THE STRENGTHS OF SOME VIOLATIONS OF COVERING

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Abstract. We show that in order to get $V_1 \subseteq V_2$ two models of ZFC with the same cofinality function and the same ω-sequences and some set in V_2 not being covered by any set in V_1 of the same cardinality an inner model of V_2 a measurable cardinal κ of Mitchell order κ^{++} is necessary.

We show that changing cardinal characteristics without changing cofinalities or ω-sequences (which was done for some characteristics in [12]) has consistency strength at least $\mathrm{o}(\kappa) = \kappa^{++}$.

From this we get that the changing of cardinal characteristics without changing *cardinals* or ω-sequences has consistency strength $\mathrm{o}(\kappa) = \omega_1$, even in the case of characteristics that do not stem from a transitive relation. Hence the known forcing constructions for such a change have lowest possible consistency strength.

We consider some stronger violations of covering which have appeared as intermediate steps in forcing constructions.

1. Introduction

We are going to prove the results listed in the abstract and some equiconsistent formulations in terms of pseudo powers and possible cofinalities. This gives some bounds on the consistency stengths of the constellations obtained in [11] and in [12].

We shall obtain our results by applying pcf theory and citing some core model theory. The reader can find the proofs for the facts we are going to cite in [19], [5], [15], [6], or [17].

Before stating the results, we need some notation.

Notation: $H_{\aleph_1}$ is the set of all hereditarily countable sets. Suppose that U_1 and U_2 are both ultrafilters on κ. Then we write $U_1 \lhd U_2$ iff $U_1 \in \mathrm{ult}(V, U_2)$, where the latter is the ultrapower V^κ modulo U_2. Mitchell [13] showed that $\lhd$, which is called the Mitchell order, is a well-founded partial order. The Mitchell

Date: August 26, 1999.

1991 *Mathematics Subject Classification.* 03E35, 03E55.

Key words and phrases. pcf theory, pseudo power, covering, large cardinals.

The author was partially supported by a Lise Meitner Fellowship of the State of North Rhine Westphalia and by a Minerva fellowship.

order of an ultrafilter U, short $\mathrm{o}(U)$, is its rank in this well-founded partial order. The Mitchell order of a cardinal κ, $\mathrm{o}(\kappa)$, is $\{\mathrm{o}(U) \mid U$ is a free ultrafilter on $\kappa\}$ (which is an ordinal).

Our notation follows [10] and [7] and [19], however, we do not presuppose the knowledge of the notions from the latter: in Section 2 we shall recall the notions from pcf theory ([19]) that we are going to use.

Now we are going to consider general syntactical forms of definitions of cardinal invariants. The goal is to make first steps in drawing conclusions on how hard it is to change the values of certain cardinal characteristics, just by looking at the syntacs of the definition of the cardinal characteristic.

The most general invariants we are going to consider are of the form:

$$\mathrm{inv}_\phi^V = \min\{|A| \mid A \subseteq H_{\aleph_1} \ \wedge \ (H_{\aleph_1}, \in, A) \models \phi\} \tag{1.1}$$

for some first order formula ϕ in the language containing $\in$ and a unary predicate symbol P, and such that $(H_{\aleph_1}, \in, A) \models \forall x \ Px \to \phi$ and such that ϕ is monotone in the unary predicate. The supscript V indicates that the invariant is computed in the universe V. All entries of Cichoń's Diagram (see, e.g., [1]) and many other of the common cardinal characteristics have the form given by (1.1).

Suppose that we have some ϕ of the above form and that we have the following constellation of evaluations:

$$(<,\phi) \qquad \begin{array}{l} V_1 \subseteq V_2 \text{ both models of ZFC,} \\ \text{and } V_1 \text{ and } V_2 \text{ have the same cofinality function,} \\ \ \text{and the same } H_{\aleph_1} \\ \text{and } \mathrm{inv}_\phi^{V_2} < \mathrm{inv}_\phi^{V_1} . \end{array}$$

Again in the Cichoń's Diagram, some examples for $(<,\phi)$ have been obtained (of course only consistently): In [12] such a scenario V_1, V_2 in constructed, with same ω-sequences, not only same $H_{\aleph_1}$ in V_1 and in V_2, where the bounding number, both additivities and both uniformities, and the covering number of the ideal of meagre sets drop. The construction of the models is based upon some bare set-theoretic premises. Here we show that these premises are not too strong from the point of view of consistency strength.

