More models of the club principle

Heike Mildenberger

Kurt Gödel Research Center for Mathematical Logic, Universität Wien
http://www.logic.univie.ac.at/~heike

Workshop on Combinatorial and Descriptive Set Theory
Research Institute for the Mathematical Sciences, Kyoto
August 25–29, 2008

Survey

Ostaszewski's club

The main result

Using Axiom A and more properties

Definition

The club principle, $\clubsuit$, is the following statement: There is some $\langle A_\alpha \mid \alpha < \omega_1, \alpha \text{ limit}\rangle$ such that for every α, A_α is cofinal in α and for every uncountable $X \subseteq \omega_1$ there is an α such that $A_\alpha \subseteq X$.

The club principle

Definition

The club principle, $\clubsuit$, is the following statement: There is some $\langle A_\alpha \mid \alpha < \omega_1, \alpha \text{ limit}\rangle$ such that for every α, A_α is cofinal in α and for every uncountable $X \subseteq \omega_1$ there is an α such that $A_\alpha \subseteq X$.

It is equivalent to say "there are stationarily many α" instead of "there is an α".

The club principle and CH

Devlin, 1977

$\clubsuit + \mathrm{CH} \Leftrightarrow \diamondsuit$.

The club principle and CH

Devlin, 1977

$\clubsuit + \mathrm{CH} \Leftrightarrow \diamondsuit$.

Shelah, Baumgartner, 1970s, two models

$\clubsuit + \neg\mathrm{CH}$ is consistent relative to ZFC.

Truss, 1973

The club principle (indeed, already the stick) implies that $\mathrm{cov}(\mathcal{M}) = \aleph_1$ or $\mathrm{cov}(\mathcal{N}) = \aleph_1$.

$$
\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[dd] & \mathrm{non}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{N}) \arrow[l] \arrow[dd] \\
 & \mathfrak{b} \arrow[d] & \mathfrak{d} \arrow[l] \arrow[d] & \\
\boxed{\mathrm{add}(\mathcal{N})} & \mathrm{add}(\mathcal{M}) \arrow[l] & \mathrm{cov}(\mathcal{M}) \arrow[l] & \mathrm{non}(\mathcal{N}) \arrow[l]
\end{tikzcd}
$$

Figure: The club principle implies that the framed entry is $\aleph_1$.

$\mathrm{add}(\mathcal{N})$ is indeed the only entry of this kind

Fuchino, Shelah, and Soukup, 1997

The club principle and $\mathrm{cov}(\mathcal{M}) = \kappa = 2^\omega$ for a regular $\kappa \geq \aleph_2$ is consistent relative to ZFC.

$\mathrm{add}(\mathcal{N})$ is indeed the only entry of this kind

Fuchino, Shelah, and Soukup, 1997

The club principle and $\mathrm{cov}(\mathcal{M}) = \kappa = 2^\omega$ for a regular $\kappa \geq \aleph_2$ is consistent relative to ZFC.

Džamonja and Shelah, 1999

The club principle and $\mathrm{add}(\mathcal{M}) = \aleph_2 = 2^\omega$ is consistent relative to ZFC.

$\mathrm{add}(\mathcal{N})$ is indeed the only entry of this kind

Fuchino, Shelah, and Soukup, 1997

The club principle and $\mathrm{cov}(\mathcal{M}) = \kappa = 2^\omega$ for a regular $\kappa \geq \aleph_2$ is consistent relative to ZFC.

Džamonja and Shelah, 1999

The club principle and $\mathrm{add}(\mathcal{M}) = \aleph_2 = 2^\omega$ is consistent relative to ZFC.

Brendle, 2006

The club principle and $\mathrm{cov}(\mathcal{N}) = \kappa$ for a regular $\kappa \geq \aleph_2$ is consistent relative to ZFC.

Two cardinals

$\mathfrak{s}$ is the splitting number.

$\mathfrak{h}$ is the distributivity number.

Two cardinals

$\mathfrak{s}$ is the splitting number.

$\mathfrak{h}$ is the distributivity number.

Question, Brendle, Hrušák

Is there a model of the club principle in which $\mathfrak{s}$ or even $\mathfrak{h}$ is larger than $\aleph_1$?

