On Shoenfield's Absoluteness Theorem

Heike Mildenberger

Universität Freiburg, Mathematisches Institut, Abteilung für Mathematische Logik

http://home.mathematik.uni-freiburg.de/mildenberger

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Steinhaus' Axiom of Determinacy (AD, compatible with ZF + dependent choice) was already there, however very few consequences of it were known.
Later the theorem found many applications. $\Sigma^1_2(a)$ and $\Pi^1_2(a)$ is optimal.

Overview over the talk

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   interspersed with steps of the proof

Upwards absoluteness

Definition

Let T be a theory. A property φ is upwards absolute for models of T if for any two models M_1 and M_2 of T such that M_1 is a substructure of M_2 and such that φ is true in M_1, the statement φ holds also in M_2.

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Example

If φ has only existential quantifiers ranging over M_i and no others then is upwards absolute.

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Example

If φ has only universal quantifiers ranging over M_i and no others then is downwards absolute.

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Definition

Let T be a weak part of set theory.

(1) A property φ is absolute for inner models of T if the following holds: It holds in one model V of T iff it holds in any inner model M of V that fulfils T.

(2) M is an inner model of V iff

  (a) M contains all the ordinals of V and

  (b) V is an end extension of M in the following sense: If $y \in M$ and $x \in V$ and $x \in y$ then $x \in M$.

Absoluteness and provability

Gödel's completeness theorem

Let T be a theory with set-sized models and let φ be a sentence of first order logic. Absoluteness of φ for all set-sized models of T coincides with provability of φ or provability of $\neg\varphi$ from T (in any calculus, e.g., the Hilbert calculus).

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In set theory we work with class sized models. Shoenfield's theorem is about any two models with the same ordinals, as it relates any model M with $a \in M$ to its inner model $L[a]$, the smallest inner model that contains a as an element.

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You will see from the proof sketch where the premise about "the same ordinals" is used.

From algebra and model theory

Theories T like the first order theory of algebraically closed fields of characteristic 0 (e.g.) are complete, that is for any sentence φ, T proves φ or T proves $\neg\varphi$.
Complete theories have absoluteness for first order properties.

Examples are T that allow elimination of quantifiers have Nullstellensaetze. Often then there only few completions.

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The theory from Shoenfield's theorem: Zermelo Fraenkel set theory and the axiom of dependent choice

The axiom of dependent choice, DC

Let R be a relation such that $\forall x \exists y R(x,y)$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(\forall n \in \mathbb{N}) R(x_n, x_{n+1})$.

Recall: The Zermelo–Fraenkel axioms, ZF

(Existence), extensionality, infinity, pairing, union, powerset, well-foundedness of $\in$, replacement, comprehension.

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Overview over the talk, third and last time

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Arithmetical sets

Definition

A set $X \subseteq \mathbb{N}$ is called arithmetical iff there are a first order formula $\varphi(x, \bar{p})$ in the language of arithmetic and a finite tuple $\bar{p} = (p_1, \ldots, p_k)$ of socalled parameters $p_i \in \mathbb{N}$ such that

$$X = \{n \in \mathbb{N} \mid (\mathbb{N}, +, \cdot, 0, 1) \text{ fulfils } \varphi(n, \bar{p})\}$$

Fact

Arithmetical sets are absolute for models of ZF.

Example

The twin prime set $\{n \mid n \text{ and } n+2 \text{ are prime}\}$ is arithmetical.

Computable sets, also called recursive sets

A subset of the arithmetical sets are the computable sets:

Definition

$X \subseteq \mathbb{N}$ is computable (or recursive or decidable) iff there is a Turing machine M_X such that for all $n \in \mathbb{N}$, the question whether $n \in X$ is decided by a Turing machine M_X.

Counterexample

The halting problem. Let $(T_n)_{n \in \mathbb{N}}$ be a computable enumeration of all Turing machines. The halting problem

$$H = \{n \mid \text{ the Turingmachine } T_n\text{, run on the empty input, stops after finitely many steps}\}$$

is arithmetical and not computable.