If $(<,\phi)$ holds for some universes V_1 and V_2 and some cardinal θ, then we have

$$\mathrm{inv}_\phi^{V_1} > \theta = \mathrm{inv}_\phi^{V_2} \geq \aleph_1.$$

So, looking at the witness A for the evaluation of the invariant in V_2 we get

$$(*)_{\theta,V_1,V_2} \qquad \begin{array}{l} V_1 \subseteq V_2 \text{ both are models of ZFC,} \\ \text{and } V_1 \text{ and } V_2 \text{ have the same cofinality function} \\ \text{ and the same hereditarily countable sets,} \\ \text{in } V_2\text{, there is some set } A \subseteq V_1 \cap 2^\omega \\ \text{ of cardinality } \theta \text{ that cannot be covered} \\ \text{ by any set in } V_1 \text{ of cardinality } \theta. \end{array}$$

The symbol pp stands for pseudo power from pcf theory and will be explained in the next section, where we shall prove:

Theorem 1.1. *Let θ be an uncountable cardinal.*

Let V_1, V_2 be given. If θ is minimal with $(*)_{\theta,V_1,V_2}$ then we have in V_2 there is some κ such that $2^\omega \geq \kappa > \mathrm{cf}(\kappa) = \theta$ and $\mathrm{pp}(\kappa) > \kappa^+$.

We conclude this section by discussing the impact of the theorem.

Remark on the reverse: As shown in the proof of [12, Theorem 7.1], $(*)_{\theta,V_1,V_2}$ can be obtained from the assumption $\exists\kappa\ \mathrm{o}(\kappa) = \kappa^{++} + \omega_1$. An intermediate step in the forcing construction there is to get to a pair $V_1 \subset V_2$ of models of the conclusion of Theorem 1.1 together with covering above κ and the additional property, that $V_2 = V_1[G]$ for some P-generic G over V_1 and some $P \in V_1$.

Shelah proved an analogon of Silver's result about cardinal exponentiation, which shows that pseudo powers resemble powers:

Theorem 1.2. *(Shelah [19, Theorem II,2.4]) The minimum κ such that κ is singular and $\mathrm{pp}_{\mathrm{cf}(\kappa)}(\kappa) > \kappa^+$ has cofinality ω.*

The important result on consistency strengths that we are going to use is:

Theorem 1.3. *(Gitik [5] for $\kappa > 2^\omega$ and arbitrary cofinality, Mitchell [17] for arbitrary κ of cofinality ω) The property "$\exists\kappa > \omega\ (\mathrm{cf}(\kappa) = \omega \wedge \mathrm{pp}_{\mathrm{cf}(\kappa)}(\kappa) > \kappa^+)$" has strength at least $\exists\kappa\ \mathrm{o}(\kappa) = \kappa^{++}$.*

Putting 1.1, 1.2 and 1.3 together yields the corollary

Corollary 1.4. *$(*)_{\theta,V_1,V_2}$ implies that in V_2 there is an inner model (namely Mitchell's $\mathbf{K}(\mathcal{F})$) where $\exists\kappa\ \mathrm{o}(\kappa) = \kappa^{++}$.*

(Maybe such a κ can be found even in V_1. We need that $\mathbf{K}(\mathcal{F})^{V_1} = \mathbf{K}(\mathcal{F})^{V_2}$ for certain pairs (V_1, V_2). The equation is true if V_2 is a forcing extension of V_1 by a set-sized forcing.)