Two cardinals

$\mathfrak{s}$ is the splitting number.

$\mathfrak{h}$ is the distributivity number.

Question, Brendle, Hrušák

Is there a model of the club principle in which $\mathfrak{s}$ or even $\mathfrak{h}$ is larger than $\aleph_1$?

In ZFC, $\mathfrak{h} \leq \mathfrak{s}$.

Theorem

Any countable support iteration of Axiom A iterands of tree form or of a linear form with the finiteness property for $\leq_n$ over a ground model of Jensen's diamond yields a model of the club principle. In particular, the club principle holds in the Laver model, the Miller model, the Blass-Shelah model, the Mathias model, the Matet model.

The main result

Theorem

Any countable support iteration of Axiom A iterands of tree form or of a linear form with the finiteness property for $\leq_n$ over a ground model of Jensen's diamond yields a model of the club principle. In particular, the club principle holds in the Laver model, the Miller model, the Blass-Shelah model, the Mathias model, the Matet model.

This was formerly known for the Sacks model.

Corollaries

Theorem
The club principle together with $\mathfrak{h} = \aleph_2$ is consistent relative to ZFC*.*

Proof: Mathias forcing increases $\mathfrak{h}$.

Corollaries

Theorem

The club principle together with $\mathfrak{h} = \aleph_2$ *is consistent relative to* ZFC.

Proof: Mathias forcing increases $\mathfrak{h}$.

Theorem

The club principle together with $\mathfrak{u} < \mathfrak{g}$ *is consistent relative to* ZFC.

Proof: Matet forcing and Blass-Shelah forcing give $\mathfrak{u} < \mathfrak{g}$.

Axiom A

Our work builds on Baumgartner and Laver's "Perfect Set Forcing" and on Hrušák's "Life in the Sacks Model", which builds on work by Steprāns.

Axiom A

Our work builds on Baumgartner and Laver's "Perfect Set Forcing" and on Hrušák's "Life in the Sacks Model", which builds on work by Steprāns.

Definition

A notion of forcing $(\mathbb{P}, \leq_{\mathbb{P}})$ satisfies Axiom A if there are relations $\leq_n$, $n \in \omega$, with the following properties:

(1) $p \leq_0 q$ implies $p \leq_{\mathbb{P}} q$,

(2) $p \leq_{n+1} q$ implies $p \leq_n q$,

(3) if $p_n \leq_n p_{n+1}$ for $n \in \omega$, then there is some $q \in \mathbb{P}$ such that for all n, $p_n \leq_n q$,

(4) for every $p \in \mathbb{P}$ and every n and every antichain A in $\mathbb{P}$ there is some $q \geq_n p$ such that $\{r \in A \mid r \not\leq q\}$ is countable.

$< \omega_1$-properness is not enough

Example, joint work with Shelah

There is a $< \omega_1$-proper forcing specialising all Aronszajn trees and not adding reals and hence destroying the club principle.

Example, joint work with Shelah

There is a $< \omega_1$-proper forcing specialising all Aronszajn trees and not adding reals and hence destroying the club principle.

However this forcing does not have the finiteness property.

Definition

We give a version for tree forcings: A notion of forcing $(\mathbb{P}, \leq_{\mathbb{P}})$ whose elements p are subsets of $2^{<\omega}$ or in $\omega^{<\omega}$ has the finiteness property for $\leq_n$ iff there is a function $f\colon \mathbb{P} \times \omega \to \omega$ such that for every n, p, q:

$$
\begin{aligned}
p \leq_n q \ \text{ if } \ & p \leq q \text{ and} \\
& q \cap f(p,n)^{f(p,n)} = p \cap f(p,n)^{f(p,n)}.
\end{aligned}
$$

In the case of $2^{<\omega}$ we can write $2^{f(p,n)}$ instead of $f(p,n)^{f(p,n)}$.

