More results on non-elementary proper forcings

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Outline

A brief introduction to non-elementary proper forcing

Definition
$(\mathbb{P}, \leq_{\mathbb{P}})$ is proper iff: For any regular $\chi > 2^{2^{|\mathbb{P}|}}$, for any $p \in P$ and $N \prec (H(\chi), \in, <)$ with $P, p \in N$ there is a stronger condition q such that q is (N, P)-generic.
q is (N, P)-generic iff the following holds: For any $D \in N$: If

$$N \models D \text{ is dense in } \mathbb{P}$$

then

$$q \Vdash \underset{\sim}{G}_{\mathbb{P}} \cap D \neq \emptyset.$$

Requiring generics for $N \subseteq H(\chi)$

Take the first definition is now strengthened: Existence of generic conditions is required for more countable $\in$-structures $N \models \mathrm{ZFC}^*$.

For example we think of $M' = M[g]$ for some g that "makes things more convenient" and is not related to $\mathbb{P}$. $N \prec H(\chi)$, $M = \pi_N(M)$, the collapse. So M is as usual.

$N[g] \prec H(\chi)[g]$, and $M[g'] \subseteq H(\chi)$ if g' is small.
Still: $N[g], M[g'] \models \mathrm{ZFC}^*$.

$N \cap \mathbb{P} = \mathbb{P}^N$ might be lost

We consider definable forcings:
$(\mathbb{P}, \leq_{\mathbb{P}}) = (\varphi_{\mathbb{P}}, \varphi_{\leq_{\mathbb{P}}})$.

$N^{\mathbb{P}}$ is the interpretation of $(\varphi_{\mathbb{P}}, \varphi_{\leq_{\mathbb{P}}})$ in N.
Now, of course $N \cap \mathbb{P} \neq \mathbb{P}^N$ is now possible.

We add absoluteness requirements: $\varphi_{\mathbb{P}}$ and $\varphi_{\leq_{\mathbb{P}}}$ are upwards absolute.
Then $(\mathbb{P}^N, \leq_{\mathbb{P}}^N) = (N \cap \mathbb{P}, \leq_{\mathbb{P}} \cap N \times N)$.

How to compute a generic condition

Given $D \in N$ that is dense in $\mathbb{P}$, from outside we can find a maximal antichain $\langle p_n \mid n \in \omega \rangle$ in D. Then "q is generic" implies that $\langle p_n \mid n \in \omega \rangle$ is predense above q. Let us put this fact to two fomulae

$\varphi(\langle p_n \mid n \in \omega \rangle)$ and
$\varphi^+(\langle p_n \mid n \in \omega \rangle \hat{\ } q)$

that hold in V. Now the aim is, given p and
$\langle p_n \mid n \in \omega \rangle$ to compute such a q in an absolute way, ideally Borel.

Then the outcome of the computation is a condition in N, and the computation is repeated with this starting point and with the next dense set D'. The chain of results should have a common strengthening, an $(N, \mathbb{P})$ generic condition.

For all D together

$q_{n+1} \geq_n q_n$ such that $\varphi^+(D_n, q_{n+1})$.

Reading in various models

Given N, $\mathbb{P}$, p we compute in some $N' \supseteq N$, $N \in N'$, $N' \models$ ZFC*, $N' \subseteq V$, and get in N' a result q.

We compare the computation to that in V, and want:

$N' \models q$ is $(N, \mathbb{P}, p)$-generic.
implies
q is $(N, \mathbb{P}, p)$-generic.

An application: Collapsing Souslin trees to countable objects

Let $T \in N \prec H(\chi)$ be a Souslin tree. $\mathbb{P} \in N$.

Idee:
We look at the question whether $\mathbb{P}$ preserves T not in $\pi_N(N)$ but in the Levy extension that changes the height of T to ω.

Preserving Souslin trees

Now let $\mathbb{P}$ be a nep forcing.
We want to find a an easy criterion when $\mathbb{P}$ preserves Souslin trees:
Let $(T, <_T)$ be a Souslin tree. $\mathbb{P}$ preserves T, if in for any $\mathbb{P}$-generic Filter $G_{\mathbb{P}}$,
$V[G_{\mathbb{P}}] \models (T, <_T)$ is a Souslin tree.

We consider only normal Souslin trees. Adding a branch amounts to adding an uncountable antichain. So the Souslin tree can be destroyed by destroying ω_1 or by adding an uncountable antichain.