We may conclude: Changing cardinal characteristics of the reals without changing cofinalities nor $H_{\aleph_1}$ has consistency strength at least $\exists\kappa\ \mathrm{o}(\kappa) = \kappa^{++}$ and hence has higher consistency strength than changing cardinal characteristics of the reals without changing *cardinalities* nor ω-sequences, where the upper bound is $\mathrm{o}(\kappa) = \omega_1$ (for some procedure that is based upon a cofinality change). For characteristics of the form

$$\mathrm{inv}^V_{\psi,B} = \min\{|A| \mid A \subseteq H_{\aleph_1} \ \wedge \forall x \in B\ \exists y \in A\ (H_{\aleph_1}, \in) \models \psi(x,y)\}$$

for some $\psi \in \mathcal{L}(\in)$ and some $B \subseteq H_{\aleph_1}$ with transitive ψ a change of cofinality is necessary by [12, 1.4 and 1.5], where ψ is transitive if $\forall x, y, z,\ ((\psi(x,y) \wedge \psi(y,z)) \to \psi(x,z))$. So, in this case, $\exists\kappa\ \mathrm{o}(\kappa) = \omega_1$ was known to be optimal. For invariants of the general form (1.1), Theorem 1.4 shows that there is no procedure without changing *cardinalities* nor ω-sequences, of strength less than $\mathrm{o}(\kappa) = \omega_1$.

2. Proof of Theorem 1.1

We shall need some of the most basic and most important definitions from pcf theory [19] which we collect here for the reader's convenience.

Products

Let $\mathfrak{a}$ denote a set of regular cardinals such that $|\mathfrak{a}| = \mathrm{ordertype}(\mathfrak{a}) < \min \mathfrak{a}$. Let $\langle a_i \mid i \in |\mathfrak{a}|\rangle$ be the increasing enumeration of $\mathfrak{a}$.

$$\prod \mathfrak{a} = \{f \mid f \colon \mathfrak{a} \to \bigcup \mathfrak{a} \wedge \forall a \in \mathfrak{a}\ f(a) \in a\}.$$

This is often identified with

$$\prod_{i \in |\mathfrak{a}|} a_i = \{f \mid f \colon |\mathfrak{a}| \to \bigcup \mathfrak{a} \wedge \forall i \in |\mathfrak{a}|\ f(i) \in a_i\}.$$

An ideal I is a family of subsets of its domain, $\mathrm{dom}(I)$, closed under union and subsets; usually I is proper, i.e. $\mathrm{dom}(I) \notin I$.

For $f, g \in \prod \mathfrak{a}$ and a proper ideal I on $\mathfrak{a}$ we have the partial order

$$f \leq_I g \text{ iff } \{a \in \mathfrak{a} \mid \neg f(a) \leq g(a)\} \in I.$$

(This would make sense even if $(\bigcup \mathfrak{a}, \leq)$ were only a partial order.) Of course, the ideal I and the order can naturally be read as if they were living on the set of functions on the set of indices $\{i \mid i \in |\mathfrak{a}|\}$ (and we shall do so).

Cofinalities

The cofinality $\mathrm{cf} \prod \mathfrak{a}/J$ or $\mathrm{cf}(\prod \mathfrak{a}, <_J)$, where J is an ideal on $\mathfrak{a}$, is the minimal power of a subset F of $\prod \mathfrak{a}$ such that for every $g \in \prod \mathfrak{a}$ there is some $f \in F$ such that $g \leq_J f$.

The true cofinality, $\mathrm{tcf} \prod \mathfrak{a}/J$ or $\mathrm{tcf}(\prod \mathfrak{a}, <_J)$, is well defined if we can choose F as above being well-ordered by $\leq_J$ (the point is, that the order is linear), and then the true cofinality is the cofinality of this linear order.

$\mathrm{pcf}(\mathfrak{a})$ is the set $\{\mathrm{tcf} \prod \mathfrak{a}/J \mid J$ is a maximal ideal on $\mathfrak{a}\}$.