Definition

We give a version for forcings $\mathbb{P}$ whose conditions are of the form $p = (a, c_0, c_1, \dots)$, with $c_i \subseteq [n_i, n_{i+1})$ for some strictly increasing sequence n_i, $i < \omega$. Then $(\mathbb{P}, \leq_{\mathbb{P}})$ has the finiteness property for $\leq_n$ iff there is a function $f\colon \mathbb{P} \times \omega \to \omega$ such that for every n, p, q:

$$
\begin{aligned}
p = (s, \bar{c}) \leq_n q = (t, \bar{d}) \text{ if} \\
& p \leq_{\mathbb{P}} q \text{ and} \\
& (s \cap f(p,n), c_0 \cap f(p,n), c_1 \cap f(p,n), \dots) = \\
& (t \cap f(p,n), d_0 \cap f(p,n), d_1 \cap f(p,n), \dots).
\end{aligned}
$$

Definition

If $p, q \in \mathbb{P}_\beta$, $F \subseteq \beta$, F finite, $\vec{n} \in {}^F\omega$, we write $q \geq_{(F,\vec{n})} p$ iff $F = \{\beta_0, \ldots, \beta_r\}$, $\vec{n} = (n(\beta_0), \ldots, n(\beta_r))$ and $(\forall i \leq r)((q \restriction \beta_i) \Vdash q(\beta_i) \geq_{n(\beta_i)} q(\beta_i))$.

Definition

Let $p \in \mathbb{P}_\alpha$, $F = \{\beta_0, \ldots, \beta_r\} \in [\mathrm{supp}(p)]^{<\omega}$ and $\vec{\sigma} = (\sigma(\beta_0), \ldots, \sigma(\beta_r))$, $\sigma(\beta_i) \in k(\beta_i)^{k(\beta_i)}$ or $\sigma(\beta_i) \subseteq k(\beta_i)$ (in the creature case), $\vec{k} = (k(\beta_0), \ldots, k(\beta_r)) \in {}^F\omega$. By induction on $\beta \in F$ we define when $(p \restriction \vec{\sigma}) \restriction \beta$ is consistent with p and then we define $(p \restriction \vec{\sigma})(\beta)$. The first step in the induction is split into two cases:

Continuation of the definition

1. For tree forcings. We write for $s \in {}^k k$, $p_s = \{t \in p \mid t \triangleleft \vee t \trianglerighteq s\}$ in the tree case. If p_s is defined, i.e., if $s \in p$, we say that s is consistent with p.
2. For linear forcings. For $p = (a, c_0, c_1, \dots)$, $s \subseteq k$, we let $p_s = (a, c_0 \cap s, c_1 \cap s, \dots, c_{m-1} \cap s, c_m, c_{m+1} \dots)$ for $p = (a, \bar{c})$ and $a = s \cap (\max a + 1)$ and $s \subseteq a \cup c_0 \cup \dots \cup c_{m-1}$ for some m (or, in general, when s is in the set of possibilities given by $(a, c_0, \dots c_{m-1})$ for a more general creature forcing), otherwise p_s is undefined. If p_s is defined we say that s is consistent with p.

Continuation of the definition, 2

Now we continue the induction: Suppose that $(p \restriction \vec{\sigma}) \restriction \beta$ is defined. If $(p \restriction \vec{\sigma}) \restriction \beta \Vdash$ "$\sigma(\beta)$ is consistent with $p(\beta)$", then we say that $\vec{\sigma} \restriction (\beta + 1)$ is consistent with p and we have the condition $(p \restriction \vec{\sigma}) \restriction \beta + 1$ defined by

$$(p \restriction \vec{\sigma})(\beta) = \begin{cases} p(\beta) & \text{if } \beta \notin F; \\ p(\beta)_{\sigma(\beta)} & \text{if } \beta \in F. \end{cases}$$

Definition

Let F be a finite subset of α. A condition $p \in \mathbb{P}_\alpha$ is said to be $(F, \vec{k})$-determined if $\forall \sigma(\beta_i) \in {}^{k(\beta_i)}k(\beta_i)$, $i \leq r$, either $\vec{\sigma} = (\sigma(\beta_0), \ldots, \sigma(\beta_r))$ is consistent with p or $\exists \beta \in F$ so that $\vec{\sigma} \upharpoonright (F \cap \beta)$ is consistent with p and $(p \upharpoonright \vec{\sigma}) \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} \sigma(\beta)$ is not consistent with $p(\beta)$.

