Weak diamonds and clubs

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"Young set theory workshop",

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"Young set theory workshop",

Outline

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Ostaszewski's guessing principle ♣

Definition

♣ is the abbreviation of the following statement:

$(\exists \langle A_\alpha : \alpha \in \omega_1, \lim(\alpha) \rangle)$

$(A_\alpha$ is cofinal in α and

$\forall X \subseteq_{\mathrm{unc}} \omega_1 \{\alpha \in \omega_1 : A_\alpha \subseteq X\}$ is stationary$)$.

Theorem, Devlin

♣ + $CH \leftrightarrow \Diamond$.

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♣ + CH ↔ ◇.

Separating ♣ from ◇

Theorem, Shelah, Baumgartner

$\clubsuit + \neg CH$ is consistent relative to ZFC.

In the recent years, more models of

$$\clubsuit + ci = \mathfrak{c} > \aleph_1,$$

for some cardinal characteristics ci have been found by Fuchino, Shelah, Soukup, Džamonja and Shelah, Brendle.
No particularly easy construction has yet come up.

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Separating ♣ from ♢

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Two weakenings of the ♣-principle

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$♣_w$ is the abbreviation of the following statement:

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Definition

$♣_{w^2}$ is the abbreviation of the following statement:

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A weakening of ♣$_w$ in another direction

Definition

❗ is the abbreviation of the folllowing statement:

$(\exists \langle A_\alpha \, : \, \alpha \in \omega_1, \lim(\alpha) \rangle)$

$(A_\alpha$ is countably infinite and $\forall X \subseteq_{\mathrm{unc}} \omega_1 \exists \alpha A_\alpha \subseteq X)$.

❗ follows from the CH.

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A weakening of ♣ₓ in another direction

Definition

♣ is the abbreviation of the folllowing statement:

$$(\exists \langle A_\alpha \,:\, \alpha \in \omega_1, \lim(\alpha) \rangle)$$

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♣ follows from the CH.

A longstanding open question

Question, Juhász

Does ♣ imply the existence of a Souslin tree?

Stronger version of the question if heading for a negative answer
Is ♣ together with all Aronszajn trees are special consistent relative to
ZFC?

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Recall, a specialisation of an Aronszajn tree $\mathbf{T} = (\omega_1, <_{\mathbf{T}})$ is a function $f : \omega_1 \to \mathbb{Q}$ such that for any $s, t \in \omega_1$, $s <_{\mathbf{T}} t \to f(s) < f(t)$. We call such a function monotone.

A special Aronszajn tree has an uncountable antichain. "All Aronszajn trees are special" is strictly stronger than "there are no Souslin trees".

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Suppose that $\mathbb{P} = \langle \mathbb{P}_\beta, \mathbb{Q}_\alpha \; : \; \alpha < \omega_2, \beta \leq \omega_2 \rangle$ is a countable support iteration of proper forcings with the $\aleph_2$-c.c. Suppose that it forces "$\uparrow + \neg CH$" or $\clubsuit_{\mathrm{w}}$. Then by the properties of names for objects in $\mathscr{P}(\omega_1)$, the guessing sequence is already in some intermediate model.

However, then no Souslin trees can be destroyed any more, because uncountable branches and uncountable antichains can be guessed in the intermediate model and hence cannot come in only later in the iteration. So any Souslin tree existing in the intermediate model will be Souslin in the final extension.

Bad news about forcings with good reflection properties

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♣_{w²} does not have this obstruction

Theorem

"♣_{w²} + CH + all Aronszajn trees are special" is consistent relative to ZFC.

Based on techniques from MdSh:848.

♣$_{w^2}$ does not have this obstruction

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Outline

Weak diamonds

Definition, Moore, Hrušák, Džamonja

Let $A, B \subseteq \mathbb{R}$ be Borel and let $E \subseteq A \times B$ be Borel in $\mathbb{R}^2$. $\Diamond(A, B, E)$ is the following principle:

$$(\forall F : 2^{<\omega_1} \to A)(\exists g_F : \omega_1 \to B)(\forall f : \omega_1 \to 2)$$
$$\{\alpha \in \omega_1 \; : \; F(f \restriction \alpha) E g_F(\alpha)\} \text{is stationary.}$$

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Observation

The weak diamond for the reaping relation, i.e.,

$\diamondsuit(2^\omega, [\omega]^\omega,$ is almost constant on$)$, implies ♣$_{w^2}$.