Pseudo powers

For κ a limit cardinal, and $\theta < \kappa$ I an ideal on θ, and Γ a class of ideals, let

$$\begin{aligned} \mathrm{pp}^*_I(\kappa) = \sup\{\mathrm{tcf}(\textstyle\prod_{i<\theta}\kappa_i, <_I) \mid \langle \kappa_i \mid i < \theta\rangle & \text{ is increasing and} \\ & \kappa_i = \mathrm{cf}(\kappa_i) < \kappa = \sup_{i<\theta} \kappa_i \text{ and} \\ & \text{for each } \mu < \kappa, \{i \mid \kappa_i < \mu\} \in I \text{ and} \\ & (\textstyle\prod_{i<\theta}\kappa_i, <_I) \text{ has true cofinality}\}; \end{aligned}$$

so $\mathrm{pp}^*_I(\kappa)$ is undefined if for all choices of κ_i the true cofinality does not exist. Next we recall

$$\mathrm{pp}_I(\kappa) = \sup\{\mathrm{pp}^*_J(\kappa) \mid I \subseteq J \text{ and } \mathrm{dom}(I) = \mathrm{dom}(J)\}.$$

Since for maximal ideals the reduced structure is an ultraproduct and linearly ordered, $\mathrm{pp}_I(\kappa)$ is always defined. Finally we have

$$\begin{aligned} \mathrm{pp}_\Gamma(\kappa) &= \sup\{\mathrm{pp}^*_I(\kappa) \mid \exists \theta \leq \kappa \ \mathrm{dom}(I) = \theta \text{ and } I \text{ is an ideal in } \Gamma\}, \\ \mathrm{pp}_\gamma(\kappa) &= \mathrm{pp}_\Gamma(\kappa), \text{ for } \Gamma = \{I \mid I \text{ is an ideal on some cardinal } \theta \leq \gamma\}, \\ \mathrm{pp}(\kappa) &= \mathrm{pp}_{\mathrm{cf}(\kappa)}(\kappa). \end{aligned}$$

Again, $\mathrm{pp}_\Gamma(\kappa)$ might be undefined, depending on Γ. However, the latter two pseudo powers are always defined.

A frequently used class Γ is (for some regular σ')

$$\begin{aligned} \Gamma &= \Gamma(\theta', \sigma') \\ &= \{I \mid \text{ for some cardinal } \theta_I < \theta', I \text{ is a } \sigma'\text{-complete proper ideal on } \theta_I\} \end{aligned}$$

A prominent rôle is played by the ideals

$$J^{bd}_\theta = \{B \mid B \text{ is a bounded subset of } \theta\}.$$

Covering numbers

We assume that $\lambda \geq \theta' \geq \sigma > 1$, $\kappa \geq \aleph_0$, $\theta' > 1$, and

$$\kappa \geq \theta' \vee (\kappa^+ = \theta' \wedge \mathrm{cf}(\theta') < \sigma).$$

Then $\mathbf{cov}(\lambda,\kappa,\theta',\sigma)$ is the first cardinal μ such that there is a family $\mathcal{P}$ of μ subsets of λ, each of cardinality $<\kappa$, such that

$$\forall t\Big((t \subseteq \lambda \ \wedge\ |t| < \theta') \Rightarrow (\exists \mathcal{P}')\Big(\mathcal{P}' \subseteq \mathcal{P} \ \wedge\ |\mathcal{P}'| < \sigma \ \wedge\ t \subseteq \bigcup_{A \in \mathcal{P}'} A\Big)\Big).$$

Later we will apply the covering numbers with $\theta' = \theta^+$. This is the end of the definition part, and now we prove Theorem 1.1.

Proof of 1.1. Suppose that we have $(*)_{\theta,V_1,V_2}$. We take κ minimal such that there is in V_2 some witness $A \subseteq \kappa$ for $(*)_{\theta,V_1,V_2}$. Then we have that in V_2, $\kappa > \mathrm{cf}(\kappa) = \theta = |A|$, A is cofinal in κ. We claim that in V_2

$$\mathrm{pp}_{\Gamma(\theta^+,\theta)}(\kappa) = \mathrm{pp}(\kappa) > \kappa^+.$$

In order to prove the claim, we shall modify the proof of the following special case of [19, Chapter II, Theorem 4.5(1), harder inequality], the so-called "cov versus pp Theorem". For $\lambda \geq \kappa \geq \theta' > \sigma \geq \omega_1$, σ regular, the harder inequality says

$$\begin{aligned}&\mathbf{cov}(\lambda,\kappa,\theta',\sigma) + \lambda \leq \\ &\sup\{\mathrm{pp}_{\Gamma(\theta',\sigma)}(\lambda^*) \mid \lambda^* \in [\kappa,\lambda] \text{ and } \sigma \leq \mathrm{cf}(\lambda^*) < \theta'\} + \lambda.\end{aligned} \tag{2.1}$$

The modification is as follows: we do not start from a from the covering number on the left-hand side but from the premise given by $(*)_{\theta,V_1,V_2}$.

