On Shoenfield’s Absoluteness Theorem

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Theorem, J. Shoenfield, 1961

Every $\Sigma^1_2(\alpha)$ relation and every $\Pi^1_2(\alpha)$ relation is absolute for inner models $M$ of the Zermelo–Fraenkel axioms and dependent choice that contain the real number $\alpha$ as an element.
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Remarks: 1961 was before the advent of the forcing era. People had not even two different models of ZFC, just Gödel’s constructible $L$ from 1938 was there, not known whether $V \neq L$ is relatively consistent.
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Steinhaus’ Axiom of Determinacy (AD, compatible with ZF + dependent choice) was already there, however very few consequences of it were known.

Later the theorem found many applications. $\Sigma^1_2(a)$ and $\Pi^1_2(a)$ is optimal.
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   Absoluteness for a class of models
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The complexity classes $\Sigma^1_2(a)$ and $\Pi^1_2(a)$, interspersed with steps of the proof
Definition
Let $T$ be a theory. A property $\varphi$ is **upwards absolute for models of $T$** if for any two models $M_1$ and $M_2$ of $T$ such that $M_1$ is a substructure of $M_2$ and such that $\varphi$ is true in $M_1$, the statement $\varphi$ holds also in $M_2$. 
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Example
If $\varphi$ has only existential quantifiers ranging over $M_i$ and no others then is upwards absolute.
Definition
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Let $T$ be a theory. A property $\varphi$ is **downwards absolute** for models of $T$ if for any two models $M_1$ and $M_2$ of $T$ such that $M_1$ is a substructure of $M_2$ and such that $\varphi$ is true in $M_2$, the statement $\varphi$ holds also in $M_1$. 
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Example
If $\varphi$ has only universal quantifiers ranging over $M_i$ and no others then is downwards absolute.
Absoluteness

Definition
Let $T$ be a theory. A property $\varphi$ is absolute for models of $T$ if the following holds: It holds in one model of $T$ iff it holds in any model of $T$. 
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Definition
Let $T$ be a weak part of set theory.

(1) A property $\varphi$ is absolute for inner models of $T$ if the following holds: It holds in one model $V$ of $T$ iff it holds in any inner model $M$ of $V$ that fulfils $T$.

(2) $M$ is an inner model of $V$ iff

(a) $M$ contains all the ordinals of $V$ and

(b) $V$ is an end extension of $M$ in the following sense: If $y \in M$ and $x \in V$ and $x \in y$ then $x \in M$. 
Absoluteness and provability

Gödel’s completeness theorem

Let $T$ be a theory with set-sized models and let $\varphi$ be a sentence of first order logic. Absoluteness of $\varphi$ for all set-sized models of $T$ coincides with provability of $\varphi$ or provability of $\neg \varphi$ from $T$ (in any calculus, e.g., the Hilbert calculus).
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In set theory we work with class sized models. Shoenfield’s theorem is about any two models with the same ordinals, as it relates any model \( M \) with \( a \in M \) to its inner model \( L[a] \), the smallest inner model that contains \( a \) as an element.
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In set theory we work with class sized models. Shoenfield’s theorem is about any two models with the same ordinals, as it relates any model $M$ with $a \in M$ to its inner model $L[a]$, the smallest inner model that contains $a$ as an element.

You will see from the proof sketch where the premise about “the same ordinals” is used.
Theories $T$ like the first order theory of algebraically closed fields of characteristic 0 (e.g.) are complete, that is for any sentence $\varphi$, $T$ proves $\varphi$ or $T$ proves $\neg\varphi$.

Complete theories have absoluteness for first order properties.

Examples are $T$ that allow elimination of quantifiers have Nullstellensäetze. Often then there only few completions.
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The axiom of dependent choice, DC
Let $R$ be a relation such that $\forall x \exists y R(x, y)$. Then there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $(\forall n \in \mathbb{N}) R(x_n, x_{n+1})$.