Proof: For $\alpha < \omega$, fix some $h_\alpha : \omega \to \alpha$ such that range(h_α) has ordertype ω and is cofinal in α.

Define $F \restriction 2^\alpha : 2^\alpha \to 2^\omega$ by

$$F(f \restriction \alpha)(n) = f(h_\alpha(n)).$$

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Define $F \upharpoonright 2^\alpha \colon 2^\alpha \to 2^\omega$ by

$$F(f \upharpoonright \alpha)(n) = f(h_\alpha(n)).$$

Shifting realms of constancy

We assume that $g_F \colon \omega_1 \to [\omega]^\omega$ is a guessing function as in the weak diamond for the reaping relation.

Then for every $f \colon \omega_1 \to 2$,

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$\{\alpha < \omega_1 \ : \ \beta \mapsto f(\beta) \text{ is almost constant on } h''_\alpha g_F(\alpha)\}.$

So

$A_\alpha = h''_\alpha g_F(\alpha)$

gives a $\clubsuit_{\omega^2}$-sequence.

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$\{\alpha < \omega_1 : n \mapsto f(h_\alpha(n)) \text{ is almost constant on } g_F(\alpha)\} =$

$\{\alpha < \omega_1 : \beta \mapsto f(\beta) \text{ is almost constant on } h_\alpha'' g_F(\alpha)\}.$

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An extension of 848

Theorem

"$\diamondsuit$(reaping) + CH + all Aronszajn trees are special" is consistent relative to ZFC.

Cichoń's diagram of weak diamonds

Figure: The framed weak diamonds imply the existence of a Souslin tree. The arrows indicate implications.

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Outline

Connections: Weak diamonds, CH or not, and Souslin trees

Theorem, Moore, Hrušák, Džamonja

$\Diamond(\mathcal{M}, \mathbb{R}, \not\ni)$ implies that there is a Souslin tree.

Theorem, Laver plus a bit Moore, Hrušák, Džamonja

$\Diamond(\mathcal{N}, \mathbb{R}, \not\ni) + \neg CH +$ "all Aronszajn trees are special" is consistent.

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Non-Connections: The club and weak diamonds

> **Theorem, Fuchino, Shelah, Soukup**
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> ♣ and $\mathrm{cov}(\mathcal{M}) = \aleph_2$ is consistent relative to ZFC.

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Weak diamond versus "cardinal equals $\aleph_1$"

> **Fact**
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Example, Brendle

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Just force with an iteration of all Souslin trees in a countable support iteration giving each of them a cofinal branch. So there is no Souslin tree. Because of the Sacks property, $\mathrm{cof}(\mathcal{M})$ stays small. By Moore Hrušák and Džamonja, $\Diamond(\mathcal{M}, \mathcal{M}, \subseteq)$ would also imply that there is a Souslin tree.

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Connections: The club principle and Souslin trees

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Outline

Specialising Aronszajn trees adding no reals with a very good completeness system. New techniques to compute in a Borel manner generics filters over countable models that have suprema.

Recall, $p \in P$ is (M, P)-generic if for every P-generic filter G over V with $p \in G$, $p \Vdash M[G] \cap On = M \cap On$.

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Recall, $p \in P$ is **(M, P)-generic** if for every P-generic filter G over V with $p \in G$, $p \Vdash M[\underset{\sim}{G}] \cap On = M \cap On$.

An (M, P_γ)-generic filter G is called bounded if there is a $q \in P_\gamma$ such that $G = \{p \in M \cap P_\gamma : p \leq q\}$.

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Forcings with side conditions

Now we work with monotone functions f, that specialise only a part of $\mathbf{T}$, namely the union of countably many of its levels, so that the indices of the levels form a closed set C. We call such a pair (f, C) an *approximation*.

Definition

$H \subseteq \mathbb{Q}^{[T_\gamma]^n}$ is called *dispersed* iff for each $t \in [T_\gamma]^{<\omega}$, there is some $h \in H$ such that $t \cap \mathrm{dom}(h) = \emptyset$.