Important variation: Arithmetical sets relative to a real parameter (oracle)

Definition

Let $a \subseteq \mathbb{N}$. A set $X \subseteq \mathbb{N}$ is called arithmetical in a iff there are a first order formula $\varphi(x, \bar{p})$ in the language of arithmetic expanded by an extra unary predicate for a and a finite tuple $\bar{p}$ of natural numbers such that

$$X = \{n \in \mathbb{N} \mid (\mathbb{N}, +, \cdot, a, 0, 1) \text{ fulfils } \varphi(n, \bar{p})\}$$

Fact

Sets that are arithmetical in a are absolute for models of ZF *that contain a as an element.*

Oracle-computable sets

A subset of the arithmetical sets in a are the sets that are computable relative to a.

Definition

We say "X is computable relative to a" iff $n \in X$ is decided by a Turing machine M_X that has an infinite auxiliary Turing tape on which a is written. The real a is also called an oracle.

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Example

Example of a non-arithmetical set: Let a be the first order theory of $(\mathbb{N}, +, \cdot, 0, 1)$.

Is the halting problem computable relative to a?

The "ugly monster of independence" (Erdős) is lurking around the corner

The power set of $\mathbb{N}$, $\mathscr{P}(\mathbb{N})$, depends on the respective model M of ZF + DC.

Although $\mathbb{R} \cong \mathscr{P}(\mathbb{N}) \cong \mathbb{N}^{\mathbb{N}}$, the set of real numbers $\mathbb{R}$ is a very successful notion (as a complete, ordered field, with Archimedean order). In each model of ZF + DC, these axioms give up to isomorphism exactly one $\mathbb{R}$.

As $\mathscr{P}(\mathbb{N})$ is already variable, the set $\mathscr{P}(\mathbb{R})$ varies even more with the background model of ZF + DC.

Definition

By induction on $1 \leq \alpha < \aleph_1$ we simultaneously define $\mathbf{\Sigma}^0_\alpha(\mathbb{R})$ and $\mathbf{\Pi}^0_\alpha(\mathbb{R})$ as follows:

$$\begin{aligned}
\mathbf{\Sigma}^0_1(\mathbb{R}) &= \{Y \subseteq \mathbb{R} \mid Y \text{ open}\},\\
\mathbf{\Pi}^0_\alpha(\mathbb{R}) &= \{\mathbb{R} \setminus Y \mid Y \in \mathbf{\Sigma}^0_\alpha(\mathbb{R})\},\\
\mathbf{\Sigma}^0_\alpha(\mathbb{R}) &= \{\bigcup_{n\in\mathbb{N}} B_n \mid (\forall n \in \mathbb{N}) B_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(\mathbb{R})\}.
\end{aligned}$$

$\mathrm{Borel}(\mathbb{R}) = \bigcup_{\alpha<\aleph_1} \mathbf{\Sigma}^0_\alpha(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$.

Remark

Lemma, Freiburg wisdom from the 1980's

By the way, in ZFC, $|\mathrm{Borel}(\mathbb{R})| = |\mathbb{R}|$.

The lightface analytical sets, also known as Σ^1_1-sets

Definition

$X \subseteq \mathbb{R}$ is Σ^1_1 iff there is a computable closed set $Z \subseteq \mathbb{R} \times \mathbb{R}$ such that

$$X = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}(x, y) \in Z\} =: p(Z).$$

Equivalently, iff there is a computable set R such that

$$X = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} \forall n \in \mathbb{N}(x \upharpoonright n, y \upharpoonright n) \in R\}.$$

Allow real parameters

We replace computability by computablily relative to a for a real number $a \in \mathbb{R}$ and get $\Sigma^1_1(a)$.

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The set of analytical sets is $\mathbf{\Sigma}^1_1(\mathbb{R}) = \bigcup\{\Sigma^1_1(a) \mid a \in \mathbb{R}\}$.

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The Lusin separation theorem

Theorem, Lusin, 1927

In ZF + DC *we have*

$$\mathrm{Borel}(\mathbb{R}) = \mathbf{\Sigma}^1_1(\mathbb{R}) \cap \mathbf{\Pi}^1_1(\mathbb{R}).$$

A weaker absoluteness theorem

Mostowski's absoluteness theorem, 1959

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Sketch of proof:
$\exists y \in \mathbb{R} \forall n \ldots$ is an existence, so upwards absolute.

