THE FILTER DICHOTOMY PRINCIPLE DOES NOT IMPLY THE SEMIFILTER TRICHOTOMY PRINCIPLE

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ABSTRACT. We answer Blass' question from 1989 of whether the inequality $\mathfrak{u} < \mathfrak{g}$ is strictly stronger than the filter dichotomy principle [6, page 36] affirmatively. We show that there is a forcing extension in which every non-meagre filter on ω is ultra by finite-to-one and the semifilter trichotomy does not hold. This trichotomy says: every semifilter is either meagre or comeagre or ultra by finite-to-one. The trichotomy is equivalent to the inequality $\mathfrak{u} < \mathfrak{g}$ by work of Blass and Laflamme. Combinatorics of block sequences is used to establish forcing notions that preserve suitable properties of block sequences.

1. INTRODUCTION

We separate two useful combinatorial principles: We show the filter dichotomy principle is strictly weaker than the semifilter trichotomy principle. Consequences of the latter and equivalent statements to the latter in the realm of measure, category, rarefication orders are investigated in [20, 18, 7]. Paul Larson proves in [21] a long-standing question about medial limits: The filter dichotomy implies that there are none. Our result on the combinatorical side thus separates two powerful principles in analysis.

We first recall the definitions: For $B \subseteq \omega$ and $f \colon \omega \to \omega$, we let $f''B = \{f(b) : b \in B\}$ and $f^{-1}{}''B = \{n : f(n) \in B\}$. By a *filter* we mean a proper filter on ω. We call a filter *non-principal* if it contains all cofinite sets. Let $\mathcal{F}$ be a non-principal filter on ω and let $f \colon \omega \to \omega$ be finite-to-one (that means that the preimage of each natural number is finite). Then also $f(\mathcal{F}) = \{X : f^{-1}{}''X \in \mathcal{F}\}$ is a non-principal filter. From now on we consider only non-principal filters. Two filters $\mathcal{F}$ and $\mathcal{G}$ are *nearly coherent*, if there is some finite-to-one $f \colon \omega \to \omega$ such that $f(\mathcal{F}) \cup f(\mathcal{G})$ generates a filter. The set of all infinite subsets of ω is denoted by $[\omega]^\omega$. A *semifilter* $\mathcal{S}$ is a subset of $[\omega]^\omega$ that contains ω as an element and that is closed under almost supersets, i.e., $(\forall X \in \mathcal{S})(\forall Y \in [\omega]^\omega)(X \setminus Y \text{ finite } \to Y \in \mathcal{S})$. In particular, $[\omega]^\omega$ is a semifilter.

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The *filter dichotomy principle*, abbreviated FD, says that for every filter there is a finite-to-one function f such that $f(\mathcal{F})$ is either the filter of cofinite sets (also called the *Fréchet filter*) or an ultrafilter. In the latter case we call $\mathcal{F}$ *ultra by finite-to-one* or *nearly ultra*. A semifilter $\mathcal{S}$ is called *meagre/comeagre* if the set of the characteristic functions of the members of $\mathcal{S}$ is a meagre/comeagre subset of the space 2^ω.

The *semifilter trichotomy principle*, abbreviated SFT, says that for every semifilter $\mathcal{S}$ either $\mathcal{S}$ is meagre or $f(\mathcal{S})$ is ultra or $f(\mathcal{S}) = [\omega]^\omega$ for some finite-to-one f. The latter is equivalent to $\mathcal{S}$ being comeagre, for an explicit proof see [21, Th. 4.1].

The semifilter trichotomy can also formulated in terms of two cardinal characteristics: Let $\mathcal{F}$ be a filter on ω. $\mathcal{B} \subseteq \mathcal{F}$ is a *base* for $\mathcal{F}$ if for every $X \in \mathcal{F}$ there is some $Y \in \mathcal{B}$ such that $Y \subseteq X$. The character of $\mathcal{F}$, $\chi(\mathcal{F})$, is the smallest cardinality of a base of $\mathcal{F}$. The cardinal $\mathfrak{u}$ is the smallest character of a non-principal ultrafilter. We denote by $\mathfrak{g}$ be the *groupwise density number*, that is the smallest number of groupwise dense sets whose intersection is empty. A set $\mathcal{G} \subseteq [\omega]^\omega$ is called *groupwise dense* if it is closed under almost subsets and for every strictly increasing function $f \colon \omega \to \omega$ there is an infinite A such that $\bigcup_{n \in A}[f(n), f(n+1)) \in \mathcal{G}$. Laflamme [20, Theorem 8] showed that $\mathfrak{u} < \mathfrak{g}$ implies SFT, and Blass [7] showed that SFT implies $\mathfrak{u} < \mathfrak{g}$. The purpose of this paper is to show the following:

Main Theorem. *“FD and the negation of SFT” is consistent relative to* ZFC.

A groupwise dense family that is closed under finite unions is called a groupwise dense ideal. The *groupwise density number for filters*, $\mathfrak{g}_f$, is the smallest number of groupwise dense ideals with empty intersection. From [10] and Blass [7], just read for groupwise dense ideals, it follows that $\mathfrak{u} < \mathfrak{g}_f$ is equivalent to FD. Moreover, FD implies $\mathfrak{b} = \mathfrak{u} < \mathfrak{g}_f = \mathfrak{d} = \mathfrak{c}$ [7]. Hence FD and and not SFT is equivalent to $\mathfrak{g} \le \mathfrak{u} < \mathfrak{g}_f$. Brendle [13] constructed a c.c.c. extension with $\kappa = \mathfrak{g} < \mathfrak{g}_f = \mathfrak{b} = \kappa^+$, and asked whether $\mathfrak{b} = \mathfrak{g} < \mathfrak{g}_f$ is consistent. By Shelah's $\mathfrak{g}_f \le \mathfrak{b}^+$ in ZFC [31], the only constellation for $\mathfrak{b} \le \mathfrak{g} < \mathfrak{g}_f$ is $\mathfrak{b} = \mathfrak{g} < \mathfrak{g}_f = \mathfrak{b}^+$. Since in any model of the dichotomy and $\mathfrak{u} = \mathfrak{g}$ we have the cardinal constellation $\mathfrak{b} = \mathfrak{u} = \mathfrak{g} < \mathfrak{g}_f = \mathfrak{c}$, the main theorem also answers a question by Brendle [13, Question 10] about separating $\mathfrak{g}$ and $\mathfrak{g}_f$ above $\mathfrak{b}$.

For $S, X \in [\omega]^\omega$ we say S splits X iff $X \cap S$ and $X \setminus S$ are both infinite. A set $SP \subseteq [\omega]^\omega$ is called *splitting* or a splitting family iff for every $X \in [\omega]^\omega$ there is some $S \in SP$ splitting X. The smallest cardinal of a splitting family is called the *splitting number* and denoted by $\mathfrak{s}$. Necessarily the splitting number $\mathfrak{s}$ must be bounded by $\mathfrak{u}$ for FD and $\mathfrak{u} \ge \mathfrak{g}$, because by [22, Cor. 4.4.], FD together with $\mathfrak{s} > \mathfrak{u}$ implies $\mathfrak{u} < \mathfrak{g}$.

The same argument shows:

Proposition 1.1. $\mathfrak{g}_f \le \mathfrak{s}$ *implies* $\mathfrak{g} = \mathfrak{g}_f$.

Proof. Assume that we have groupwise dense families $\mathcal{G}_\alpha$, $\alpha < \kappa$ for some $\kappa < \mathfrak{g}_f$. Then there is a diagonalisation D of the generated ideals, that is for every $\alpha < \kappa$ there are $n_\alpha \in \omega$ and $A_{\alpha,i} \in \mathcal{G}_\alpha$, $i \leq n_\alpha$, such that $D \subseteq A_{\alpha,0} \cup \dots \cup A_{\alpha,n_\alpha}$. Since $\kappa < \mathfrak{g}_f \leq \mathfrak{s}$, these $A_{\alpha,i} \cap D$, $\alpha < \kappa$, $i < n_\alpha$, are not a splitting family in $[D]^\omega$. Hence there is some infinite $D' \subseteq D$ and there are i_α, $\alpha < \kappa$, such that $(\forall \alpha < \kappa)(D' \subseteq A_{\alpha,i_\alpha})$. So $D' \in \bigcap_{\alpha<\kappa} \mathcal{G}_\alpha$ and $\mathfrak{g} > \kappa$. □

A third principle is strictly weaker than FD: The principle of near coherence of filters, short NCF, says that for any two filters (recall: they contain the Fréchet filter) are nearly coherent. The filter dichotomy implies NCF by [10]; the converse does not hold by [24].

Often in set theory of the reals, combinatorial work in the Baire space and its subsets isolates indicators to non-implication, and work in the set of the hereditarily countable sets allows to construct a proper forcing just in the $\aleph_1$-$\aleph_2$-scenario that proves the non-implication. Also this paper is of this kind. Only a few mathematical reasons for smallness and cardinal spread one are known, among them the Raisonnier filter [28], which implies NCF $\to$ $\mathrm{add}(\mathcal{N}) = \aleph_1$ and the base matrix tree [2], from which Shelah derived the already mentioned bound $\mathfrak{g}_f \leq \mathfrak{b}^+$ [31].

The paper is organised as follows: In Section 2 we explain Matet forcing with centred systems. In Section 3 we recall Matet forcing with stable ordered-union ultrafilters and Eisworth's work. In Section 4 we define block-swallowing and prove preservation theorems for it. In Section 5 we define iterated forcing orders and a name for a semifilter that serves as a counterexample to the trichotomy. The final section contains some related results on cardinal characteristics.