Promises

Definition

(See Definition 4.1 (4) in [AbSh:403].) Γ is a **T-promise** iff $\mathrm{dom}(\Gamma)$ is club in ω_1 and $\Gamma = \langle \Gamma(\gamma) : \gamma \in \mathrm{dom}(\Gamma) \rangle$ has the following properties:

(a) For each $\gamma \in \mathrm{dom}(\Gamma)$, $\Gamma(\gamma)$ is a countable set of requirements of height γ.

(b) $(\forall \gamma \in \mathrm{dom}(\Gamma))(\forall H \in \Gamma(\gamma))$ H is dispersed.

(c) $(\forall \alpha_0 < \alpha_1 \in \mathrm{dom}(\Gamma))(\Gamma(\alpha_0) \supseteq \{H{\restriction}\alpha_0 : H \in \Gamma(\alpha_1)\})$. Here, $H{\restriction}\alpha_0 = \{h{\restriction}\alpha_0 : h \in H\}$, and for $h \colon T_{\alpha_1} \to \mathbb{Q}$ we let $\mathrm{dom}(h{\restriction}\alpha_0) \subseteq T_{\alpha_0}$ and $h{\restriction}\alpha_0(x) = \min\{h(y) : y{\restriction}\alpha_0 = x, y \in \mathrm{dom}(h)\}$.

The iterands

Definition

(Def. 4.2 [AbSh:403]) $Q_{\mathbf{T}}$ is the set of (f, C, Γ) such that (f, C) is an approximation, and Γ is a promise and (f, C) fulfils Γ. The partial order is defined as "(f_1, C_1, Γ_1) is stronger than (f_0, C_0, Γ_0)" iff

(1) f_1 extends f_0,

(2) C_1 is an end-extension of C_0 and $C_1 \smallsetminus C_0 \subseteq \mathrm{dom}(\Gamma_0)$, and

(3) $(\forall \gamma \in \mathrm{dom}(\Gamma_0 \smallsetminus \mathrm{last}(f_1))(\gamma \in \mathrm{dom}(\Gamma_1) \text{ and } \Gamma_0(\gamma) \subseteq \Gamma_1(\gamma))$.

If $p = (f, C, \Gamma)$, we write $f = f^p$, $C = C^p$ and $\Gamma = \Gamma^p$, and we write $\mathrm{last}(p) = \mathrm{last}(f^p) = \max(C^p)$.

A reflection property

Why does every Aronszajn tree in $\mathbf{V}^{P_{\omega_2}}$ have a P_α-name for some $\alpha < \omega_2$? We have $|Q_{\mathbf{T}}| = \aleph_2$, so that we cannot work with the $\aleph_2$-chain condition for each iterand. Now Chapter VIII, Section 2 of [Sh:f] helps: Each $Q_{\mathbf{T}}$ has the $\aleph_2$-p.i.c. (proper isomorphism condition), see Chapter VIII, Def. 2.2 of [Sh:f], and hence by Chapter VIII, Lemma 2.4 [Sh:f], P_{ω_2} has the $\aleph_2$-c.c, *if* $\mathbf{V}_0$ fulfils the CH.

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Completeness systems

Definition

(Chapter V, 5.5 [Sh:f])

(1) We call $\mathbb{D}$ a **completeness system** if for some μ, $\mathbb{D}$ is a function defined on the set of triples $\langle M, P, p \rangle$, $p \in M \cap P$, $P \in M$, $M \prec (H(\mu), \in)$, M countable such that $\mathbb{D}(M, P, p)$ is a family of non-empty subsets of

$$\mathrm{Gen}(M, P, p) = \{ G \; : \; G \subseteq M \cap P, G \text{ is directed and } p \in G$$
$$\text{and } G \cap \mathcal{I} \neq \emptyset$$
$$\text{for every dense subset } \mathcal{I} \text{ of } P \text{ which belongs to } M \}.$$

(2) We call $\mathbb{D}$ a λ-completeness system if each family $\mathbb{D}(M, P, p)$ has the property that the intersection of any i elements is non-empty for $i < 1 + \lambda$ (so for $\lambda \geq \aleph_0$, $\mathbb{D}(M, P, p)$ generates a filter). $\aleph_1$-completeness systems are also called countably closed completeness systems.