Also its negation is an existence:
There is no such y then any search tree for such an y does not have an infinite branch, so it (or rather its turn-over) is well-founded and has a rank function. "There is a rank function" is an existential clause.

Rank functions on well-founded trees

Let T be a tree whose nodes are finite sequences (of members of a fixed set, here: $\mathbb{N}$) and whose tree order $\leq_T$ is end extension. Also assume that T has no infinite branch. (Then T, or rather the mirror $(T, \geq_T)$ is sometimes called “well-founded”.)

Definition

We define by recursion on the well-founded relation $\geq_T$ a function $\mathrm{rk}\colon T \to \mathrm{Ordinals}$ as follows:

$$\begin{array}{rcl} \mathrm{rk}(t) & = & 0 \text{ iff } t \text{ is a leaf of } T; \\ \mathrm{rk}(s) & = & \sup\{\mathrm{rk}(t) + 1 \mid t \text{ is an immediate } \leq_T\text{-successor of } s\}. \end{array}$$

Once rk is defined, we let $\mathrm{Rk}(T) = \sup\{\mathrm{rk}(t) \mid t \in T\}$.

$\Pi^1_2(a)$-sets and $\Sigma^1_2(a)$-sets

Definition

Let $a \in \mathbb{R}$. $X \subseteq \mathbb{R}$ is $\Pi^1_2(a)$ if there is a closed set $Z \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ such that $X = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R} \exists z \in R(x, y, z) \in Z\}$.

(iff there is a computable set C such that $X = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R} \exists z \in \mathbb{R} \forall n \in \mathbb{N} (x \restriction n, y \restriction n, z \restriction n, a \restriction n) \in C\}$.)

Definition

$\Sigma^1_2(a)$ sets are complements in $\mathbb{R}$ of $\Pi^1_2(a)$ sets.

Carrying rank functions further: Shoenfield trees

Lemma

If A is $\Sigma^1_2(a)$ then $A = p([T])$ where T is a tree on $\mathbb{N} \times \aleph_1$ and $T \in L[a]$ and $[T]$ is the set of branches of T.

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Proof sketch: There is a tree $U \subseteq (\mathbb{N}^{<\mathbb{N}})^3$ constructible from a such that

$$x \in A \leftrightarrow \exists y \forall z \exists n (x \upharpoonright n, y \upharpoonright n, z \upharpoonright n) \notin U$$

that is

$$x \in A \leftrightarrow \exists y U(x,y) := \{s \mid s \in U(x \upharpoonright |s|, y \upharpoonright |s|, s)\} \text{ is well-founded}$$

$$x \in A \leftrightarrow \exists y \exists f : U(x,y) \to \aleph_1 \text{ s.t., if } s \subset s' \text{ then } f(s') < f(s)$$

Well-foundedness is absolute for models of ZF + DC.

Consequences

The theorem has corollaries that Σ^1_3 sentences are upward absolute (if such a sentence holds in L then it holds in V) and Π^1_3 sentences are downward absolute (if they hold in V then they hold in L).

Since any two transitive models of set theory with the same ordinals have the same constructible universe, Shoenfield's theorem shows that two such models must agree about the truth of all Π^1_3 sentences.

The rôle of the axiom of choice, and assumptions beyond ZF + DC

Gödel proved that every model V of ZF has an inner model, namely L, that satisfies ZFC.

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Shoenfield's theorem shows that Π^1_3-statements are downwards absolute for inner models. If there is a model of ZF in which a given Σ^1_3 statement φ is false, then φ is also false in the constructible universe of that model.

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In particular:
If ZFC + $2^{\aleph_0} = \aleph_1$ proves a Σ^1_3 sentence then that sentence is also provable in ZF.

Independence is not a panacea

It is not possible to use forcing to change the truth value of arithmetical sentences, as forcing does not change the ordinals of the model to which it is applied.

Many famous open problems can be expressed as Π^1_2 sentences or Σ^1_2 sentences (or sentences of lower complexity), and thus cannot be proven independent of ZFC by forcing.

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References

Thomas Jech, Set Theory, The Millenium Edition. Springer 2003.
Here, I used in particular Chapter 25.