Undefined notation on cardinal characteristics can be found in [4, 9]. Undefined notation about forcing can be found in [19, 30]. In the forcing, we follow the Israeli style that the stronger condition is the *larger* one. A good background in proper forcing is assumed.

2. A variant of Matet forcing

We define a variant of Matet forcing. For this purpose, we first introduce some notation about block sequences. Our nomenclature follows Blass [5] and Eisworth [15].

We let $\mathbb{F}$ be the collection of all finite non-empty subsets of ω. For $a, b \in \mathbb{F}$ we write $a < b$ if $(\forall n \in a)(\forall m \in b)(n < m)$. A filter over $\mathbb{F}$ is a subset of $\mathscr{P}(\mathbb{F})$ that is closed under intersections and supersets. A sequence $\bar{a} = \langle a_n : n \in \omega \rangle$ of members of $\mathbb{F}$ is called unmeshed if for all n, $a_n < a_{n+1}$. An infinite unmeshed sequence $\bar{a}$ is called a block sequence, and the a_n are called blocks of $\bar{a}$. The set $(\mathbb{F})^\omega$ denotes the collection of block sequences. If X is a subset of $\mathbb{F}$, we write $\mathrm{FU}(X)$ for the set of all finite unions of members of X. We write $\mathrm{FU}(\bar{a})$ instead of $\mathrm{FU}(\{a_n : n \in \omega\})$.

We write $\mathrm{set}(\bar{b}) = \bigcup\{b_n : n \in \omega\}$. We say $E \subseteq (\mathbb{F})^\omega$ generates $\mathcal{C}$ iff $\mathcal{C} = \{\bar{a} : (\exists \bar{b} \in E)(\bar{b} \sqsubseteq^* \bar{a})\}$.

Definition 2.1. Given $\bar{a}$ and $\bar{b}$ in $(\mathbb{F})^\omega$, we say that $\bar{b}$ *is a condensation of* $\bar{a}$ and we write $\bar{b} \sqsubseteq \bar{a}$ if $\{b_n : n \in \omega\} \subseteq \mathrm{FU}(\bar{a})$. We say $\bar{b}$ is *stronger than* $\bar{a}$ and we write $\bar{b} \sqsubseteq^* \bar{a}$ iff there is an n such that $\langle b_t : t \geq n\rangle$ is a condensation of $\bar{a}$.

Definition 2.2. A set $\mathcal{C} \subseteq (\mathbb{F})^\omega$ is called *centred*, if for any finite $C \subseteq \mathcal{C}$ there is $\bar{a} \in \mathcal{C}$ that is stronger than any $\bar{c} \in C$.

Definition 2.3. In the *Matet forcing*, $\mathbb{M}$, the conditions are pairs $(a, \bar{c})$ such that $a \in \mathbb{F}$ and $\bar{c} \in (\mathbb{F})^\omega$ and $a < c_0$. The forcing order is $(b, \bar{d}) \geq (a, \bar{c})$ (recall the stronger condition is the larger one) iff $a \subseteq b$ and $b \setminus a$ is a union of finitely many of the c_n and $\bar{d}$ is a condensation of $\bar{c}$.

Definition 2.4. Given a centred system $\mathcal{C} \subseteq (\mathbb{F})^\omega$, the notion of forcing $\mathbb{M}(\mathcal{C})$ consists of all pairs $(s, \bar{a}')$, such that $s \in \mathbb{F}$ and there is $\bar{a} \in \mathcal{C}$ such that $\bar{a}'$ is an end-segment of $\bar{a}$, i.e., $\bar{a}' = \langle a_n : n \geq k\rangle$. The forcing order is the same as in the Matet forcing.

A poset $\mathbb{P}$ is called centred if for any finite $F \subseteq \mathbb{P}$ there is q stronger than any of the $p \in F$. $\mathbb{P}$ is σ-centred if it is the union of countably many centred sub-posets. The forcing orders $\mathbb{M}(\mathcal{C})$ are σ-centred, and hence proper.

Here is more notation for handling block sequences.

Definition 2.5.

(1) The set of finite-to-finite relations is

$$\mathcal{R}^* = \{R \subseteq \omega \times \omega :$$
$$(\forall m)(\text{there are finitely many and at least one } n)(mRn) \wedge$$
$$(\forall n)(\text{there are finitely many and at least one } m)(mRn)\}.$$

We let the letter R range over elements of $\mathcal{R}^*$.

(2) For $a \subseteq \omega$, $R \in \mathcal{R}^*$ we let $R(a) = \{n : mRn, m \in a\}$.

(3) For $\bar{c} = \langle c_n : n \in \omega\rangle \in (\mathbb{F})^\omega$, $R \in \mathcal{R}^*$ we let $R(\bar{c}) = \langle R(c_n) : n \in \omega\rangle$. This can be not unmeshed or even be not pairwise disjoint, but it does not matter. When we use it, we will first look whether it is a block sequence.

The purpose of $R \in \mathcal{R}^*$ is to increase infinite sets in a gentle manner, as with finite-to-one functions: If $f''x \subseteq f''y$, then $x \subseteq Ry$ for $R = \{(m, n) : f(n) = f(m)\}$. Another use is: For a finite-to-one f, $f(\mathcal{F}) = \{X : f^{-1}{}''X \in \mathcal{F}\} = \{X : R(X) \in \mathcal{F}\}$, where xRy iff $f(y) = x$. Since f is a finite-to-one function, we have $R \in \mathcal{R}$.

Remark 2.6. There are many $R \in \mathcal{R}^*$: For any two sequences $\bar{c}$, $\bar{d}$ in $(\mathbb{F})^\omega$ there is $R_{\bar{c},\bar{d}}$ such that $R_{\bar{c},\bar{d}}(\bar{c}) = \bar{d}$, e.g., $R_{\bar{c},\bar{d}} = \bigcup\{c_n \times d_n : n \in \omega\}$.

3. Preserving a P-point from the ground model

In this section we specialise the partial orders $\mathbb{M}(\mathcal{C})$ further. The results we collect in this section are Hindman's and Eisworth's (see [15]).

An ultrafilter $\mathcal{U}$ is called a *P-point* if for every for every sequence A_n, $n \in \omega$, of elements of $\mathcal{U}$, there is some $A \in \mathcal{U}$ such that for all n, $A \subseteq^* A_n$; such an A is called a *pseudo-intersection* of the A_n. Let $\mathbb{P}$ be a notion of forcing. We say that $\mathbb{P}$ preserves an ultrafilter $\mathcal{U}$ if $\Vdash_{\mathbb{P}}$ "$(\forall X \in [\omega]^\omega)(\exists Y \in \mathcal{U})(Y \subseteq X \vee Y \subseteq \omega \setminus X))$" and in the contrary case we say "$\mathbb{P}$ destroys $\mathcal{U}$". If $\mathbb{P}$ is proper and preserves $\mathcal{U}$ and $\mathcal{U}$ is a P-point, then $\mathcal{U}$ stays a P-point [11, Lemma 3.2].

Definition 3.1. A non-principal filter $\mathcal{F}$ over $\mathbb{F}$ is said to be an *ordered-union* filter if it has a basis of sets of the form $\mathrm{FU}(\bar{d})$ for $\bar{d} \in (\mathbb{F})^\omega$. Let μ be an uncountable cardinal. An ordered-union filter is said to be *$< \mu$-stable* if, whenever it contains $\mathrm{FU}(\bar{d}_\alpha)$ for $\bar{d}_\alpha \in (\mathbb{F})^\omega$, $\alpha < \kappa$, for some $\kappa < \mu$, then it also contains some $\mathrm{FU}(\bar{e})$ for some $\bar{e}$ that is almost a condensation of $\bar{d}_\alpha$ for $\alpha < \kappa$. For "$< \omega_1$-stable" we say "stable". Stable ordered-union ultrafilters are also called Milliken–Taylor ultrafilters.

Ordered-union ultrafilters need not exist, as their existence implies the existence of Q-points [5] and there are models without Q-points [27]. With the help of Hindman's theorem one shows that under $\mathsf{MA}(\sigma\text{-centred})$ stable (even $< 2^\omega$-stable) ordered-union ultrafilters exist [5]. We recall Hindman's theorem:

Theorem 3.2. *(Hindman,* [17, Corollary 3.3]*) If the set $\mathbb{F}$ is partitioned into finitely many pieces then there is a set $\bar{d} \in (\mathbb{F})^\omega$ such that $\mathrm{FU}(\bar{d})$ is included in one piece.*

The theorem also holds if instead of $\mathbb{F}$ we partition only $\mathrm{FU}(\bar{c})$ for some $\bar{c} \in (\mathbb{F})^\omega$, the homogeneous sequence $\bar{d}$ given by the theorem is then a condensation of $\bar{c}$.

Corollary 3.3. *Under CH for every $\bar{a} \in (\mathbb{F})^\omega$ there is a stable ordered-union ultrafilter $\mathcal{U}$ such that $\mathrm{FU}(\bar{a}) \in \mathcal{U}$.*

In order to state a preservation property of $\mathbb{M}(\mathcal{U})$, we need the following definition.