We apply the above inequality (2.1) in the special case of $\mathrm{cf}(\kappa) = \theta < \kappa = \lambda$, and $\sigma = \theta$, $\theta' = \theta^+$. Since the left-hand summands on both sides are $\geq \kappa$, the addition of λ to each side does not matter. By $(*)_{\theta,V_1,V_2}$ and by the fact that $\theta \geq \aleph_1$ is regular, , we have that $\mathcal{P} = \{A \mid A \subseteq \kappa, A \in V_1, |A| < \kappa\}$ does not cover $\{A \mid A \subseteq \kappa, A \in V_2, |A| = \theta\}$ in the sense of $\mathbf{cov}(\kappa,\kappa,\theta^+,\sigma)^{V_2}$ and hence can be used like the $\mathcal{P}$ in Shelah's original proof. So $\kappa^+ \leq |\mathcal{P}| < |\mathcal{P}|^+ =: \mu \leq \mathbf{cov}(\kappa,\kappa,\theta^+,\sigma)^{V_2}$ and we are to show that the right hand side of the inequality (2.1) is greater than or equal to μ, i.e. for some $I \in \Gamma(\theta,\sigma)$ and for some $\langle \lambda_\alpha \mid \alpha \in \mathrm{dom}(I)\rangle$ we have that

$$\begin{aligned}&\mathrm{tlim}_I\langle \lambda_\alpha \mid \alpha \in \mathrm{dom}(I)\rangle = \kappa, \text{ and}\\ &\mathrm{tcf}(\textstyle\prod \lambda_\alpha, <_I) \geq \mu.\end{aligned}$$

Since $\mathcal{P}$ is not a witness for the computation of $\mathbf{cov}(\kappa,\kappa,\theta^+,\sigma)^{V_2}$, in V_2 there is some $\theta^* \in [\sigma,\theta^+)$ and a function

$$f^* \colon \theta^* \to \kappa$$

such that $\neg\exists \zeta^* < \sigma$, $A_\zeta \in \mathcal{P}$, $\zeta < \zeta^*$ such that $\mathrm{range}(f^*) \subseteq \bigcup_{\zeta<\zeta^*} A_\zeta$.

Now we can just continue as in Shelah's proof of [19, Chapter II, Theorem 5.4(1), page 89 ff.]. So we have that in V_2,

$$\mathrm{pp}(\kappa) \geq |\mathcal{P}|^+ \geq \kappa^{++}.$$

$\Box_{1.1}$

3. A stronger violation of covering

In this section we consider another situation of violation of covering, which appears at an intermediate step in the construction from [12]. It is not known whether such a constellation is necessarily connected with the change of cardinal charactersistics from the former sections. We look at the covering properties of the pair (V_1, V_2) in the following diagram of ZFC-models.

$$\begin{tikzcd}
 & \tilde{V}_1 = V_1[G] \arrow[r] & \tilde{V}_2 = V_2[G] & \\
V \arrow[r] & V_1 \arrow[u] \arrow[r] & V_2 \arrow[u] \arrow[r] & V^{P_{\mathrm{Gitik}}}
\end{tikzcd}$$

Diagram 1

The arrows in the diagram denote inclusion.

