} M \prec (H(\chi), \in, <^*_\chi) \\ & \text{such that } \underset{\sim}{f}, \underset{\sim}{F}, \underset{\sim}{C}, P_{\omega_2}, p \in M \\ & \text{and there is an isomorphism } h^\delta \text{ from } N^\delta \text{ onto } M \\ & \text{mapping} P^\delta_{\omega_2} \text{to} P_{\omega_2}, \underset{\sim}{f}^\delta \text{to} \underset{\sim}{f}, \\ & \underset{\sim}{F}^\delta \text{to } \underset{\sim}{F}, \underset{\sim}{C} \text{ to } \underset{\sim}{C}^\delta, p^\delta \text{ to } p, <^\delta \text{ to } <^*_\chi \restriction M\} \end{aligned}$$

is a stationary subset of ω_1.
(e) Choose $\langle \mathbf{B}_{\gamma(\delta)} : \delta \in S\rangle$ such that $\gamma(\delta) = \mathrm{otp}(N^\delta \cap \omega_2)$ and

$$\begin{aligned} \mathbf{B}_{\gamma(\delta)} & : (\omega^\omega)^{\gamma(\delta)} \times \mathcal{P}(\omega) \to \mathrm{Gen}^+(P^\delta_{\omega_2}) \\ & = \{G \subseteq P^\delta_{\omega_2} \cap N^\delta : G \text{ is } P^\delta_{\omega_2} - \text{generic over} N^\delta \text{and bounded}\} \end{aligned}$$

be as in Theorem 4.4 with $U^\delta = U(N^\delta, P^\delta_{\omega_2}, p^\delta, \bar{\beta}^\delta)$.

We do not require uniformity, $\langle \nu_\varepsilon, \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ is indeed $\langle \nu^\delta_\varepsilon, \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ since we have the dependence on the δ in the definition of $\mathbf{B}_{\gamma(\delta)}$. We assume that $N^\delta \cap \omega_1 = \delta$. Since this holds on a club set of $\delta \in \omega_1$, this is no restriction.

Now assume the $p \in G$ and $\underset{\sim}{F}$, $\underset{\sim}{f}$, $\underset{\sim}{C}$ are as in (d).

We define a function $\mathbf{B}'_{\delta,U_\delta}$ with domain $(\omega^\omega)^{\gamma(\delta)}$.

$$\mathbf{B}'_{\delta,U^\delta}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle) = \begin{cases} \underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta \restriction \delta)[\mathbf{B}_{\gamma(\delta)} & \\ \quad (\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta)], & \text{if the argument is good;} \\ \langle 0, 0, \dots, \rangle \in 2^\omega, & \text{otherwise.} \end{cases}$$

Here, we call $\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ a good argument if there is a play $\langle \nu_\varepsilon, \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ in the game $\eth_{(N^\delta, P^\delta, p^\delta)}$ from Theorem 4.4 in which the generic player plays ν_ε according his winning strategy and the antigeneric player plays η_ε according to the rules. Goodness is a Borel predicate because the ν_ε are irrelevant, just check whether the η_ε are large enough in the sense of Lemma 4.3 in the respective iteration step. So $\mathbf{B}'_{\delta,U^\delta}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ is a Borel function. Now we choose a "very good" argument $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ that player IN plays with his strategy in the game $\eth_{(\gamma(\delta), \mathbf{B}'_{\delta,U^\delta})}$ from Lemma 5.1 applied to $\mathbf{B}'_{\delta,U_\delta}$, answering to a good argument $\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ played by player OUT.

For verifying that the guessing function g that we are going to derive guesses right at stationarily many points in $S(p, \underset{\sim}{f}, \underset{\sim}{F})$, it suffices to note that by density arguments and by the closedness of the club set C that $q^\delta = \bigcup \mathbf{B}_{\gamma(\delta)}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle)$ forces that $\delta \in \underset{\sim}{C}^\delta$.