(3) We say $\mathbb{D}$ is on μ if $M \prec (H(\mu), \in)$. We do not always distinguish strictly between $\mathbb{D}$ and its definition.

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If all iterands are $\mathbb{D}$-complete and $< \omega_1$ proper, then no reals are added in the countable support iteration.

Abraham's handbook article on "Proper forcing".

Adding no reals in the limit steps

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Outline

Many generic filters over small models

Definition

Suppose that $\mathbb{D}$ is a completeness system on χ. We say P is $\mathbb{D}$-complete, if for every countable $M \prec (H(\chi), \in)$ with $P \in M$, $\mathbb{D} \in M$, $p \in P \cap M$, the following set contains as a subset a member of $\mathbb{D}(M, P, p)$:

$$\mathrm{Gen}^+(M, P, p) = \{G \in \mathrm{Gen}(M, P, p) :$$
$$\text{there is an upper bound for } G \text{ in } P\}.$$

Simplicity: All is given by real parameters

Definition

(Chapter V, 5.5 [Sh:f]) A completeness system $\mathbb{D}$ is called simple if there is a first order formula ψ such that

$$\mathbb{D}(M, P, p) = \{A_x : x \text{ is a finitary relation on } M, \text{i.e.,}$$
$$x \subseteq M^k \text{ for some } k \in \omega\},$$

where

$$A_x = \{G \in \mathrm{Gen}(M, P, p) : (M \cup \mathcal{P}(M), \in, p, M, P) \models \psi(x, G)\}.$$

Our completeness system

Lemma

Q_T is $\mathbb{D}$-complete for the simple $\aleph_1$-completeness system $\mathbb{D}$ given by $\psi(x, G) = \psi_0(x) \wedge \psi_1(x, G)$, with

$$\psi_0(x) \equiv x = (x_1, x_2, \bar{\beta}) \wedge \bar{\beta} = \langle \beta_n \, : \, n \in \omega \rangle \text{ increasing}$$
$$\wedge \, M \cap \omega_1 = \bigcup\{\beta_n \, : \, n < \omega\}$$

and

$$\psi_1(x, G) \equiv (\forall \varepsilon > 0)(\exists m < \omega)(\forall n_1 < n_2 \in [m, \omega))(\forall t \in T_\mu)(\forall y_1, y_2 <_T t)$$
$$\left((y_1 \in T_{\beta_{n_1}} \wedge y_2 \in T_{\beta_{n_2}} \wedge y_1 <_T y_2 \rightarrow \underline{f}[G](y_2) < \underline{f}[G](y_1) + \frac{\varepsilon}{2^{n_2}} \right)$$
$$\wedge \text{ ``}G \text{ is a filter''}$$
$$\wedge \, p \in G \wedge \forall D \in M((D \subseteq P \wedge D \text{ dense in } P) \rightarrow D \cap G \neq \emptyset)$$
$$\wedge \, (\forall H \in x_2)(\forall n)(\forall t \in [T_{\beta_n}]^{<\omega})(\exists h \in H)$$
$$(\text{dom } h \lceil \beta_n \cap t = \emptyset \wedge \underline{f}[G] \upharpoonright T_{\beta_n} \text{ fulfils } h \lceil \beta_n).$$

Here M, P, x and G appear in the formulas as (names for) predicates and p is a constant. To ease readability, we write T_μ instead of x_1 (though T_μ is not a subset of M) and $\bigcup_{\gamma \geq \mu} \lceil^P(\gamma) \lceil \mu$ instead of x_2.

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Computing bounded generic filters by Borel functions

Lemma

Let $p \in Q_{\mathbf{T}} \cap M$. Let $\mu = \mathrm{otp}(M \cap \omega_{\mathbf{1}}) = \sup\langle \beta_{\mathbf{n}} : n < \omega \rangle$, $\beta_{\mathbf{n+1}} > \beta_{\mathbf{n}}$. Let $c \colon \omega \to M$ be a bijection with $c(0) = Q_{\mathbf{T}}$, $c(1) = p$, $c(2n+2) = \beta_{\mathbf{n}}$, and let

$$U = U(M, Q_{\mathbf{T}}, p) = \{2e(n_{\mathbf{1}}, n_{\mathbf{2}}) : c(n_{\mathbf{1}}) \in c(n_{\mathbf{2}})\} \cup \{2e(n_{\mathbf{1}}, n_{\mathbf{2}}) + 1 : c(n_{\mathbf{1}}) <^{*}_{\chi} c(n_{\mathbf{2}})\}.$$