$(T, <_T)$ is considered a notion of forcing

In this criterion, the Souslin tree $(T, <_T)$ is considered as a forcing $\mathbb{Q}$ adding a branch to T. Stronger conditions in $\leq_{\mathbb{Q}}$ are nodes higher up in the Souslin tree.

Definition

Let $Y \subseteq T$. We say T is $(Y, \mathscr{S})$-proper iff $Y \subseteq T$ and $\mathscr{S} \subseteq [\omega_1]^\omega$ and for every sufficiently large χ for every countable $N \prec \mathscr{H}(\chi)$ with $\{T, \mathscr{S}\} \subset N$ and $N \cap \omega_1 \in \mathscr{S}$, $\delta = N \cap \omega_1$ for every $t \in Y \cap T_\delta$,

$$T_{<t} := \{s \mid s <_T t\}$$

is (N, T) generic.

Every Souslin tree T is $(T, [\omega_1]^\omega)$-proper and every (Y, S)-proper (for a stationary S and stationarily many levels in Y) tree T is Souslin.

Preserving generic branches and hence Souslinness

Definition

We say $\mathbb{P}$ is $(T, Y, \mathscr{S})$-preserving iff the following holds: Let $\mathscr{S} \subseteq \omega_1$ be stationary and let T be a Souslin tree, $Y \subseteq T$. For every $N \prec \mathscr{H}(\chi)$ with $\{Y, T, \mathbb{P}, \mathscr{S}\} \subseteq N$ and $p \in \mathbb{P} \cap N$: if $\sup(N \cap \omega_1) = \delta$, $N \cap \omega_1 \in \mathscr{S}$, and for every $t \in Y \cap T_\delta$, $T_{<t}$ is $(N, \mathbb{P}, p)$-generic, then there is $q \geq_{\mathbb{P}} p$ such that q is $(N, \mathbb{P})$-generic and

$$q \Vdash_{\mathbb{P}} (\forall t \in Y \cap T_\delta)(T_{<t} \text{ is } (N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T)\text{-generic}).$$

Manipulating the countable ground model

Let $N \prec H(\chi)$ and $(T, <T) \in N$, $\mathbb{P}, p \in N$. P shall be nep in a strong sense.
We add a suitable generic g of the Levy collapse of ω_1 to ω to M.
$M = \pi_N(N)$.

Forcing with a normal Souslin tree can look like Cohen forcing

In $M = \pi_N{}''N$, $N \prec H(\chi)$, $(T, <_T) \in N$.

Then $T \cap M$ is
$(T_{<\delta}, <_T)$ where $\delta = \omega_1 \cap M$.

In $M[g]$, g a $\mathrm{Coll}(\omega, \delta)$-generic reals over M, $(T, <_T)$ looks like the Cohen partial order.

Preserving the Cohen genericity of $T_{<t}$ over $M[g]$ follows from preserving any Cohen real

Let $t \in T_\delta$. Then $T_{<t}$ is a branch through T in N and hence if T is c.c.c in N, $T_{<t}$ is (N, T) generic.
Let $\mathbb{R} = \mathrm{Col}(\omega, \delta)$, $\delta = \omega_1 \cap M$

Now: There is a Levy collapse-generic g over M such that $T_{<t}$ is $(M[g], (T, <_T))$-generic, so Cohen generic.

A density argument

Let $\underset{\sim}{\mathcal{I}}$ be a an $\mathbb{R}$-name for a dense subset of T. Then

$$\{q \in \mathbb{R} \mid \exists \nu \in T q \not\Vdash_{\mathbb{R}} \nu \notin \underset{\sim}{\mathcal{I}}\}$$

is dense in $\mathbb{R}$.

$\mathbb{P}$ preserves the collapsed situation

$M[g] \models T_{<t}$ is Cohen generic, $p \in \mathbb{P}^{M[g]}$.

Wish: There is an $(M[g], \mathbb{P})$-generic $q \geq p$ such that $q \Vdash_{\mathbb{P}}$ "$M[g][G_{\mathbb{P}}] \models T_{<t}$ is Cohen generic."