Pinning down finite parts of names of conditions

Lemma

(Baumgartner and Laver) Let $p \in \mathbb{P}_\alpha$*,* $F \in [\alpha]^{<\omega}$ *and* $\vec{n} \in {}^F\omega$*. There is* $q \geq_{(F,\vec{n})} p$ *and there is* $\vec{k}(p, F, \vec{n}) = \vec{k}$ *such that* q *is* $(F, \vec{k})$*-determined and* $F = \{\beta_0, \dots \beta_r\}$*, and for all* $i \leq r$*,* $k(\beta_i) \geq f(p \restriction \beta_i, F, \vec{n})$*.*

Definition

Given an $\mathbb{P}_\alpha$-name $\underset{\sim}{X}$ for an uncountable subset of ω_1 a condition $p \in \mathbb{P}_\alpha$ and $F \in [\alpha]^{<\omega}$ and $\vec{m} \in {}^F\omega$ we let

$$A_{F,\vec{m}}(p, \underset{\sim}{X}) = \{\gamma \in \omega_1 \mid (\exists q \in \mathbb{P}_\alpha)(q \geq_{F,\vec{m}} p \wedge q \Vdash \gamma \in \underset{\sim}{X})\}.$$

Definition

Given an $\mathbb{P}_\alpha$-name $\underset{\sim}{X}$ for an uncountable subset of ω_1 a condition $p \in \mathbb{P}_\alpha$ and $F \in [\alpha]^{<\omega}$ and $\vec{m} \in {}^F\omega$ we let

$$A_{F,\vec{m}}(p, \underset{\sim}{X}) = \{\gamma \in \omega_1 \mid (\exists q \in \mathbb{P}_\alpha)(q \geq_{F,\vec{m}} p \wedge q \Vdash \gamma \in \underset{\sim}{X})\}.$$

Definition

A condition $p \in \mathbb{P}_\alpha$ is said to be $(\underset{\sim}{X}, F, \vec{m})$-good if p is $(F, \vec{k}(F, p, \vec{m}))$-determined and $\forall q \geq_{F,\vec{m}} p$, $|A_{F,\vec{m}}(q, \underset{\sim}{X})| = \aleph_1$.

Stepping up

Lemma

If $p = (s, C) \in \mathbb{M}$, $\underset{\sim}{X}$ *a* $\mathbb{M}$*-name for an uncountable subset of* ω_1 *and let* $m \in \omega$. *If* p *is* $(\underset{\sim}{X}, m)$*-good then there is* $q \in \mathbb{M}$, *such that* $q \geq_m p$ *and* q *is* $(\underset{\sim}{X}, m+1)$*-good.*

Skip proof

Proof

Suppose that the lemma fails. We construct a sequence $\langle p_n \mid n \in \omega \rangle$ with the following properties:

(1) $p_n \in \mathbb{M}$, $p_0 = p = (s, C)$,

(2) $p_{n+1} \geq_m p_n$,

(3) $p_n = (s, C_n)$ and the $(m+1)$-st element of C_n is k_n, the $(m+2)$-nd element of C_n is ℓ_n,

(4) $|A_{m+1}(p_n, \underset{\sim}{X})| < \aleph_1$.

To do this, suppose that p_n has been chosen.

Proof, part 2

At step $n+1$ we find
$p_{n+1} \geq_{m+1} p'_n = (s, \{c_i \mid i < m\} \cup \{\ell_n\} \cup rest)$ such that
$|A_{m+1}(p_{n+1}, \underset{\sim}{X})| < \aleph_1$. Here c_i is the i-th element of C. If this were not possible then the lemma holds with p'_n as desired condition.

Now we have the countable set

$$A = \bigcup\{A_{m+1}(p_{n+1}, \underset{\sim}{X}) \mid n \in \omega\}.$$

Proof, part 3

We take $p_\omega = (s, \{c_i \mid i < m\} \cup \{\ell_n \mid n \in \omega\})$. Since p is $(\underset{\sim}{X}, m)$-good, and $p_\omega \geq_m p$, the set $A_m(p_\omega, \underset{\sim}{X})$ is uncountable. We choose $\gamma \in A_m(p_\omega, \underset{\sim}{X}) \setminus A$. Then there is a $p' \geq_m p_\omega$ such that $p' \Vdash \gamma \in \underset{\sim}{X}$. Let $p' = (s, C')$. The $(m+1)$-st element of C' is ℓ_n for some n. So we have $p' \geq_{m+1} p_{n+1}$. Then, however, $\gamma \in A_{m+1}(p_{n+1}, \underset{\sim}{X})$, which is impossible.