We let η stand for function from ω to ω and we let the functions $h_{\mathbf{y}, \bar{\beta}}$ and $h_{\mathbf{p}, H_{\mathbf{n}}}$ be defined as to code the levels and the promises on the levels.
There is a Borel function $B_{\mathbf{1}} \colon \omega^{\omega} \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, such that if

$$(\forall y \in T_{\mu})(h_{\mathbf{y}, \bar{\beta}} \leq^{*} \eta) \tag{3.1}$$

and

$$(\forall x \in T_{\mathbf{last}(\mathbf{p})})(\forall n)(h_{\mathbf{p}, H_{\mathbf{n}}}(I(\cdot)) \leq^{*} \eta) \tag{3.2}$$

for

$$G = \{c(n) : n \in B_{\mathbf{1}}(\eta, U)\}$$

the following holds: G is $(M, Q_{\mathbf{T}})$-generic and $p \in G$ and there is an upper bound r of G.

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Version of the previous lemma for iterated forcing

Theorem

Let $P_{\omega_2} = \langle P_\alpha, Q_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration of iterands of the form Q_T. If χ is sufficiently large and regular and if $M \prec (H(\chi), \in, <^*_\chi)$ is a countable elementary model and

(a) $P_\gamma \in M$, $\gamma \leq \omega_2$,

(b) $p \in P_\gamma \cap M$,

(c) $\alpha = \mathrm{otp}(M \cap \gamma)$,

(d) Let $\bar{\beta}$ be cofinal in $M \cap \omega_1$. Let $c \colon \omega \to M$ be a bijection with $c(0) = P_\gamma$, $c(1) = p$, $c(2n+2) = \beta_n$, and let

$$U \;=\; U(M, P_\gamma, p) \;=\; \{2e(n_1, n_2) \;:\; c(n_1) \;\in\; c(n_2)\} \;\cup\; \{2e(n_1, n_2) + 1 \;:\; c(n_1) \;<^*_\chi\; c(n_2)\}.$$

Then there is a Borel function $\mathbf{B} = \mathbf{B}_\alpha \colon (\omega^\omega)^\alpha \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, such that in the following game $\partial_{(M, P_\gamma, p)}$ the generic player has a winning strategy σ, which depends only on the isomorphism type of $(M, \in, <^*_\chi, P_\gamma, p, \bar{\beta})$:

(α) a play lasts α moves,

(β) in the ε-th move the generic player chooses some real ν_ε and the antigeneric player chooses some $\eta_\varepsilon \in \omega^\omega$, such that $\eta_\varepsilon \not\leq^* \nu_\varepsilon$,

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Continuation

Continuation

(γ) in the end the generic player wins iff the following is true:

$$G_\gamma = \{c(n) \, : \, n \in \mathbf{B}_\alpha(\langle \eta_\varepsilon \, : \, \varepsilon < \alpha \rangle, U)\} \text{ is } (M, P_\gamma)\text{-generic and}$$

$$p \in G_\gamma \text{ and}$$

$$(\exists q \in P_\gamma)(p \leq q \text{ and } q \text{ bounds } G_\gamma).$$

Choosing a suitable argument $\bar{\eta}$

Lemma

Suppose that

(α) $\gamma < \omega_1$, and

(β) $\mathbf{B}'$ is a Borel function from $(\omega^\omega)^\gamma$ to 2^ω,

Then we can find some $\mathcal{C} = \mathcal{C}_{\mathbf{B}'}$ such that

(a) $\mathcal{C} \in [\omega]^\omega$,

(b) in the following game $\eth_{(\gamma, \mathbf{B}')}$ between two players, IN and OUT, the player IN has a winning strategy, the play lasts γ moves and in the ε-th move OUT chooses $\nu_\varepsilon \in \omega^\omega$ and then IN chooses $\eta_\varepsilon \not\leq^* \nu_\varepsilon$. In the end IN wins iff $\mathbf{B}'(\langle \eta_\varepsilon : \varepsilon < \gamma \rangle)$ is almost constant on $\mathcal{C}$.