So we consider for every $\delta \in S$ a very good argument $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$. We assume that G is P_{ω_2}-generic over $\mathbf{V}$ and that $p \in G$. Then we also have by the rules of the game $\eth_{(N^\delta, P^\delta, p^\delta)}$ that

$$\mathbf{B}_{\gamma(\delta)}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta) \text{has an upper bound } q^\delta.$$

Lemma 5.1 gives an infinite set $C_{\mathbf{B}'_{\delta,U^\delta}}$, such that for $\delta \in S$, and we have

$$\mathbf{B}'_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle) \text{is almost constant on } C_{\mathbf{B}'_{\delta,U^\delta}}. \tag{5.1}$$

Note that $C_{\mathbf{B}'_{\delta,U^\delta}}$ does not depend on $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$. So (5.1) also holds for $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ that are the answers of player IN in the game from Lemma 5.1 to any good sequence $\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ given by the generic player that is so fast growing ν^δ_ε that $\mathbf{B}_{\gamma(\delta)}(\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta)$ computes a bounded generic filter over M like $c'' \mathbf{B}_{\gamma(\delta)}$ in Theorem 4.4. This is important, since the isomorphism h^δ does not preserve the knowledge (that is which branches are continued and what are the values of the promises in these continuations) about the trees' level δ for the Aronszajn trees involved in $P \cap M$.

We set

$$C_{\mathbf{B}'_{\delta,U^\delta}} =: g(\delta).$$

Since $\omega \subseteq M$ and $\omega \subseteq N^\delta$ we have that $h^\delta\left(C_{\mathbf{B}'_{\delta,U^\delta}}\right) = C_{\mathbf{B}'_{\delta,U^\delta}}$. We show that g is a diamond function.

Since P_{ω_2} is proper, $S(p, \underset{\sim}{f}, \underset{\sim}{F})$ is also stationary in $\mathbf{V}[G]$. Now we take a very good sequence $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ that is suitable so that $\mathbf{B}_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle)$ computes a bounded (M, P)-generic filter for M that witnesses that $\delta \in S$. So now we take the game $\eth_{(M,P,p)}$ for the choice of the $\langle \nu^\delta_\eta : \eta < \gamma_\delta\rangle$ and then again we take the winning strategy in the game $\eth_{(\gamma(\delta),\mathbf{B}'_{\delta,U_\delta})}$, which is unchanged by the collapse, for choosing $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma_\delta\rangle$. We take q to be a bound of $\mathbf{B}_{\gamma(\delta)}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta)$. Now we have that $q \geq p$ and

$$q \Vdash \text{“}\mathbf{B}_{\gamma(\delta)}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta) \text{ is } (M, P)\text{-generic and bounded by q”}.$$

Now for $\delta \in S(p, \underset{\sim}{f}, \underset{\sim}{F})$ we have by the isomorphism property of h^δ and by (5.1),

$$q \Vdash h^{\delta\prime\prime}\underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta \restriction \delta) = \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \ \wedge \ \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \in g(\delta) \ \wedge \ \delta \in \underset{\sim}{C}.$$

So we have that p forces that $\{\alpha \in S : F(f \restriction \delta)$is almost constant on$g(\delta)\}$ contains a stationary subset of $S(p, \underset{\sim}{f}, \underset{\sim}{F})$. Note that the stationary subset depends on F (and f of course), but the guessing function g does not. So actually we proved a diamond of the kind:

There is some $g: \omega_1 \to B$ such that for every Borel map $F: 2^{<\omega_1} \to A$ and for every $f: \omega_1 \to 2$ the set

$$\{\alpha \in \omega_1 : F(f \restriction \alpha) E g(\alpha)\}$$

is stationary. □

The forcing from Definition 3.6 could easily be mixed with proper $\aleph_2$-p.i.c. iterands, for example iterands with $|Q_\alpha| \leq \aleph_1$ (by [23, Lemma VIII 2.5] this is sufficient for the $\aleph_2$-p.i.c.) that add reals. Still we specialise all Aronszajn trees in such a mixed iteration. However, the parallel of our main technique for the weak diamonds does not work any more, since the completeness systems are no longer in the ground model. In future work with Shelah we plan to extend these techniques in order to interweave the addition of ω^ω-bounding reals into the iteration.

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