L-spaces and the P-ideal dichotomy

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A known consistency result

Survey

A known consistency result

Different conditions

Survey

A known consistency result

Different conditions

More cases covered

Countable tightness

All topological spaces considered are regular.

Definition

A topological space (X, τ) is called countably tight if for every $A \subseteq X$ and $x \in A$ if X is in the closure of A, then there is a countable subset B of A such that x is in the closure of B.

L-spaces

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A topological space (X, τ) is called an L-space if it is hereditarily Lindelöf and not separable.

Existence

Theorem, Moore

There is an L-space.

Additional wishes

Theorem, Todorčević

Under the assumption $(\mathcal{K})$,

any regular space Z with countable tightness such that Z^n is Lindelöf for all $n \in \omega$ has no L-subspace. $\boxtimes$

Todorčević 1989

The combinatorial principle $\mathcal{K}$

Todorčević 1989

Definition

Let S be an uncountable set and let $[S]^{<\omega} = K_0 \cup K_1$ be a partition. Then this is called a c.c.c. partition of all singletons are in K_0 and K_0 is closed under subsets and every uncountable subset of K_0 has two elements whose union is in K_0.

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Definition

The principle $(\mathcal{K})$ says: For any c.c.c. partition (S, K_0, K_1) there is an uncountable $H \subseteq S$ such that $[H]^2 \subseteq K_0$. If we replace 2 by the finite number m in the dimension (still partitioning into two parts) we get $(\mathcal{K}_m)$.

Theorem, Szentmiklóssy 1977

Assume MA_{ω_1}. Let Z be a compact space with countable tightness. Then Z has no L-subspaces.

P-ideals

An ideal $\mathscr{I}$ on a set S is a P-ideal, if for every countable $\mathscr{J} \subseteq \mathscr{I}$ there exists $I \in \mathscr{I}$ such that $J \subseteq^* I$ for all $J \in \mathscr{J}$.

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We say that $T \subset S$ is locally in (resp. orthogonal to) the ideal $\mathscr{I}$, if $[T]^\omega \subseteq \mathscr{I}$ (resp. $\mathscr{P}(T) \cap \mathscr{I} = [T]^{<\omega}$).

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We consider only ideals $\mathscr{I} \subseteq [S]^{\leq \omega}$ containing all singletons.

Todorčević

Two P-ideal dichotomy principles

Todorčević

WPID

For every P-ideal on an uncountable set S, either S contains an uncountable subset locally in $\mathscr{I}$, or an uncountable subset orthogonal to $\mathscr{I}$.

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PID

For every P-ideal on an uncountable set S, either S contains an uncountable subset locally in $\mathscr{I}$, or S can be decomposed into countably many pieces orthogonal to $\mathscr{I}$.

Restrictions

We write PID_κ for the restriction of PID for sets S of size at most κ, We write PID_κ for the restriction of PID for sets S of size at most κ,

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Similarly with WPID (here "W" stands for "weak"), but WPID_κ is obviously equivalent to WPID_{ω_1} for every cardinal κ.

A new premise

Theorem

$\boxtimes$ follows from $\mathfrak{p} > \omega_1$ and the WPID.

Strengths?

An iterated forcing adding $\aleph_2$ reals and doing the Abraham-Todorčević forcing from with a suitable book-keeping shows that PID restricted to ω_1-generated P-ideals together with $2^\omega = \aleph_2$ is consistent relative to ZFC.

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But the proof of our first Theorem uses the WPID restricted to P-ideals on ω_1 which are not necessarily ω_1-generated in the final model, and we do not know the strength of the condition.

Not stronger than ZFC

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Theorem

The conjunction of MA_{ω_1}, PID for ω_1-generated ideals on ω_1, and $\boxtimes$ is consistent relative to the consistency of ZFC.

As the following theorem shows, the previous Theorem adds more cases as compared with Todorčević's theorem.

Getting $\neg\mathcal{K}$

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Theorem

The following is consistent: PID(ω_1-generated) and $\mathfrak{p} > \omega_1$, $\boxtimes$, and there is a non-special Aronszajn tree. Therefore, PID(ω_1-generated) and $\mathfrak{p} > \omega_1$ does not imply $(\mathcal{K})$.
If we assume the existence of a supercompact cardinal then the same is true about the (full version) of PID.

Definition IX.4.5 of Shelah Proper and Improper Forcing

Usually S is costationary. We call a forcing notion P (T, S)-preserving if the following holds: T is an Aronszajn tree, $S \subseteq \omega_1$, and for every $\lambda > (2^{|P|+\aleph_1})^+$ and countable $N \prec H(\lambda, \in)$ such that $P, T, S \in N$ and $\delta = N \cap \omega_1 \notin S$, and every $p \in N \cap P$ there is some $q \geq p$ (bigger conditions are stronger) such that

(1) q is (N, P) generic; and

(2) for every $x \in T_\delta$, if $(x \in A \rightarrow (\exists y <_T x) y \in A)$ for all $A \in \mathscr{P}(T) \cap N$, then $q \Vdash (x \in \underset{\sim}{A} \rightarrow (\exists y < x) y \in \underset{\sim}{A})$ for every P-name $\underset{\sim}{A} \in N$ such that $\Vdash_P \underset{\sim}{A} \subset T$.

Lemma, Mi, Zdomskyy

Let T be an Aronszajn tree, S costationary, and $\mathscr{D}$ be a centred subfamily of $[\omega]^\omega$ of size ω_1. Then the forcing $\mathbb{P}_{\mathscr{D}}$ for increasing $\mathfrak{p}$ is (T, S)-preserving.

Preserving a non-special Aronszajn tree

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Theorem, Hirschorn

Let T be an Aronszajn tree and let S be costationary. The Abraham-Todorčević forcing is (T, S)-preserving.

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Theorem, Abraham

Let T be a Souslin tree and let S be costationary. No countable support iteration of (T, S)-preserving proper iterands specialises T on the levels in S.

The last slide

Thank you for your attention.