Definition 3.4. Let $\mathcal{U}$ be a filter over $\mathbb{F}$. The *core of $\mathcal{U}$* is the filter $\Phi(\mathcal{U})$ such that

$$X \in \Phi(\mathcal{U}) \text{ iff } (\exists\, \mathrm{FU}(\bar{c}) \in \mathcal{U})(\bigcup_{n \in \omega} c_n \subseteq X).$$

If $\mathcal{U}$ is ultra over $\mathbb{F}$, then $\Phi(\mathcal{U})$ does not have a pseudointersection (see [15, Prop. 2.3]) and also all finite-to-one images of $\Phi(\mathcal{U})$ do not (same proof). So $\Phi(\mathcal{U})$ is not meagre.

Blass proved that the filter $\Phi(\mathcal{U})$, though, is not ultra by finite-to-one [8]. The reason is: There are two ultrafilters $\min(\mathcal{U}) = \{\{\min(d) : d \in D\} : D \in \mathcal{U}\}$, $\max(\mathcal{U}) \supset \Phi(\mathcal{U})$ that are not nearly coherent.

The *Rudin-Blass ordering* on semifilters is defined as follows: Let $\mathcal{F} \leq_{RB} \mathcal{G}$ iff there is a finite-to-one f such that $f(\mathcal{F}) \subseteq f(\mathcal{G})$. Usually only filters are considered. We use this order also for semifilters.

The following property of stable ordered-union ultrafilters $\mathcal{U}$ will be important for our proof:

Theorem 3.5. *(Eisworth* [15, "$\leftarrow$" Theorem 4, "$\rightarrow$" Cor. 2.5, this direction works also with non-P ultrafilters]*) Let $\mathcal{U}$ be a stable ordered-union ultrafilter on $\mathbb{F}$ and let $\mathcal{V}$ be a P-point. The forcing $\mathbb{M}(\mathcal{U})$ preserves $\mathcal{V}$ iff $\mathcal{V} \not\geq_{RB} \Phi(\mathcal{U})$.*

Let ${}^\omega\omega$ denote the set of functions from ω to ω. For $f, g \in {}^\omega\omega$ we say g eventually dominates f and write $f \leq^* g$ iff $(\exists n_0 \in \omega)(\forall n \geq n_0)(f(n) \leq g(n))$.

The forcing $\mathbb{M}(\mathcal{U})$ adds an unbounded real, i.e., a real such that is not eventually dominated by any real of the ground model. Indeed, any pseudointersection of $\Phi(\mathcal{U})$ is unbounded over the ground model. Since $\Phi(\mathcal{U})$ is not meagre, this claim follows from the following result due to Talagrand [32]:

Lemma 3.6. *For every semifilter $\mathcal{S}$ the following are equivalent*

(1) There is a finite-to-one function such that $\{X : (\exists S \in \mathcal{S})(f''S \subseteq X)\}$ is the Fréchet filter.

(2) $\mathcal{S}$ is meagre.

(3) The set of enumerating functions of members of $\mathcal{S}$ is $\leq^*$-bounded.

4. Preserving block-swallowing families

From now on we consider only the simple case that $\mathcal{C}$ is generated by the range of a $\sqsubseteq^*$-descending sequence $\langle \bar{c}_\varepsilon : \varepsilon < \delta \rangle$.

Definition 4.1. Suppose that G is an $\mathbb{M}(\mathcal{C})$-generic filter over $\mathbf{V}$. The generic real is $s(\mathcal{C}) = \bigcup\{s : \exists \bar{c} \in \mathcal{C}(s, \bar{c}) \in G\}$.

We intend to arrange that later in an iteration no forcing $\mathbb{M}(\mathcal{C}')$ appears that adds a generic real that is a subset of $s(\mathcal{C})$. For this we introduce block-swallowing families. The quantifier $\exists^\infty n$ says that there are infinitely many n.

Definition 4.2. (1) We say $\bar{x}$ *swallows* $\bar{a}$ iff

$$(\exists^\infty n)(\exists k)(x_k \supseteq a_n).$$

(2) Let $\mathcal{X}$ be a set of block sequences. We say $\mathcal{X}$ *is block-swallowing* iff for any block sequence $\bar{a}$ there is $\bar{x} \in \mathcal{X}$ that swallows $\bar{a}$.

Remark 4.3. Let $\mathcal{X}$ be a block-swallowing family. Then, given any countable sequence $(\bar{a}_j)_j$ of block sequences there is $\bar{x} \in \mathcal{X}$ such that

$$(\exists^\infty n)(\exists k)(\exists i_0 \dots i_n)(x_k \supseteq \bigcup\{a_{j,i_j} : j \leq n\}),$$

i.e., for any n, $\bar{x}$ swallows $\bar{a}_n$.

In order to see this, diagonally interweave the sequences $(\bar{a}_j)_j$: As a first block we take any block of $\bar{a}_0$, then we take a block of $\bar{a}_0$ and one of $\bar{a}_1$ such that both of them lie after the first block and merge them and declare the outcome as the second block, an so on. This interweaving procedure can in particular be applied to all block-seqeunces in a countable elementary submodel M. Then we get one block sequence that swallows all of them. (End of remark)

Now we want to preserve block-swallowing families in later iteration steps and create new block-swallowing families.

We denote by $\mathsf{MA}_{<\kappa}(\sigma\text{-centred})$ Martin's axiom for σ-centred posets and $< \kappa$ dense sets. Let Γ be a class of forcings. $\mathsf{MA}_{<\kappa}(\Gamma)$ says: For any $\mathbb{P} \in \Gamma$ for any collection $\mathcal{D}$ of size strictly less than κ of dense sets there is a filter $G \subseteq \mathbb{P}$ such that $(\forall D \in \mathcal{D})(D \cap G \neq \emptyset)$.

The following is a cornerstone:

Lemma 4.4. *Let $\kappa = 2^\omega$. We assume CH or $\mathsf{MA}_{<\kappa}(\sigma\text{-centred})$. Let $\mathcal{X}_\zeta$, $\zeta < \kappa$, be block-swallowing families. There are a sequence $\mathcal{C} = \langle \bar{c}_\varepsilon : \varepsilon < \kappa \rangle$ and a sequence $\langle \bar{d}_{\varepsilon+1} : \varepsilon < \kappa \rangle$ such that the following hold:*

(A) After forcing with $\mathbb{M}(\mathcal{C})$

(1) for each $\zeta < \kappa$, $\mathcal{X}_\zeta$ is still block-swallowing, and

(2) $\mathrm{out}(\mathcal{C}) := \{\bar{d}_{\varepsilon+1} : \varepsilon < \kappa\}$ is block-swallowing.

(B) For each $\varepsilon < \kappa$, $\mathrm{set}(\bar{d}_{\varepsilon+1}) \cap \mathrm{set}(\bar{c}_{\varepsilon+1})$ is finite.

The proof of the lemma we use a technique called "sealing antichains" or "processing names". This method has been used in the set theory of the reals [1, 12, 14, 16, 23] and possibly elsewhere and also in constructing forcings under the assumption of large cardinals.

Let $f : \omega \to H(\omega)$. As usual, $H(\omega)$ denotes the set of hereditarily finite sets. Suppose that $\mathbb{P}$ is a c.c.c forcing order. A standardised name for f is

$$f = \{\langle (n, k_{n,m}), p_{n,m} \rangle : n, m \in \omega\},$$

such that $\{p_{n,m} : m \in \omega\}$ is predense in $\mathbb{P}$ and $p_{n,m} \Vdash_{\mathbb{P}} f \restriction n = k_{n,m}$, $k_{n,m} \in H(\omega)$, and such that $k_{n',m'} \restriction n = k_{n,m}$ if $p_{n',m'}$ and $p_{n,m}$ are compatible and $n' \geq n$.

We write $\mathbb{P} \subseteq_{ic} \mathbb{P}'$ iff $\mathbb{P} \subseteq \mathbb{P}'$ and for any $p, q \in \mathbb{P}$, if p and q are incompatible in $\mathbb{P}$ then they are also incompatible in $\mathbb{P}'$. If $\mathbb{P} \subseteq_{ic} \mathbb{P}'$ then not every standardised $\mathbb{P}$-name for a real is also $\mathbb{P}'$-name for a real. This happens, however, if any maximal antichain $\{p_{n,m} : m \in \omega\}$ in $\mathbb{P}$ stays maximal in $\mathbb{P}'$. In the end we evaluate only names in the final order, and each name that made it to the final stage appears at some stage of countable cofinality (see Lemma 4.5) and then can be construed also as a name of a forcing order of any later stage.

If $\mathcal{C} \subseteq \mathcal{C}'$ are centred systems, then $\mathbb{M}(\mathcal{C}) \subseteq_{ic} \mathbb{M}(\mathcal{C}')$. If f is an $\mathbb{M}(\mathcal{C})$-name and an $\mathbb{M}(\mathcal{C}')$-name for a function from ω to $H(\omega)$, $\mathcal{C} \subseteq \mathcal{C}'$, $p \in \mathbb{M}(\mathcal{C})$, and $k \in H(\omega)$, then $p \Vdash_{\mathbb{M}(\mathcal{C})} f(n) = k$ is equivalent to $p \Vdash_{\mathbb{M}(\mathcal{C}')} f(n) = k$.

We write $\mathbb{M}(\bar{c})$ for $\mathbb{M}(\{\bar{c}\})$. In general $\mathbb{M}(\mathcal{C})$ is not a complete suborder of $\mathbb{M}(\mathcal{C}')$. For example, there are $\mathbb{M}(\mathcal{C})$ that preserve an ultrafilter from the ground model, as in Eisworth's theorem, and on the other hand, $\mathbb{M}(\bar{c})$ is Cohen forcing.