This is a somewhat known property

Definition

Let $\mathbb{P}$ be a proper forcing notion. We say $\mathbb{P}$ is ω-*Cohen preserving* iff the following holds: For every $N \prec \mathscr{H}(\chi)$ such that $\mathbb{P} \in N$, for every $p \in \mathbb{P} \cap N$ for every $\{x_n \mid n \in \omega\}$ such that every x_n is a Cohen real over N, there is an $(N, \mathbb{P})$-generic condition $q \geq p$ such that

$$q \Vdash (\forall n \in \omega)(x_n \text{ is Cohen over } N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}]).$$

Definition

Let $\mathbb{P}$ be a proper forcing notion. We say $\mathbb{P}$ is *ω-Cohen preserving over candidates* iff the following holds: For every candidate $N \subseteq \mathscr{H}(\chi)$ such that $\mathbb{P} \in N$, for every $p \in \mathbb{P} \cap N$ for every $\{x_n \mid n \in \omega\}$ such that every x_n is a Cohen real over N, there is an $(N, \mathbb{P})$-generic condition $q \geq p$ such that

$$q \Vdash (\forall n \in \omega)(x_n \text{ is Cohen over } N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}]).$$

back to the uncountable forcings

Preserving ω Cohen generic reals and preserving Souslinity

M, p $\qquad$ $T_{<t},\ t \in T_\delta$

$T_{<t},\ t \in T_\delta$

Figure: Comparing computations of generic conditions

$$M, p \qquad\qquad\qquad\qquad T_{<t},\ t \in T_\delta$$

$$q', M[G_{\mathbb{P}}] \qquad T_{<t},\ t \in T_\delta$$

Figure: Comparing computations of generic conditions

M, p $\quad$ $M[g]$ $\quad$ $T_{<t},\ t \in T_\delta$

$T_{<t},\ t \in T_\delta$

Figure: Comparing computations of generic conditions

Preserving ω Cohen generic reals and preserving Souslinity

M, p $\quad$ $M[g]$ $\quad$ $q, M[g][G_{\mathbb{P}}]$ $\quad$ $T_{<t},\ t \in T_\delta$

$T_{<t},\ t \in T_\delta$

Figure: Comparing computations of generic conditions

Preserving ω Cohen generic reals and preserving Souslinity

$$M, p \qquad M[g] \qquad q, M[g][G_{\mathbb{P}}] \qquad T_{<t},\ t \in T_\delta$$

$$q', M[G_{\mathbb{P}}] \qquad T_{<t},\ t \in T_\delta$$

Figure: Comparing computations of generic conditions

Generic conditions in candidates

If M_1 is a $(\bar{\varphi}, \mathfrak{B}, \mathrm{ZFC}^*)$-candidate and $M_1 \models$ "M_0 is a $(\bar{\varphi}, \mathfrak{B}, \mathrm{ZFC}^*)$-candidate and $p \in \mathbb{P}^{M_0}$" then then there is $q \in \mathbb{P}^{M_1}$, $q \geq p$ such that $M_1 \models$"q is $(M_0, \mathbb{P})$-generic" and such that in $\mathbf{V}$, q is $(M_0, \mathbb{P})$-generic

The wish from the previous slide is not exaggerated

Many definable forcings (definitions with parameters in $H(\omega_1)$) fulfil the criterion.
Examples: Tree forcings, creature forcings.
Counterexamples: Cohen forcing, random forcing, Blass-Shelah forcing.

Using the collapse

Theorem

Suppose $(\bar{\varphi}, \mathfrak{B}, \mathrm{ZFC}^*)$ *is a definition of* $\mathbb{P}$ *that is non-elementary proper and fulfils the criterion on existence of generics in candidates.*
Suppose that $\mathbb{P}$ *is* ω*-Cohen preserving for* $(\bar{\varphi}, \mathfrak{B}, \mathrm{ZFC}^*)$*-candidates.*
Then $\mathbb{P}$ *preserves Souslin trees.*

Larger forcings $\mathbb{P}$

Let $\mathbb{P}$ be a forcing destroying Souslin trees and not adding reals, for example

the NNR forcing from the Proper and Improper Forcing book

Jensen's forcing for the relative consistency of SH and CH.

These forcings are proper and do not add reals. So for elementary submodels N, they are Cohen preserving.
They are non-elementary proper to some extent.
Cohen preserving over candidates. back to Cohen preserving over candidates

Transferring both wishes

$$q \Vdash (\forall t \in \pi_N(Y(\delta)))(\pi_N(T_{<_T t}) \text{ is } (M[\mathbf{G}_{\mathbb{P}}], \pi_N(T))\text{-generic}) \\ \text{and } q \text{ is } (M, \mathbb{P})\text{-generic.} \quad (2.1)$$

Now we get from the latter

$$(\exists q_3 \geq \pi_N(p))(q \Vdash \text{``}(\forall t \in \pi_N(Y(\delta))) \\ \pi_N(T_{<_T t}) \text{ is } (N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], \pi_N(T))\text{-generic'' and } q_3 \text{ is } (N, \mathbb{P})\text{-generic}). \quad (2.2)$$