The $\tilde{V}_i$ are the models witnessing a change in cardinal characteristics as in Sections one and two. The V_i are the models used as the starting points of the forcing in [12], the ones with the following violation of covering:

$$(**)_{\theta,\kappa,V_1,V_2} \qquad \begin{array}{l} V_1 \subseteq V_2 \text{ both are models of ZFC,} \\ \text{and } V_1 \text{ and } V_2 \text{ have the same cofinality function} \\ \text{ and the same } \omega\text{-sequences in } \kappa > 2^\omega, \\ \text{in } V_2, \text{ there is some set } A \subseteq \kappa, \\ \text{of cardinality } \theta > \omega,\ A \text{ cofinal in } \kappa, \\ \text{that cannot be covered by any set in } V_1 \text{ of cardinality } \theta. \end{array}$$

The outer models V and $V^{P_{\mathrm{Gitik}}}$ are the classical "non SCH at an uncountable cofinality" models. The consistency strength to obtain them is almost pinned down by work of Gitik, Mitchell [6], and Woodin, namely between $\mathrm{o}(\kappa) = \kappa^{++}$ and $\mathrm{o}(\kappa) = \kappa^{++} + \mathrm{cf}^{V_2}(\kappa)$, and for $\mathrm{cf}^{V_2}(\kappa) \geq \aleph_2$ it coincides with the upper bound.

Now we use a modification of a Theorem of [6, 3.1]. It is not known whether the following works for violations of covering of sets A cofinal in some $\kappa \leq 2^\omega$.

Theorem 3.1. *Let θ be an uncountable cardinal.*

Let V_1, V_2 be given. If κ is minimal with $(**)_{\theta,\kappa,V_1,V_2}$ and $\kappa > 2^\omega$, then we have in V_2:

$\kappa > \mathrm{cf}(\kappa) = \theta$, *and*

there is some cardinal ρ and there is a sequence $\mathfrak{a} \subseteq \rho$ of regular cardinals, cofinal in ρ, with $\mathrm{cf}(\rho) \leq \theta$ and $\mathrm{otp}(\mathfrak{a}) = |\mathfrak{a}| = \theta$, such that

(1) $\mathrm{tcf}(\Pi\mathfrak{a}/J^{bd}_\theta) \geq \rho^{++}$, *and*

(2) Any strictly increasing sequence from $\Pi\mathfrak{a}$ of length less than $\mathrm{tcf}(\Pi\mathfrak{a}/J^{bd}_\theta)$ and cofinality greater than 2^θ has a least upper bound.

Proof. For (1) we claim: Suppose $(**)_{\theta,\kappa,V_1,V_2}$ and κ is minimal such that for V_1, V_2, θ the property $(**)_{\theta,\kappa,V_1,V_2}$ holds. Under the premises of Theorem 3.1, there are a singular cardinal ρ and an increasing sequence $\langle \rho_i \mid i < \theta \rangle$ of regular cardinals such that the sequence is cofinal in ρ and such that

$$\mathrm{tcf}(\textstyle\prod \rho_i, <_{J^{bd}_\theta}) \geq \rho^{++}.$$

The claim comes from page 311 of [19], where we have

If $\rho > \theta \geq \mathrm{cf}(\rho) > \aleph_0$ and

$\forall \mu < \rho$ large enough of $\mathrm{cf}(\mu) \leq \theta$ we have that $\mathrm{pp}_\theta(\mu) < \rho$,

then we have that

$\otimes_1$ (α) $\forall$ regular $\gamma \in (\rho, \mathrm{pp}^+_\theta(\rho))$

there is a sequence $\langle \rho_i \mid i < \mathrm{cf}(\rho) \rangle$ of regular cardinals $< \rho$ with limit ρ such that $\mathrm{tcf}(\prod \rho_i / J^{bd}_{\mathrm{cf}(\rho)}) = \gamma$, so

(β) $\mathrm{pp}_\theta(\rho) = \mathrm{pp}_{\mathrm{cf}(\rho)}(\rho) = \mathrm{pp}^*_{J^{bd}_{\mathrm{cf}(\rho)}}(\rho)$.

We use that we have $\otimes_1$ with θ being our θ and $\rho = \min\{\kappa \mid \mathrm{pp}_\theta(\kappa) > \kappa^+ \wedge \mathrm{cf}(\kappa) \leq \theta\}$.