Lemma 4.5. *Let $\langle \bar{c}_\varepsilon : \varepsilon < \delta\rangle$, be a decreasing sequence that generates $\mathcal{C}$. Assume $\mathrm{cf}(\delta) > \omega_0$ and f is a $\mathbb{M}(\mathcal{C})$-name for a function from ω to $H(\omega)$. Then we can find an $\varepsilon_0 < \delta$ such that for every $\varepsilon \in [\varepsilon_0, \delta)$ there are $p_{n,m} \in \mathbb{M}(\bar{c}_\varepsilon)$ and $k_{n,m} \in H(\omega)$ such that $\{p_{n,m} : m < \omega\}$ is predense in $\mathbb{M}(\mathcal{C})$ and $p_{n,m} \Vdash f(n) = k_{n,m}$.*

Proof. We assume that $f = \{\langle (n, h_{n,m}), q_{n,m}\rangle : m, n < \omega\}$. Since $\mathrm{cf}(\delta) > \omega$, there is some $\varepsilon_0 < \delta$ such that all $q_{n,m}$ are in $\mathbb{M}(\bar{c}_\beta : \beta \le \varepsilon_0)$. Now, given $\varepsilon \in [\varepsilon_0, \delta)$, we take

$$I_n = \{q \in \mathbb{M}(\bar{c}_\varepsilon) : (\exists m)(q \ge q_{n,m})\}.$$

Then I_n is predense in $\mathbb{M}(\mathcal{C})$. Now let $p_{n,m}$, $m < \omega$, list I_n and choose $k_{n,m}$ such that $p_{n,m} \Vdash f(n) = k_{n,m}$. Then $k_{n,m}$, $p_{n,m}$, $n, m \in \omega$, describe f as desired. □

The purpose of the lemma is to allow to work just in $\mathbb{M}(\bar{c}_\varepsilon)$ for a single $\bar{c}_\varepsilon$.

Some notation:

(1) If $\bar{c}_\varepsilon$ is a block sequence, then we let $\bar{c}_\varepsilon = \langle c_{\varepsilon,n} : n \in \omega\rangle$.

(2) We let $\bar{a}$; past $n = \langle a_m : m \in [k, \omega)\rangle$ with the minimal k such that $n < \min(a_k)$.

(3) We let $\bar{a}$; before $n = \langle a_m : m \in [0, k]\rangle$ with the maximal k such that $\max(a_k) < n$.

Proof of Lemma 4.4:
Let $\langle \bar{b}_\varepsilon, \zeta_\varepsilon, f_\varepsilon, : \varepsilon < \kappa\rangle$ list the tuples $(\bar{b}, \zeta, f)$ such that $\bar{b} \in (\mathbb{F})^\omega$, $\zeta \in \kappa$, and $f = \{\langle (n, k_{n,m}), p_{n,m}\rangle : m, n \in \omega\}$ is a standardised $\mathbb{M}(\bar{b})$-name for a block-sequence. We assume that each triple $(\bar{b}, \zeta, f)$ appears in the list κ many times.

We choose by induction on $\varepsilon < \kappa$ a $\sqsubseteq^*$-descending sequence $\bar{c}_\varepsilon \in (\mathbb{F})^\omega$.

For $\varepsilon = 0$ we let $\bar{c}_0 = \langle \{n\} : n < \omega\rangle$.

Limit step:
Let $\varepsilon < \kappa$ be a limit ordinal. We apply $\mathsf{MA}_{<\kappa}(\sigma\text{-centred})$ to the σ-centred forcing notion $\{(\bar{c}, n, F) : \bar{c}$ is a finite block sequence of subsets of n and F is a finite subset of $\varepsilon\}$, ordered by $(\bar{c}', n', F') \ge (\bar{c}, n, F)$ iff $n' \ge n$, $F' \supseteq F$, $\bar{c}' \sqsubseteq \bar{c}$, $c'_i = c_i$ for $i < n$ and $(\forall \gamma \in F)(\forall k)(c'_k \subseteq [n, n') \rightarrow c'_k \in \mathrm{FU}(\bar{c}_\gamma))$, and the dense sets $\mathscr{I}_{\delta,n} = \{(\bar{c}, m, F) : \mathrm{set}(\bar{c}) \setminus n \ne \emptyset \wedge \delta \in F \wedge m > n\}$, $\delta < \varepsilon$, $n < \omega$, and thus we get a filter G intersecting all the $\mathscr{I}_{\delta,n}$. The set $\bar{c}_\varepsilon = \bigcup\{\bar{c} : (\exists n, F)((\bar{c}, n, F) \in G)\}$ is as desired. If $\mathrm{cf}(\varepsilon) = \omega$ then we simply take a $\sqsubseteq^*$-lower bound in ZFC.

Step $\varepsilon = \delta + 1$, and $\bar{c}_\delta$ is chosen. We assume that for some $\gamma \leq \delta$, $\bar{b}_\delta = \bar{c}_\gamma$ and f_δ is a $\mathbb{M}(\bar{b}_\delta)$-name of a block sequence that has an equivalent $\mathbb{M}(\bar{c}_\delta)$-name f. Otherwise we can take $\bar{c}_{\delta+1} = \bar{c}_\delta$.

By our coding, $f = \{\langle\langle n, k_{n,m}\rangle, p_{n,m}\rangle \,:\, n, m \in \omega\}$, $k_{n,m}$ is an unmeshed sequence of n blocks.

By induction on $r \in \omega$ we first choose $c^+_{\delta,r} \in \mathbb{F}$, $b(r) \in \omega$, and $u_r \in \mathbb{F}$ such that $c^+_{\delta,r} = \bigcup\{c_{\delta,n} \,:\, n \in u_r\}$. We let $c^+_{\delta,0} = c_{\delta,0}$, $b(0) = \max(c^+_{\delta,0}) + 1$.

Suppose that $c^+_{\delta,r-1}$ and $b(r-1)$ are chosen. Let $\{w_{r,i} \,:\, i < 2^{b(r-1)}\}$ enumerate all subsets of $b(r-1)$. We write $p = (w \cup c(p), \bar{c}(p))$ to indicate components. Since we work in $\mathbb{M}(\bar{c}_\delta)$, $\bar{c}(p)$ is an end segment of $\bar{c}_\delta$.

Now by subinduction on $i < 2^{b(r-1)}$ we choose $n(r,i) = n(i)$, $m(r,i) = n(i)$ such that

(1) $n(-1) \geq r$ and $p_{n(0),m(0)} \geq (w_{r,0}, \bar{c}_\delta \restriction \text{past } b(r-1))$.

(2) $p_{n(i),m(i)} = (w_{r,i} \cup c(p_{n(i),m(i)}), \bar{c}(p_{n(i),m(i)}))$.

(3) $(c(p_{n(i),m(i)}), \bar{c}(p_{n(i),m(i)})) \leq (c(p_{n(i+1),m(i+1)}), \bar{c}(p_{n(i+1),m(i+1)}))$.

(4) For $0 \leq i \leq 2^{b(r-1)} - 1$, $p_{n(i),m(i)}$ determines $f \restriction \text{before}\, n(i)$ and forces that there is a full f-block that is a subset of $[n(i-1), n(i))$. We call this selected block $f(p_{n(i),m(i)})\,\text{in}[n(i-1), n(i))$.

Once the subinduction is performed, we do not drop the counter r anymore. We take the union

$$\bigcup\{f(p_{n(r,i),m(r,i)})\,\text{in}[n(r,i-1), n(r,i)) \,:\, i < 2^{b(r)}\}$$

and call this $\mathbf{f}_r$. The parts of $\mathbf{f}_r$ come from possibly incompatible conditions, because of the different $w_{r,i}$. We let $c^+_{\delta,r} = \bigcup\{c(p_{n(r,i),m(r,i)}) \,:\, i < 2^{b(r-1)}\}$. This also determines u_r. Thus the step r is finished. We start the step from r to $r+1$ with $b(r) = \max(c^+_{\delta,r}) + 1$ and $\bar{c}_\delta \restriction \text{past } b(r)$.

Since $\mathcal{X}_{\zeta_\delta}$ is a block-swallowing family there is $\bar{x} \in \mathcal{X}_{\zeta_\delta}$ and there is an infinite set $\{r_k \,:\, k \in \omega\}$ such that the $\mathbf{f}_{r_k}$, $k \in \omega$, are pairwise disjoint and

$$(\oplus_1) \qquad (\forall k \in \omega)(\bar{x} \restriction \text{past } r_k \text{ has a block that is a superset of } \mathbf{f}_{r_k})$$

In addition, after possibly thinning out the $(r_k)_k$ further, we assume that

$$(\oplus_2) \qquad [\max(c^+_{\delta,r_k}), \min(c^+_{\delta,r_{k+1}})) \supseteq \mathbf{f}_r \text{ for some } r_k \leq r < r_{k+1}$$

We let for $k \in \omega$,

$$(\oplus_3) \qquad c_{\varepsilon,k} = c^+_{\delta,r_{2k}} \cup c^+_{\delta,r_{2k+1}},$$

and

$$(\oplus_4) \qquad d_{\varepsilon,k} = [\max(c^+_{\delta,r_{2k}}), \min(c^+_{\delta,r_{2k+1}})).$$

We say $\bar{c}_\varepsilon$ *seals* f_δ *at* $\bar{c}_\delta$ *for the block-swallowing family* $\mathcal{X}_{\zeta_\delta}$ and $\bar{d}_\varepsilon$ *seals block-swallowing* f_δ *at* $\bar{c}_\delta$.