Similarly we can do for one Matet iterand.

Stepping up for the iterated forcing

Definition

Let $F = \{\beta_0 < \cdots < \beta_r\}$ and let for example $G = \{\beta_0 < \alpha_0 < \beta_1 < \cdots < \beta_r\}$, in increasing order. Let $\vec{k} \in {}^F\omega$ and let $\vec{\ell} \in {}^G\omega$. Then we write $\vec{k} \leq_{F,G} \vec{\ell}$ iff $k(\beta_i) \leq \ell(\beta_i)$ for all $i \leq r$ and we write $\vec{k} <_{F,G} \vec{\ell}$ iff one inequality is strict or if $G \supsetneq F$.

Definition

Let $F = \{\beta_0 < \cdots < \beta_r\}$ and let for example $G = \{\beta_0 < \alpha_0 < \beta_1 < \cdots < \beta_r\}$, in increasing order. Let $\vec{k} \in {}^F\omega$ and let $\vec{\ell} \in {}^G\omega$. Then we write $\vec{k} \leq_{F,G} \vec{\ell}$ iff $k(\beta_i) \leq \ell(\beta_i)$ for all $i \leq r$ and we write $\vec{k} <_{F,G} \vec{\ell}$ iff one inequality is strict or if $G \supsetneq F$.

Lemma

Let $p \in \mathbb{P}_\alpha$, let F be finite subset of α, $\underset{\sim}{X}$ be a $\mathbb{P}_\alpha$-name for an uncountable subset of ω_1, and let $\vec{m} \in {}^F\omega$ and $G \supseteq F$, $\vec{n} \in {}^G\omega$, $\vec{m} <_{F,G} \vec{n}$. If p is $(\underset{\sim}{X}, F, \vec{m})$-good then there is $q \in \mathbb{P}_\alpha$ such that $q >_{F,\vec{m}} p$ and q is $(\underset{\sim}{X}, G, \vec{n})$-good.

Translating $\mathbb{P}_{\omega_2}$-names for subsets of ω_1 into subsets of $H(\omega_1)$

Let $\mathbb{P}_\alpha$ be a countable support iteration of proper iterands of size at most $\aleph_1$. So we assume CH in the ground model.

Lemma

Let N be a countable elementary submodel of $H(\chi)$ that contains $\mathbb{P}_\alpha$. Then for every $\mathbb{P}_\alpha$-name $x \in N$ for a real there is a name y, that is hereditarily countable relative to every $(N, \mathbb{P}_\alpha)$-generic p and every $(N, \mathbb{P}_\alpha)$-generic p forces $x = y$.

Translating $\mathbb{P}_{\omega_2}$-names for subsets of ω_1 into subsets of $H(\omega_1)$

Let $\mathbb{P}_\alpha$ be a countable support iteration of proper iterands of size at most $\aleph_1$. So we assume CH in the ground model.

Lemma

Let N be a countable elementary submodel of $H(\chi)$ that contains $\mathbb{P}_\alpha$. Then for every $\mathbb{P}_\alpha$-name $x \in N$ for a real there is a name y, that is hereditarily countable relative to every $(N, \mathbb{P}_\alpha)$-generic p and every $(N, \mathbb{P}_\alpha)$-generic p forces $x = y$.

Corollary

If $x \in N$ is a $\mathbb{P}_\alpha$-name for a real then there is some hereditarily countable y such that the set of conditions that force $x = y$ is dense above conditions in N, i.e., $(\forall p \in N)(\exists q \geq p)(q \Vdash x = y)$.