Recall: The Zermelo–Fraenkel axioms, ZF
(Existence), extensionality, infinity, pairing, union, powerset, well-foundedness of $\in$, replacement, comprehension.
Riddles: Which are absolute for inner models of ZF + DC?

the Goodstein series converge
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- $x$ is a vector space over the field $y$ and $x$ has a basis
- $x$ is an arithmetical set
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A set $X \subseteq \mathbb{N}$ is called arithmetical iff there are a first order formula $\varphi(x, \bar{p})$ in the language of arithmetic and a finite tuple $\bar{p} = (p_1, \ldots, p_k)$ of so-called parameters $p_i \in \mathbb{N}$ such that

$$X = \{ n \in \mathbb{N} \mid (\mathbb{N}, +, \cdot, 0, 1) \text{ fulfils } \varphi(n, \bar{p}) \}$$

Fact
Arithmetical sets are absolute for models of ZF.

Example
The twin prime set $\{ n \mid n \text{ and } n + 2 \text{ are prime} \}$ is arithmetical.
A subset of the arithmetical sets are the computable sets:

**Definition**

$X \subseteq \mathbb{N}$ is computable (or recursive or decidable) iff there is a Turing machine $M_X$ such that for all $n \in \mathbb{N}$, the question whether $n \in X$ is decided by a Turing machine $M_X$.

**Counterexample**

The halting problem. Let $(T_n)_{n \in \mathbb{N}}$ be a computable enumeration of all Turing machines. The halting problem

$$H = \{ n \mid \text{the Turing machine } T_n, \text{ run on the empty input, stops after finitely many steps}\}$$

is arithmetical and not computable.
Important variation: Arithmetical sets relative to a real parameter (oracle)

Definition
Let \( a \subseteq \mathbb{N} \). A set \( X \subseteq \mathbb{N} \) is called arithmetical in \( a \) iff there are a first order formula \( \varphi(x, \bar{p}) \) in the language of arithmetic expanded by an extra unary predicate for \( a \) and a finite tuple \( \bar{p} \) of natural numbers such that

\[
X = \{ n \in \mathbb{N} \mid (\mathbb{N}, +, \cdot, a, 0, 1) \text{ fulfils } \varphi(n, \bar{p}) \}
\]

Fact
Sets that are arithmetical in \( a \) are absolute for models of ZF that contain \( a \) as an element.
Oracle-computable sets

A subset of the arithmetical sets in $a$ are the sets that are computable relative to $a$.

**Definition**
We say “$X$ is computable relative to $a$” iff $n \in X$ is decided by a Turing machine $M_X$ that has an infinite auxiliary Turing tape on which $a$ is written. The real $a$ is also called an oracle.
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**Example**
Example of a non-arithmetical set: Let $a$ be the first order theory of $(\mathbb{N}, +, \cdot, 0, 1)$.

Is the halting problem computable relative to $a$?
The “ugly monster of independence” (Erdős) is lurking around the corner

The power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, depends on the respective model $M$ of ZF + DC.

Although $\mathbb{R} \cong \mathcal{P}(\mathbb{N}) \cong \mathbb{N}^\mathbb{N}$, the set of real numbers $\mathbb{R}$ is a very successful notion (as a complete, ordered field, with Archimedean order). In each model of ZF + DC, these axioms give up to isomorphism exactly one $\mathbb{R}$.