Of course, any sequence stronger than $\bar{c}_\varepsilon$ would seal f_δ as well. The fact that sealed sequences cannot be broken up into sub-blocks is a technical core of the difference between $\mathfrak{g}_f$ and $\mathfrak{g}$.

We show that in the generic extension by $\mathbb{M}(\mathcal{C})$, $\mathcal{X}_\zeta$ is block-swallowing.

Assume towards a contradiction that there is a $\mathbb{M}(\mathcal{C})$-name f for a countable sequence of block sequences and there is $p \in \mathbb{M}(\mathcal{C})$ such that

$$p \Vdash (\forall \bar{x} \in \mathcal{X}_\zeta)(\bar{x} \text{ does not swallow } f).$$

Since $\mathrm{cf}(\kappa) > \omega$, by Lemma 4.5 there is some $\gamma < \kappa$ such that f is an $\mathbb{M}(\bar{c}_\gamma)$-name. Since in the enumeration every name appears cofinally often, for some $\delta \geq \gamma$ we have $(\bar{b}_\delta, \zeta_\delta, f_\delta) = (\bar{c}_\gamma, \zeta, f)$. So at stage $\varepsilon = \delta + 1$ in our construction we took care of f's equivalent $\mathbb{M}(\bar{c}_\delta)$-name, call it f as well. Let $\bar{x}$, $\{r_k : k \in \omega\}$ and $\bar{c}_\varepsilon$ be as in $\oplus_1$ of this step.

By our assumption there are $q \geq p$ and some $n(*)$ such that

$$q \Vdash \text{no block of } \bar{x} \text{ ; past } n(*) \text{ contains a full block of } f. \tag{4.1}$$

By the form of $\mathbb{M}(\mathcal{C})$, $q = (s, \bar{c}_{\varepsilon(1)})$ for some $\varepsilon(1) \geq \varepsilon$ and some s, such that $\bar{c}_{\varepsilon(1)}$ is a condensation of $\bar{c}_\varepsilon$. So there are k, j, ℓ_j, ℓ_{j+1} and $r_{2k} \geq n(*)$ (from the step δ of the construction) such that that $c_{\varepsilon(1),j} \subseteq \ell_{j+1}$ and $c_{\varepsilon(1),j} \cap [\ell_j, \ell_{j+1}) = c_{\varepsilon,k}$ and $\varepsilon = \delta + 1$. However, $c_{\varepsilon,k} = c^+_{\delta,r_{2k}} \cup c^+_{\delta,r_{2k+1}}$. We let $s' = s \cup (\bigcup \bar{c}_{\varepsilon(1)} \cap [0, \ell_j))$, and we let $q' = (s' \cup c_{\varepsilon,k}, c_{\varepsilon(1),j+1}, \dots)$.

There is $i < 2^{b(r_{2k}-1)}$ such that $s' = w_{r_{2k},i}$. Then we have $q \leq q'$ and $p_{n(r_{2k},i),m(r_{2k},i)} \leq q'$. Property $(\oplus_1)$ in the choice of $\bar{x}$ together with the definitions of $\bar{c}^+_\delta$ and $\bar{c}_\varepsilon$ yield

$$q' \Vdash \text{“}\bar{x} \text{ ; past } n(r_{2k}, i-1) \text{ has a block that is a superset of } \mathbf{f}_{r_{2k}}\text{”}.$$

Since by definition of $\mathbf{f}_{r_{2k}}$ clause (4), the sequence $\mathbf{f}_{r_{2k}}$ contains at least one f-block past $n(i-1)$ according to the opinion of $p_{n(r_{2k},i),m(r_{2k},i)}$ and since $n(r_{2k}, i-1) \geq r_{2k} \geq n(*)$, this contradicts Equation (4.1).

That $\mathrm{out}(\mathcal{C}) = \langle \bar{d}_{\varepsilon+1} : \varepsilon < \omega_1 \rangle$ is block-swallowing in the extension by $\mathbb{M}(\mathcal{C})$ is proved similarly using Equations $\oplus_2$, $\oplus_3$, and $\oplus_4$.

Conclusion (B) holds by our arrangement in Equation $\oplus_4$. $\Box_{4.4}$

Definition 4.6. Let $\bar{x}$, $\bar{a}$ be a block sequences and $n \in \omega$. We say

$\bar{a} R_n \bar{x}$ iff

there is $\ell \geq n$ such that x_ℓ is a superset of a full block of $\bar{a}$

A block sequence $\bar{x}$ swallows $\bar{a}$ iff $(\forall n)(\bar{a} R_n \bar{x})$. We let ${}^\omega\omega$ also denote the Baire space. This is the set of functions from ω to ω, endowed with the topology given by the open sets $N_s = \{f : s \subseteq f\}$ for $s \colon n \to \omega$. We fix a bijection $b \colon \mathbb{F} \to \omega$. We define $e \colon (\mathbb{F})^\omega \to {}^\omega\omega$ by letting $e(\bar{a})(n) = b(a_n)$. Thus e maps $(\mathbb{F})^\omega$ onto the G_δ-set $B' = \{f \in {}^\omega\omega : (\forall n)(b^{-1}(f(n)) < b^{-1}(f(n+1)))\}$. We equip B' with the topology it inherits from the Baire space; and we let $(\mathbb{F})^\omega$ carry the topology such that e is a homeomorphism from $(\mathbb{F})^\omega$ onto B'.

Lemma 4.7. *For any $\bar{x}$, the set $\{\bar{a} : \bar{a} R_n \bar{x}\}$ is open in B'.*

Remark 4.8. Suppose, that $\mathcal{X}$ is block-swallowing and for $\beta < \alpha$,

$$\mathbb{P}_\beta \Vdash \text{“}\mathbb{M}(\mathcal{C}_\beta) \text{ preserves that } \mathcal{X} \text{ is block-swallowing”},$$

and $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\mathcal{C}_\gamma) : \beta \leq \alpha, \gamma < \alpha\rangle$ is a countable support iteration. By citation, we show that also $\mathbb{P}_\alpha$ forces that $\mathcal{X}$ is block-swallowing.

Let M be a countable elementary submodel of some $H(\theta)$, $\theta \geq (2^{\omega_2})^+$ a regular cardinal. We work with $\bar{R} = \langle R_n : n \in \omega\rangle$ from Def. 4.6. We let $\mathbf{g}_M$ be a block sequence that block-swallows any block sequence in M. In other words, $(\forall \bar{a} \in M)(\forall n)(\bar{a} R_n \mathbf{g}_M)$. In Shelah's notation $\mathbf{g}_M$ is $(M, \bar{R})$-covering. Now the proof of Lemma 4.4 shows: $\mathbb{M}(\mathcal{C}) \Vdash (\forall \bar{a} \in M[G])(\forall n)(\bar{a} R_n \mathbf{g}_M)$. Since $\mathbb{M}(\mathcal{C})$ is proper, this says that $\mathbb{M}(\mathcal{C})$ is $(\bar{R}, \mathbf{g}_M)$-preserving in the sense of Definition [30, Ch. XVIII, Def. 3.4] [1] By Lemma 4.7 we are in so-called Case A of iteration theory: the sets $\{\bar{a} : \bar{a} R_n \mathbf{g}\}$ are open or closed subsets of a G_δ-subset of the Baire space. Hence $(\bar{R}, \mathbf{g})$-preservation is preserved in the successor step according to [30, Chapter XVIII, §3, Claim 3.5] and in the countable support limit [30, Chapter XVIII, §3, Theorem 3.6]. So each block-swallowing family that is preserved by any iterand will be preserved by a countable support iteration of these iterands.

5. Iterated forcing

We let $S^2_1 = \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$. We start with a ground model $\mathbf{V}$ that fulfils CH and $\diamondsuit(S^2_1)$ (and hence $2^{\aleph_1} = \aleph_2$). We fix a $\diamondsuit(S^2_1)$-sequence $\langle D_\alpha : \alpha \in S^2_1\rangle$.

We work with countable support iterations $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\beta, \mathbb{Q}_\gamma : \beta \leq \alpha, \gamma < \alpha\rangle$. We denote $\mathbf{V}^{\mathbb{P}_\alpha}$ by $\mathbf{V}_\alpha$. If the iterands are proper each real appears in a $\mathbf{V}_\alpha$ for some α with countable cofinality [29, Ch. III]. A reflection property ensures that each non-meagre filter $\mathcal{F}$ in the final model has ω_1-club many $\alpha \in \omega_2$ such that $\mathcal{F} \cap \mathbf{V}_\alpha$ has a $\mathbb{P}_\alpha$-name and is a non-meagre filter in $\mathbf{V}_\alpha$ (see [11, Item 5.6 and Lemma 5.10]). A subset of ω_2 is called ω_1-club if it is unbounded in ω_2 and closed under suprema of strictly ascending sequences of lengths ω_1. A subset of ω_2 is called ω_1-stationary if is has non-empty intersection with every ω_1-club. By well-known techniques based on coding $\mathbb{P}_\alpha$-names for filters as subsets of ω_2 (e.g., such a coding is carried out in [26, Claim 2.8]) and based on the maximal principle (see, e.g., [19, Theorem 8.2]) the $\diamondsuit(S^2_1)$-sequence $\langle D_\alpha : \alpha \in S^2_1\rangle$ gives ω_1-club often a $\mathbb{P}_\alpha$-name D_α for a non-meagre filter in $\mathbf{V}_\alpha$ such that for any non-meagre filter $\mathcal{F} \in \mathbf{V}_{\omega_2}$ there are ω_1-stationarily many $\alpha \in S^2_1$ with $\mathcal{F} \cap \mathbf{V}_\alpha = D_\alpha$.