The original diamond in the ground model

Lemma

Let $\mathbb{P}_{\omega_2}$ *be an iteration of Axiom A forcings that has the stepping up property for goodness from the previous technical lemma. Assume* $\diamondsuit$ *in the ground model. There is a sequence* $\langle C_\delta \mid \delta \in \lim(\omega_1)\rangle$ *such that for every* $p \in \mathbb{P}_{\omega_2}$ *and every* $\mathbb{P}_{\omega_2}$*-name* $\underset{\sim}{X}$ *for an uncountable subset of* ω_1 *there are* $q \geq p$ *and* $\delta \in \lim(\omega_1)$ *such that* $q \Vdash C_\delta \subseteq \underset{\sim}{X}$*.*

The original diamond in the ground model

Lemma

Let $\mathbb{P}_{\omega_2}$ be an iteration of Axiom A forcings that has the stepping up property for goodness from the previous technical lemma. Assume $\diamondsuit$ in the ground model. There is a sequence $\langle C_\delta \mid \delta \in \lim(\omega_1)\rangle$ such that for every $p \in \mathbb{P}_{\omega_2}$ and every $\mathbb{P}_{\omega_2}$-name $\underset{\sim}{X}$ for an uncountable subset of ω_1 there are $q \geq p$ and $\delta \in \lim(\omega_1)$ such that $q \Vdash C_\delta \subseteq \underset{\sim}{X}$.

Remark: The club sequence $\langle C_\delta \mid \delta \in \lim(\omega_1)\rangle$ is in the ground model. By the reflection properties, it appears in $\mathbb{P}_\alpha$ for some $\alpha < \omega_2$.

Claim

(see Claim IV.4 in Hrušák's Life in the Sacks Model) Under $\diamondsuit$, there is a sequence $\langle p_\delta, A_\delta, M_\delta \mid \delta \in \lim(\omega_1) \rangle$ such that if $p \in \mathbb{P}_{\omega_2}$, $\underset{\sim}{X}$ a $\mathbb{P}_{\omega_2}$-name for an uncountable subset of ω_1 such that $\underset{\sim}{X} \subseteq H(\omega_1)$ and $C \subseteq [H(\omega_2)]^{\aleph_0}$ is a closed and unbounded set of countable elementary submodels then there is an $M \in C$ and an $\delta < \omega_1$ such that $\underset{\sim}{X}, p, \mathbb{P}_\alpha \in M$, α such that $p \in \mathbb{P}_\alpha$ and $\underset{\sim}{X}$ $\mathbb{P}_\alpha$-name, $M \cap H(\omega_1) = M_\delta$, $M_\delta \cap \omega_1 = \delta$, $p_\delta \in M_\delta$ and $\underset{\sim}{X} \cap M_\delta = A_\delta$.

Finding the club sequence in the ground model

According to the stepping up lemma we may construct a sequence $\langle q_i, F_i, \vec{n}_i, \vec{k}_i, \beta_i \mid i \in \omega \rangle$ such that

(1) $F_i \subseteq F_{i+1}$, $\bigcup_{i\in\omega} F_i = \delta$,

(2) $\alpha_i < \beta_i < \delta$,

(3) $q_0 \geq p_\delta$,

(4) $q_i \in \mathbb{P}_\delta \cap M_\delta$ is $(F_i, \vec{k}_i)$ determined and $\vec{k}_i = \vec{k}(p_i, F_i, \vec{n}_i)$, wlog $\vec{n}_i$ can be the vector constant to i, at least we need that $(\forall \beta \in \mathrm{supp}(p))(\lim_{i\to\omega} \vec{n}_i(\beta) = \infty)$,

(5) $q_{i+1} >_{(F_i, \vec{n}_i)} q_i$,

(6) q_i is $(A_\delta, F_i, \vec{n}_i)$-good,

(7) $q_i \Vdash \beta_i \in \underset{\sim}{X}$ and

(8) $q_i \Vdash D_i \cap M_\delta \neq \emptyset$.

Finally set $C_\delta = \{\beta_i \mid i < \omega\}$.

Putting things together

Corollary

If Jensen's diamond holds in the ground model, then in the Sacks model, in the Laver model, in den Miller model, in the Blass-Shelah model, in the Matet-model and in the Mathias model the club principle holds.

Something different from c.s. iterations of proper iterands with the $\aleph_2$-c.c. is needed

In all our models there are Souslin trees.

Something different from c.s. iterations of proper iterands with the $\aleph_2$-c.c. is needed

In all our models there are Souslin trees.

So this work does not help at all answering Juhász' question.

The last slide

Thank you for your attention.