Now, heading for (2) of 3.1, we shall continue to free-load, this time in [19, Chapter II §1]. The statement (2) is one of the three possibilities in the so-called Shelah-trichotomy [19, Chapter II §1]. Since the other two possibilities therein are excluded by the remark following [19, Chapter II §1], (2) is true. $\square_{3.1}$

Theorem 3.2 (Gitik, Mitchell [6]). *Let θ be a regular uncountable cardinal. The strength of "$\exists \kappa$ $\mathrm{cf}(\kappa) = \theta < \kappa$ and κ is a strong limit and $2^\kappa > \kappa^+$" is* *(a) between* $\exists\kappa\ \mathrm{o}(\kappa)=\kappa^{++}$ *and* $\exists\kappa\ \mathrm{o}(\kappa)=\kappa^{++}+\omega_1$ *for* $\theta=\omega_1$*,*

(b) $\exists\kappa\,\mathrm{o}(\kappa)=\kappa^{++}+\theta$ *for* $\theta\geq\omega_2$*.*

Gitik and Mitchell derive the consistency strength in the conclusion of their theorem from (1) and (2) in Theorem 3.1. So they actually proved

Theorem 3.3 (Modification of Gitik, Mitchell). *Let* θ *be a regular uncountable cardinal. The strength of the conclusions (1) and (2) of Theorem 3.1 and* $2^\theta\leq\kappa$ *is*

(a) between $\mathrm{o}(\kappa)=\kappa^{++}$ *and* $\mathrm{o}(\kappa)=\kappa^{++}+\omega_1$ *for* $\theta=\omega_1$*,*

(b) $\mathrm{o}(\kappa)=\kappa^{++}+\theta$ *for* $\theta\geq\omega_2$*.*

Proof. Section 3 of [6] shows that only (b) (1) and (2) are used as a premise. $\square_{3.4}$

So, the forcing construction from [12, Section 7] and 3.1, 3.2, and 3.3 together yield:

Corollary 3.4. *For* $\theta\geq\omega_2$ *we have "there is an inner model* V_1 *such that* $(**)_{\theta,\kappa,V_1,V_2}$ *and* $\mathrm{cf}(\theta)\geq\aleph_2$*" has the same consistency strength as: in* V_2 *there is an inner model with* $\exists\kappa\ \mathrm{cf}(\kappa)=\theta<\kappa$ *and* κ *is a strong limit and* $2^\kappa>\kappa^+$*.*

We do not know whether in the case of $\theta=\omega_1$ the analogous coincidence is true.

Remark. We may get the strength also in V_1 if V_2 is only a set-generic extension of V_1, see the core model-theoretic technique in [2], which has to be extended to the models in [3, 14, 16, 8, 9], [18, 20].

Remark 3.5. Suppose that we have $\mathrm{o}(\kappa)=\omega_1$ (and not higher). Under this premise Gitik [4] builds a model V such that in V κ is inaccessible and there is a κ^+-c.c. forcing P that does not add new bounded subsets of κ (so P is μ-distributive for all $\mu<\kappa$) and such that in V^P, $\mathrm{cf}(\kappa)=\omega_1$.

Let $\langle\kappa_i\mid i\in\omega_1\rangle\in V^P$ be a sequence cofinal in κ.

We set $V_1=V[\langle\kappa_{2i}\mid i\in\omega_1\rangle]$. Then $\langle\kappa_{2i}\mid i\in\omega_1\rangle$ is P_1 generic for some complete suborder P_1 of P by [7]. V_1 does not have any new bounded subsets of κ because V^P does not have any new bounded subsets of κ.

Now at first sight two cases are possible:

First case: $\{\kappa_i\mid i\in\omega_1\}$ is covered by some set in V_1 of size $<\kappa$.

Second case: $\{\kappa_i\mid i\in\omega_1\}$ is not covered by any set in V_1 of size $<\kappa$. From the proof of 1.1 we get in this case that $\mathrm{pp}(\kappa)\geq\kappa^{++}$ in $V_1^{P_1}=V^P$. Now the Main

Theorem of [5] says: $(\kappa > 2^\omega \ \wedge \kappa$ singular $\ \wedge \mathrm{pp}(\kappa) \geq \kappa^{++})$ implies that there is an inner model of $\exists\alpha\ o(\alpha) \geq \alpha^{++}$.

Since we do not have this high consistency strength, the second case is excluded.

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