[1] Read in a form with one additional index running, see [25, Def. 4.5]. Unfortunately there is a misprint in [30, Ch. XVIII, Def. 3.4]. The proofs of the preservation of preservation in [30, Ch. XVIII, Claim 3.5, Theorem 3.6] work with a definition with one more running index. More explanations and rewritten proofs can be read in [25, Section 4].

In $\mathbf{V}_\alpha$, we define $\mathbb{Q}_\alpha = \mathbb{M}(\mathcal{U}_\alpha)$ and a name $s_\alpha := s(\mathcal{U}_\alpha)$.

We also fix a P-point $\mathcal{E} \in \mathbf{V}$ that will be preserved throughout our iteration. We fix an enumeration $\langle E_\varepsilon : \varepsilon < \omega_1\rangle$ of a basis of $\mathcal{E}$ such that each element appears cofinally often. For $\mathcal{X} \subseteq [\omega]^\omega$, we let $\mathrm{cl}(\mathcal{X}) = \{Y : (\exists X \in \mathcal{X})(Y \supseteq X)\}$.

We use R for elements of $\mathcal{R}^*$, and $R \in \mathbf{V}_\alpha$ means $R \in \mathcal{R}^* \cap \mathbf{V}_\alpha$. We use $\bar{b}$, $\bar{c}$, $\bar{d}$ for elements of $(\mathbb{F})^\omega$, and $\bar{c} \in \mathbf{V}_\alpha$ means $\bar{c} \in (\mathbb{F})^\omega \cap \mathbf{V}_\alpha$. We use letters B for subsets of $\mathbb{F}$ and f for a standardised $\mathbb{M}(\bar{c})$-name for block-sequence for some block sequence $\bar{c}$.

We construct by induction on $\alpha \leq \omega_2$ a countable support iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \omega_2, \alpha \leq \omega_2\rangle$ such that for any $\alpha \leq \omega_2$, the initial segment $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \beta < \alpha, \gamma \leq \alpha\rangle$ fulfils:

(I1) For all $\beta < \alpha$, $\Vdash_{\mathbb{P}_\beta}$ "$\mathbb{Q}_\beta$ is proper and $|\mathbb{Q}_\beta| \leq \aleph_1$".

(I2) $\Vdash_{\mathbb{P}_\alpha}$ "$\mathrm{cl}(\mathcal{E})$ is ultra".

(I3) If $\beta \in S^2_1 \cap \alpha$ and if D_β is a $\mathbb{P}_\alpha$-name $\mathcal{F}$ for a non-meagre filter in $\mathbf{V}^{\mathbb{P}_\beta}$, then $\Vdash_{\mathbb{P}_{\beta+1}}$ "$g_\beta(\mathcal{F}) = g_\beta(\mathrm{cl}(\mathcal{E}))$". Here $g_\beta(n) = |s_\beta \cap n|$.

(I4) For $\beta < \alpha$: $\mathbb{Q}_\beta = \mathbb{M}(\mathcal{U}_\beta)$, $\mathcal{U}_\beta$ and $\mathrm{out}(\mathcal{U}_\beta)$ are as in Lemma 4.4, and $\mathcal{U}_\beta$ is a stable ordered union ultrafilter, $\Phi(\mathcal{U}_\beta) \not\leq_{\mathrm{RB}} \mathcal{E}$ and

$$\mathbf{V}_\alpha \Vdash \mathrm{out}(\mathcal{U}_\beta) \text{ is block-swallowing.}$$

(I5) For all $\beta < \alpha$: s_β is $\leq^*$-unbounded over $\mathbf{V}_\beta$ and

$$(\forall \gamma < \beta)(\forall R \in \mathcal{R}^* \cap \mathbf{V}_\gamma)\big(\mathbb{P}_{\beta+1} \Vdash (s_\beta \not\subseteq^* R(s_\gamma))\big).$$

Property (I4) is used to construct $\mathcal{U}_\alpha$ and $\mathrm{out}(\mathcal{U}_\beta)$ so that $\mathbb{M}(\mathcal{U}_\alpha)$ forces the the statements in (I5) that forbid almost-inclusions. For deriving SFT and not FD only properties (I1), (I2), (I3), and (I5) play a role.

We first show that the existence of such an iteration implies the main theorem:

Lemma 5.1. *Assume that $\mathbb{P}$ has the properties listed above. Then in $\mathbf{V}_{\omega_2}$ the filter dichotomy holds and the semifilter*

$$\mathcal{S} = \{x \in [\omega]^\omega : (\exists \alpha \in \omega_2)(x \supseteq^* s_\alpha)\}$$

is not-meagre, not comeagre, and not ultra by finite-to-one.

Proof. By properness our iteration preserves $\aleph_1$. It preserves $\aleph_2$, because any collapse would appear at some intermediate step $\mathbb{P}_\alpha$, but $\mathbb{P}_\alpha$ has size $\aleph_1$ and the $\aleph_2$-c.c. So $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{V}_{\omega_2}}$ and $\aleph_2^{\mathbf{V}} = \aleph_2^{\mathbf{V}_{\omega_2}}$ and we write just $\aleph_1$, $\aleph_2$. The filter dichotomy holds because of (I2) and (I3).

By Talagrand's lemma 3.6 and since the enumerating functions of the s_α, $\alpha \in \omega_2$, form an $\leq^*$-unbounded family, the semifilter is not meagre.

Since FD implies NCF, the statement "$\mathcal{S}$ is not ultra by finite-to-one" is equivalent to "$\mathcal{S}$ is not nearly coherent with $\mathcal{E}$". Assume for a contradiction that $\mathcal{S}$ is nearly coherent with $\mathcal{E}$. Then a finite-to-one function f with $f(\mathcal{E}) = f(\mathcal{S})$ would appear in some $\mathbf{V}_\alpha$, $\mathrm{cf}(\alpha) = \omega$ and $\alpha < \omega_2$. We take $\beta > \alpha$. By the properties of $\mathbb{Q}_\beta$, the increasing enumeration of $f''s_\beta$ is $\leq^*$-unbounded over $\mathbf{V}_\alpha$. Hence $f''s_\beta \not\supseteq^* f''E$ for any $E \in \mathcal{E}$. So $f(\mathcal{S}) \neq f(\mathcal{E})$.

Suppose that $\mathcal{S}$ is comeagre. Then there is a finite-to-one f such that $f(\mathcal{S}) = [\omega]^\omega$. There is $\alpha \in \omega_2$ such that such an $f \in \mathbf{V}_\alpha$. Then by (I5) for $\alpha < \beta < \omega_2$, $f''s_\beta \not\subseteq^* s_\alpha$. However, s_α has $2^\omega = \aleph_2$ pairwise almost disjoint subsets, and hence $\{f''s_\beta : \beta \leq \alpha\}$, being of size at most $\aleph_1$, is not dense in $([s_\alpha]^\omega, \subseteq^*)$. So the whole set $f(\mathcal{S}) = \{f''s_\beta : \beta \in \omega_2\}$ is not $\subseteq^*$-dense below s_α and hence $f(\mathcal{S}) \neq [\omega]^\omega$. □

Lemma 5.2. *The forcing tasks (I5) follow from (I4).*

Proof. We showed at the end of Section 3 that s_β is unbounded over $\mathbf{V}_\beta$. Let $\gamma < \beta < \omega_2$ and $R \in \mathbf{V}_\gamma$ be given. The family $\mathrm{out}(\mathcal{C}_\gamma) = \{\bar{d}_{\gamma,\varepsilon+1} : \varepsilon < \omega_1\}$ is a block-swallowing family in $\mathbf{V}_{\gamma+1}$. The latter is preserved in $\mathbf{V}_\beta$ by (I5).

Now we work in $\mathbf{V}_\beta$. Let $\mathcal{U}_\beta = \langle \bar{c}_{\beta,\varepsilon} : \varepsilon < \omega_1 \rangle$. Since $R \in \mathbf{V}_\gamma$ and $\gamma < \beta$ and s_β is $\leq^*$-unbounded over $\mathbf{V}_\beta$, for sufficiently large ε, the sequence $R^{-1}(\bar{c}_{\beta,\varepsilon})$ is a block sequence, say for $\varepsilon \geq \delta$. For each $\varepsilon \in [\delta, \omega_1)$, the block sequence $R^{-1}(\bar{c}_{\beta,\varepsilon})$ is swallowed by some $\bar{d}_{\gamma,h(\varepsilon)}$, where $h(\varepsilon) < \omega_1$ is a successor ordinal. So for each $\varepsilon \in [\delta, \omega_1)$,

$$\mathbb{M}(\mathcal{U}_\beta) \Vdash \text{"}(\exists^\infty n)((R^{-1}(\bar{c}_{\beta,\varepsilon}))_n \text{ is a subset of a block of } \bar{d}_{\gamma,h(\varepsilon)})\text{"}.$$

Since $\mathrm{set}(\bar{d}_{\gamma,h(\varepsilon)}) \cap \mathrm{set}(\bar{c}_{\gamma,h(\varepsilon)})$ is finite, this implies:

$$\mathbb{M}(\mathcal{U}_\beta) \Vdash \text{“}R^{-1}(\bar{c}_{\beta,\varepsilon}) \text{ has infinitely many blocks that are not subsets of } \mathrm{set}(\bar{c}_{\gamma,h(\varepsilon)})\text{”}.$$

Since $\mathrm{set}(\bar{c}_{\gamma,\delta}) \supseteq^* s_\gamma$ for any δ, this yields:

$$\mathbb{M}(\mathcal{U}_\beta) \Vdash \text{“}R^{-1}(\bar{c}_{\beta,\varepsilon}) \text{ has infinitely many blocks that are not subsets of } s_\gamma\text{”}.$$

Since $\mathcal{U}_\beta$ is generated by the descending sequence $\langle \bar{c}_{\beta,\varepsilon} : \varepsilon < \omega_1 \rangle$ and the latter holds for cofinally many ε, we have $\mathbb{M}(\mathcal{U}_\beta) \Vdash s_\beta \not\subseteq^* R(s_\gamma)$. □

So from now on we work in order to get (I1) to (I4).