As $\mathcal{P}(\mathbb{N})$ is already variable, the set $\mathcal{P}(\mathbb{R})$ varies even more with the background model of ZF + DC.
**Definition**

By induction on $1 \leq \alpha < \aleph_1$ we simultaneously define $\Sigma^0_\alpha(\mathbb{R})$ and $\Pi^0_\alpha(\mathbb{R})$ as follows:

\[
\Sigma^0_1(\mathbb{R}) = \{ Y \subseteq \mathbb{R} | \ Y \text{ open} \}, \\
\Pi^0_\alpha(\mathbb{R}) = \{ \mathbb{R} \setminus Y | \ Y \in \Sigma^0_\alpha(\mathbb{R}) \}, \\
\Sigma^0_\alpha(\mathbb{R}) = \{ \bigcup_{n \in \mathbb{N}} B_n | \ (\forall n \in \mathbb{N})B_n \in \bigcup_{\beta < \alpha} \Pi^0_\beta(\mathbb{R}) \}.
\]

$\text{Borel}(\mathbb{R}) = \bigcup_{\alpha < \aleph_1} \Sigma^0_\alpha(\mathbb{R})$ is the Borel $\sigma$-algebra over $\mathbb{R}$. 
Remark

Lemma, Freiburg wisdom from the 1980’s

By the way, in ZFC, $|\text{Borel}(\mathbb{R})| = |\mathbb{R}|$. 
The lightface analytical sets, also known as $\Sigma^1_1$-sets

**Definition**

$X \subseteq \mathbb{R}$ is $\Sigma^1_1$ iff there is a computable closed set $Z \subseteq \mathbb{R} \times \mathbb{R}$ such that

$$X = \{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} (x, y) \in Z \} =: p(Z).$$

Equivalently, iff there is a computable set $R$ such that

$$X = \{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} \forall n \in \mathbb{N} (x \upharpoonright n, y \upharpoonright n) \in R \}. $$
We replace computability by computablility relative to $a$ for a real number $a \in \mathbb{R}$ and get $\Sigma^1_1(a)$.

**Definition**
The set of analytical sets is $\Sigma^1_1(\mathbb{R}) = \bigcup \{ \Sigma^1_1(a) \mid a \in \mathbb{R} \}$. 
We replace computability by computability relative to $a$ for a real number $a \in \mathbb{R}$ and get $\Sigma_1^1(a)$.

**Definition**

The set of **analytical sets** is $\Sigma_1^1(\mathbb{R}) = \bigcup \{ \Sigma_1^1(a) \mid a \in \mathbb{R} \}$.

**Definition**

The set of **coanalytical sets** is $\Pi_1^1(\mathbb{R}) = \{ \mathbb{R} \setminus X \mid X \in \Sigma_1^1(\mathbb{R}) \}$. 
 Allow real parameters

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Theorem, Lusin, 1927

In ZF + DC we have

\[ \text{Borel}(\mathbb{R}) = \Sigma^1_1(\mathbb{R}) \cap \Pi^1_1(\mathbb{R}). \]
Mostowski’s absoluteness theorem, 1959

$\Pi_1^1(a)$ and $\Sigma_1^1(a)$ relations are absolute for inner models of ZF + DC that contain $a$. 

Sketch of proof:

$\exists y \in R \forall n...$ is an existence, so upwards absolute.

Also its negation is an existence: There is no such $y$ then any search tree for such an $y$ does not have an infinite branch, so it (or rather its turn-over) is well-founded and has a rank function. "There is a rank function" is an existential clause.
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Rank functions on well-founded trees

Let $T$ be a tree whose nodes are finite sequences (of members of a fixed set, here: $\mathbb{N}$) and whose tree order $\leq_T$ is end extension. Also assume that $T$ has no infinite branch. (Then $T$, or rather the mirror $(T, \geq_T)$ is sometimes called “well-founded”.