We note that (I1) to (I4) are true for $\alpha = 0$.

Lemma 5.3. *Induction lemma. Assume that $\alpha \leq \omega_2$ and that $\langle \mathbb{P}_\gamma, \mathbb{Q}_\delta : \gamma < \alpha, \delta < \gamma \rangle$ is defined with properties (I1) to (I4). Then there is a continuation $\langle \mathbb{P}_\gamma, \mathbb{Q}_\delta : \gamma \leq \alpha, \delta < \alpha \rangle$ with properties (I1) to (I4).*

The proof of the induction lemma has four cases:

(a) α is a limit ordinal of uncountable cofinality
(b) α is a limit ordinal of countable cofinality
(c) α is a successor ordinal of a successor ordinal or of a limit ordinal of countable cofinality
(d) α is a successor ordinal of a limit ordinal of uncountable cofinality

We begin with the easiest case: In case (a) all statements are true, since names for reals in proper forcings are hereditarily countable objects.

In case (b) we invoke preservation theorems: Preservation of properness [29], preservation of P-points [11, Theorem 4.1]. Preservation of "out($\mathcal{U}_\beta$) is block-swallowing" follows from Remark 4.8.

In the successor cases $\alpha = \beta + 1$ we construct $\mathcal{U}_\beta$ and out($\mathcal{U}_\beta$) in $\mathbf{V}_\beta$. In order to organise the construction, we introduce two helpers.

Definition 5.4. We call a sequence $\langle B_\varepsilon, R_\varepsilon, E_\varepsilon, \zeta_\varepsilon, f_\varepsilon \, : \, \varepsilon < \omega_1 \rangle \in \mathbf{V}_\beta$ a *book-keeping in* $\mathbf{V}_\beta$ if any (B, R, E, ζ, f) is named cofinally often, where $B \subseteq \mathbb{F}$, $B \in \mathbf{V}_\beta$, $R \in \mathcal{R}^* \cap \mathbf{V}_\beta$, $E \in \mathcal{E}$, $\zeta \in \omega_1$, and finally, for some $\bar{c} \in \mathbf{V}_\beta$, the sequence $f \in \mathbf{V}_\beta$ is a standardised $\mathbb{M}(\bar{c})$-name for a block sequence (see page 7 in Lemma 4.4).

Since $\mathbf{V}_\beta \models \mathsf{CH}$, there is a book-keeping. We use standardised names in order to have only $\aleph_1$ names.

Definition 5.5. A pair $\langle \bar{c}_\varepsilon \, : \, \varepsilon < \omega_1 \rangle$, $\langle \bar{d}_{\varepsilon+1} \, : \, \varepsilon < \omega_1 \rangle$ is called a good pair for $\mathbf{V}_\beta$ if the following holds: The sequence $\langle \bar{c}_\varepsilon \, : \, \varepsilon < \omega_1 \rangle \in \mathbf{V}_\beta$, is descending and there is a book-keeping $\langle B_\varepsilon, R_\varepsilon, E_\varepsilon, \zeta_\varepsilon, f_\varepsilon \, : \, \varepsilon < \omega_1 \rangle$ in $\mathbf{V}_\beta$ such for each $\varepsilon < \omega_1$ there are $\bar{c}^1_\varepsilon$, $\bar{c}^2_\varepsilon$, $\bar{c}^3_\varepsilon$ with the following properties:

(1) (The Hindman tasks) $\mathrm{FU}(\bar{c}^1_\varepsilon)$ is included in B_ε or disjoint from B_ε and $\bar{c}^1_\varepsilon \sqsubseteq^* \bar{c}_\varepsilon$.

(2) (The Eisworth tasks) $\omega \smallsetminus R_\varepsilon(\mathrm{set}(\bar{c}^2_\varepsilon)) \in R(\mathrm{cl}^{\mathbf{V}_\beta}(\mathcal{E}))$ and $\bar{c}^2_\varepsilon \sqsubseteq^* \bar{c}^1_\varepsilon$.

(3) (The Blass–Laflamme tasks) If possible we take $\bar{c}^3_\varepsilon \sqsubseteq^* \bar{c}^2_\varepsilon$ such that $\bar{c}^3_\varepsilon \Vdash g_\beta{}'' E_\varepsilon \in g_\beta(\mathcal{F})$, for the finite-to-one function $g_\beta(n) = |s_\beta \cap n|$. If there is no such $\bar{c}^3_\varepsilon$, we let $\bar{c}^3_\varepsilon = \bar{c}^2_\varepsilon$. Here $\mathcal{F}$ is a non-meagre filter handed down by the diamond. (This item is only relevant in the case of a successor of an ordinal of uncountable cofinality.)

(4) (Preservation of block-swallowing and creating the block-swallowing out($\mathcal{U}_\beta$) and disjointness) If f_ε has a equivalent $\mathbb{M}(\bar{c}_\varepsilon)$-name f, the successor $\bar{c}_{\varepsilon+1} \sqsubseteq^* \bar{c}^3_\varepsilon$ seals f_ε at $\bar{c}^3_\varepsilon$ for the block-swallowing family out($\mathcal{U}_{\zeta_\varepsilon}$). The block-sequence $\bar{d}_{\varepsilon+1}$ seals block-swallowing f_ε at $\bar{c}^3_\varepsilon$.

Continuation the proof of the induction lemma:
Case (c): Let $\alpha = \beta + 1$ and $\mathrm{cf}(\beta) \leq \omega$. We let $\bar{c}_0 = \langle \{n\} \, : \, n \in \omega \rangle$, and we construct a good pair $\langle \bar{c}_\varepsilon \, : \, \varepsilon < \omega_1 \rangle$, $\langle \bar{d}_{\varepsilon+1} \, : \, \varepsilon < \omega_1 \rangle$. We let $\mathcal{U}_\beta$ be generated by $\langle \bar{c}_\varepsilon \, : \, \varepsilon < \omega_1 \rangle$, and let out($\mathcal{U}_\beta$) $= \{bd_{\varepsilon+1} \, : \, \varepsilon < \omega_1\}$.

We show that all properties (Ix) follow from goodness and the induction hypotheses: Since decreasing countable sequences in $\mathcal{U}_\beta$ have lower bounds and since there are the Hindman tasks, the centred system $\mathcal{U}_\beta$ generated by the first sequence in the good pair is a stable ordered-union ultrafilter.

The forcing $\mathbb{M}(\mathcal{U}_\beta)$ is σ-centred, hence proper. It has size $\aleph_1$. Forcing with $\mathbb{M}(\mathcal{U}_\beta)$ preserves $\mathcal{E}$ by the Eisworth tasks and Eisworth's Theorem 3.5.

So (I1), (I2), (I3) (vacuously) are carried on. By induction hypothesis, $\mathbb{P}_\beta$ preserves the block-swallowing families out($\mathcal{U}_\gamma$), $\gamma < \beta$. By the sealing tasks and by Lemma 4.4, the forcing $\mathbb{M}(\mathcal{U}_\beta)$ preserves the block-swallowing families $\mathrm{out}(\mathcal{U}_\gamma)$, $\gamma < \beta$. Now $\mathbb{P}_\alpha = \mathbb{P}_\beta * \mathbb{M}(\mathcal{U}_\beta)$ preserves them as well, by Shelah's successor step lemma [30, Ch. XVIII, Claim 3.5]. By construction, $\mathbb{P}_{\beta+1} \Vdash \mathrm{out}(\mathcal{U}_\beta)$ is block-swallowing and $(\forall\varepsilon)(\mathrm{set}(\bar{d}_{\varepsilon+1}) \cap \mathrm{set}(\bar{c}_{\varepsilon+1}) = \emptyset)$. This finishes the proof of (I4) and concludes case (c).

Case (d): $\alpha = \beta + 1$ and $\mathrm{cf}(\beta) = \omega_1$. Tasks can be fulfilled only in stages in which all the inputs are evaluated. Let $\langle B_\varepsilon, R_\varepsilon, E_\varepsilon, \zeta_\varepsilon, f_\varepsilon \,:\, \varepsilon < \omega_1\rangle \in \mathbf{V}_\beta$ be a book-keeping. Let $\langle \beta_\varepsilon \,:\, \varepsilon < \omega_1\rangle$ be a continuously increasing sequence with supremum β. By continuouity there is a continuous subsequence α_ε, $\varepsilon < \omega_1$, such that for any $\gamma \in \alpha_\varepsilon$, we have $B_\gamma, R_\gamma, E_\gamma, \zeta_\gamma, f_\gamma \in \mathbf{V}_{\alpha_\varepsilon}$. We assume that $\alpha_0 = 0$. Now we construct a good pari.

We start with $\bar{c}_0 = \langle \{n\} \,:\, n \in \omega\rangle$.

At limit steps ε we take the $\bar{c}_\varepsilon \sqsubseteq^* \bar{c}_\zeta$ for all $\zeta < \varepsilon$.

We carry out the successor step, $\varepsilon = \delta + 1$. Suppose $\bar{c}_\delta \in \mathbf{V}_{\alpha_\delta}$ is given. We work until further notice in $\mathbf{V}_{\alpha_\delta}$. We strengthen $\bar{c}_\delta$ four times in order to fulfil the current instance of the Hindman task, the Eisworth task, the Blass–Laflamme task, the sealing task and we call the outcome $\bar{c}_{\delta+1} \sqsubseteq \bar{c}_\delta$, $\bar{d}_{\delta+1}$. The names B_δ, R_δ, E_δ (in $\mathbf{V}_0$), ζ_δ, f_δ and the handed down names for members of $\mathcal{F}$ are elements of $\mathbf{V}_{\alpha_\delta}$ and all the strengthening is done in $\mathbf{V}_{\alpha_\delta}$. Now we leave $\mathbf{V}_{\alpha_\delta}$ and go to $\mathbf{V}_{\alpha_{\delta+1}}$. Thus we have a good pair. We showed in case (c) that goodness implies that (I1), (I2), and (I4) are carried on. We show now that (I3) follows from goodness: Since $\mathcal{F}$ is not meagre, the set

$$\begin{aligned} \mathcal{G}_1(E_\varepsilon, \mathcal{F}) = \big\{Z \in [\omega]^\omega \cap \mathbf{V}_\beta \;:\, & (\exists Y \in \mathcal{F})(\forall a, b \in Z) \\ & \big([a,b) \cap Y \neq \emptyset \to [a,b) \cap E_\varepsilon \neq \emptyset\big)\big\} \end{aligned} \tag{5.1}$$

is groupwise dense. For details see [9, Section 9]. So $(\forall \bar{c} \in (\mathbb{F})^\omega \cap \mathbf{V}_\beta)(\exists \bar{b} \sqsubseteq^* \bar{c})(\mathrm{set}(\bar{b}) \in \mathcal{G}_1(E_\varepsilon, \mathcal{F}))$. If already in $\mathbf{V}_{\alpha_\varepsilon}$ there is such a $\bar{b} \sqsubseteq \bar{c}^2_\varepsilon$ then we let $\bar{c}^3_\varepsilon \sqsubseteq^* \bar{b}$ in the relevant intermediate step. Recall $g_\beta(n) = |s_\beta \cap n|$. Blass and Laflamme [10] showed that $\bar{c}^3_\varepsilon$ ensures that $\mathbb{P}_{\beta+1} \Vdash$ "$g''_\beta E_\varepsilon = g''_\beta Y \in g_\beta(\mathcal{F})$", for a witness $Y \in \mathcal{F}$ that is as in Equation (5.1). Since $\mathcal{F} \in \mathbf{V}_\beta$ and every task $E \in \mathcal{E}$ appears at cofinally many stages and since fulfilling the task for E will be possible starting from some stage ε when a suitable member $Y \in \mathcal{F}$ is seen, at some stage the task will be taken care of. $\square$

So we have proved the main theorem.

6. Side results on cardinal characteristics

The negation of SFT implies $\mathfrak{g} \leq \mathfrak{u}$. From the short proof of Laflamme's [20] theorem $\mathfrak{u} < \mathfrak{g} \to \mathsf{SFT}$ in [9, Lemma 9.15, Theorem 9.22] we read off groupwise dense families that witness $\mathfrak{g} \leq \mathfrak{u}$ in our forcing extensions.

Observation 6.1. *In the ground model, we fix a basis* $\{E_\varepsilon : \varepsilon < \omega_1\}$ *for the P-point* $\mathcal{E}$. *Then we let*

$$\mathcal{G}_\varepsilon = \{Z \in [\omega]^\omega : (\exists S \in \mathcal{S})(\forall m, n \in Z)([m,n) \cap S \neq \emptyset \to [m,n) \cap E_\varepsilon \neq \emptyset) \wedge (\exists T \in [\omega]^\omega)(T^c \notin \mathcal{S} \wedge (\forall m, n \in Z)([m,n) \cap T \neq \emptyset \to [m,n) \cap E_\varepsilon \neq \emptyset))\}.$$

Since $\mathcal{S}$ *is not comeagre and not meagre, the sets* $\mathcal{G}_\varepsilon$ *are groupwise dense, The intersection* $\bigcap_{\varepsilon<\omega_1} \mathcal{G}_\varepsilon = \emptyset$ *because* $\mathcal{S}$ *is not equal to* $\mathcal{E}$ *by finite-to-one.*

Now we renounce the P-point and the symmetry between $\mathcal{S}$ and its dual $\{X^c : X \in [\omega]^\omega \setminus \mathcal{S}\}$ that comes with the ultrafilter $\mathcal{E}$ and go for a finite support construction of length κ^+, κ regular.

Theorem 6.2. *Let* κ *be a regular cardinal and assume* $\diamondsuit(\{\alpha \in \kappa^+ : \mathrm{cf}(\alpha) = \kappa\})$. *Then there is a finite support iteration of c.c.c. forcings that forces* $\mathfrak{g} = \mathfrak{b} = \mathfrak{s} = \kappa$ *and* $\mathfrak{g}_f = 2^\omega = \kappa^+$.

Proof. We start with a ground model in which $2^\omega = \kappa$. There is a finite support iteration of length κ^+ of a variant of the iterands of the main theorem in which all properties of $\mathcal{E}$ and the Eisworth tasks are dropped. Equation $\oplus_2$ guarantees that $\mathbb{M}(\mathcal{C})$ adds an unbounded real (formerly we argued with the ultrafilter $\mathcal{U}$). If $\kappa \geq \omega_2$, we force to get the lower bounds in the limit steps of uncountable cofinality (less than κ) in the limit steps of construction in the proof of Lemma 4.4, see page 8. These intermediate forcings have size strictly less than κ, and hence preserve well-ordered block-swallowing families by [3, Lemma 2.1]. Our families, coming from descending sequences, are well-ordered: $x \prec y$ iff $x = \bar{d}_\delta$ and $y = \bar{d}_\varepsilon$ and $\delta < \varepsilon$. The requirements onto this well-order are: At any point δ, the family $\{\bar{c}_\varepsilon : \varepsilon \geq \delta\}$ is still block-swallowing. In addition we can arrange the descending sequences so that any splitting family of size less than κ gets destroyed. We iterate with finite support. By [4, Theorem 6.4.13], finite support limits of forcings that preserve a block-swallowing family preserve that the family is block-swallowing.

Now we evaluate the invariants in the extension: The diamond hands down groupwise dense ideals and thus as in case(d) of the induction lemma we can arrange $\mathfrak{g}_f = \kappa^+$.

It is easy to see that any block-swallowing family has size at least $\mathfrak{b}$. Thus the block-swallowing family $\mathrm{out}(\mathcal{C}_0)$, being of size κ, witnesses $\mathfrak{b} \leq \kappa$.

The bounding number is κ, by Shelah's result that $\mathfrak{g}_f \leq \mathfrak{b}^+$ [31]. The splitting number is at least κ by arrangement.

We let $[\omega]^\omega \cap \mathbf{V}_0$ be enumerated as $\{X_\varepsilon : \varepsilon < \kappa\}$. We let

$$\mathcal{G}_\varepsilon = \{Z \in [\omega]^\omega : (\exists S \in \mathcal{S})(\forall m, n \in Z)([m,n) \cap S \neq \emptyset \to [m,n) \cap X_\varepsilon \neq \emptyset)\}.$$

Since $\mathcal{S}$ is not comeagre, the family $\mathcal{G}_\varepsilon$ is groupwise dense. Suppose for a contradiction that $Z \in \bigcap_{\varepsilon<\kappa} \mathcal{G}_\varepsilon$, $Z \in \mathbf{V}_\alpha$. Hence $\mathcal{S}$ is mapped by the finite-to-one function f_Z that takes the elements of the n-th Z-interval to n, into the set of supersets of elements of $[\omega]^\omega \cap \mathbf{V}_0$. This implies that $\mathcal{S}$ is generated by $[\omega]^\omega \cap \mathbf{V}_\alpha$. However, the Matet real s_α is $\leq^*$-unbounded over $\mathbf{V}_\alpha$, and hence s_α is not a superset of any infinite set in $\mathbf{V}_\alpha$. Contradiction. The groupwise dense families $\mathcal{G}_\varepsilon$, $\varepsilon < \kappa$, witness $\mathfrak{g} \leq \kappa$. By Proposition 1.1 we get from $\mathfrak{g}_f = \kappa^+$ and $\mathfrak{s} \geq \kappa$ and $\mathfrak{g} \leq \kappa$ that $\mathfrak{g} = \kappa$ and $\mathfrak{s} = \kappa$. □

In research it is a good custom to conclude with open questions.

Question 6.3. *Is $\mathfrak{b} < \mathfrak{u} < \mathfrak{d}$ consistent relative to* ZFC*?*

Beside that we refer to the plethora of riddles in the third but last paragraph of the introduction.

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