**Definition**

We define by recursion on the well-founded relation $\geq_T$ a function \( \text{rk}: T \rightarrow \text{Ordinals} \) as follows:

\[
\begin{align*}
\text{rk}(t) &= 0 \text{ iff } t \text{ is a leaf of } T; \\
\text{rk}(s) &= \sup\{\text{rk}(t) + 1 \mid t \text{ is an immediate } \leq_T\text{-successor of } s\}.
\end{align*}
\]

Once \( \text{rk} \) is defined, we let \( \text{Rk}(T) = \sup\{\text{rk}(t) \mid t \in T\} \).
\( \Pi^1_2(a) \)-sets and \( \Sigma^1_2(a) \)-sets

**Definition**

Let \( a \in \mathbb{R} \). \( X \subseteq \mathbb{R} \) is \( \Pi^1_2(a) \) if there is a closed set \( Z \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) such that 
\[
X = \{ x \in \mathbb{R} \mid \forall y \in \mathbb{R} \exists z \in R(x, y, z) \in Z \}.
\]

(iff there is a computable set \( C \) such that 
\[
X = \{ x \in \mathbb{R} \mid \forall y \in \mathbb{R} \exists z \in \mathbb{R} \forall n \in \mathbb{N}(x \upharpoonright n, y \upharpoonright n, z \upharpoonright n, a \upharpoonright n) \in C \}.
\]

**Definition**

\( \Sigma^1_2(a) \) sets are complements in \( \mathbb{R} \) of \( \Pi^1_2(a) \) sets.
Lemma

If $A$ is $\Sigma_2^1(a)$ then $A = p([T])$ where $T$ is a tree on $\mathbb{N} \times \aleph_1$ and $T \in L[a]$ and $[T]$ is the set of branches of $T$. 
Lemma

If $A$ is $\Sigma^1_2(a)$ then $A = p([T])$ where $T$ is a tree on $\mathbb{N} \times \aleph_1$ and $T \in L[a]$ and $[T]$ is the set of branches of $T$.

Proof sketch: There is a tree $U \subseteq (\mathbb{N}^{<\mathbb{N}})^3$ constructible from $a$ such that

$$x \in A \iff \exists y \forall z \exists n (x \upharpoonright n, y \upharpoonright n, z \upharpoonright n) \notin U$$

that is

$$x \in A \iff \exists y U(x, y) := \{ s \mid s \in U(x \upharpoonright |s|, y \upharpoonright |s|, s) \} \text{ is well-founded}$$

$$x \in A \iff \exists y \exists f : U(x, y) \rightarrow \aleph_1 \text{ s.t., if } s \subseteq s' \text{ then } f(s') < f(s)$$

Well-foundedness is absolute for models of ZF + DC.
The theorem has corollaries that $\Sigma^1_3$ sentences are upward absolute (if such a sentence holds in $L$ then it holds in $V$) and $\Pi^1_3$ sentences are downward absolute (if they hold in $V$ then they hold in $L$).

Since any two transitive models of set theory with the same ordinals have the same constructible universe, Shoenfield’s theorem shows that two such models must agree about the truth of all $\Pi^1_3$ sentences.
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Shoenfield’s theorem shows that $\Pi^1_3$-statements are downwards absolute for inner models. If there is a model of ZF in which a given $\Sigma^1_3$ statement $\varphi$ is false, then $\varphi$ is also false in the constructible universe of that model.
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In contrapositive, this means:
If ZFC + $V = L$ proves a $\Sigma^1_3$ sentence then that sentence is also provable in ZF.
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In contrapositive, this means:
If $\text{ZFC} + V = L$ proves a $\Sigma^1_3$ sentence then that sentence is also provable in ZF.

In particular:
If $\text{ZFC} + 2^{\aleph_0} = \aleph_1$ proves a $\Sigma^1_3$ sentence then that sentence is also provable in ZF.
Independence is not a panacea

It is not possible to use forcing to change the truth value of arithmetical sentences, as forcing does not change the ordinals of the model to which it is applied.

Many famous open problems can be expressed as $\Pi^1_2$ sentences or $\Sigma^1_2$ sentences (or sentences of lower complexity), and thus cannot be proven independent of ZFC by forcing.
Answers: Which are absolute for inner models of ZF + DC?

Goodstein series converge: a (not provable in Peano arithmetic)
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Goodstein series converge: not provable in Peano arithmetic

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\( x \) is an arithmetical set: a
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