Near coherence of filters and filter dichotomy

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Outline

1. The definitions of FD and of NCF
   - The filter dichotomy principle
   - Near Coherence of Filters
2. Is the implication an equivalence?
3. The main result
   - A sketch of the proofs

Mappings between filters

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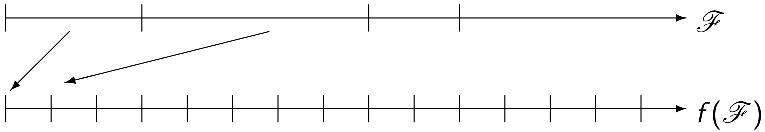
$f(\mathscr{F})$ contains less information than $\mathscr{F}$:



The Rudin-Blass ordering

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A filter $\mathscr{F}$ is Rudin-Blass less or equal a filter $\mathscr{G}$ (written $\mathscr{F} \leq_{RB} \mathscr{G}$) iff there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{F}) \subseteq f(\mathscr{G})$.

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Definition

A filter $\mathscr{F}$ is called nearly ultra if there is a finite-to-one function f such that $f(\mathscr{F})$ is ultra.

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Example (Talagrand): A filter is meagre in 2^ω iff there is some finite-to-one f such that $f(\mathscr{F})$ is the Fréchet filter. The meagre filters are minimal.

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Theorem, Blass and Shelah 1987

FD is consistent relative to ZFC.

NCF

Definition

Two filters $\mathscr{F}$ and $\mathscr{G}$ on ω are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a proper filter.

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Definition

The principle of near coherence of filters (NCF) says that any two filters are nearly coherent.

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Proof: Show FD $\Rightarrow$ NCF.

Let two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ be given. Then $\mathscr{U} \cap \mathscr{V}$ is not meagre: Plewik showed (see Blass' handbook article 9.12) that the intersection of fewer than $\mathfrak{c}$ ultrafilters is not meagre. Hence by FD there is a finite-to-one function f such that $f(\mathscr{U} \cap \mathscr{V})$ is ultra. But then $f(\mathscr{U} \cap \mathscr{V}) = f(\mathscr{U}) = f(\mathscr{V})$.

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Question

Can we reverse this implication?

NCF $\not\Rightarrow$ FD

Theorem, Mi,Shelah, 2006 [894]

NCF and not FD relative to ZFC.

Semifilter Trichotomy

A set $\mathscr{S} \subseteq [\omega]^{\aleph_0}$ that is closed under almost supersets is called a semifilter.

SFT says that each semifilter is either meager or mapped by a finite to-one function to an ultrafilter or to the whole $[\omega]^{\aleph_0}$.

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Theorem, Mi, 2000

FD and $\mathfrak{s} > \mathfrak{u}$ implies $\mathfrak{u} < \mathfrak{g}$.

Theorem, Laflamme, 1999, Blass, Laflamme 1999

SFT and $\mathfrak{u} < \mathfrak{g}$ are equivalent.

P-points

Definition

An ultrafilter $\mathscr{U}$ is a P-point if for any X_n, $n \in \omega$, such that $X_n \in \mathscr{U}$ there is some $X \in \mathscr{U}$ such that $X \subseteq^* X_n$ for all n. Such an X is called a pseudointersection of X_n, $n \in \omega$.

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We start with a ground model of CH. Under CH there is a P-point.

Preserving one P-point

First: we preserve only one arbitrary P-point $\mathscr{E} \in \mathbf{V}_0$ that will be fixed forever, and destroy many others.

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A non-complete subforcing of Matet forcing will do this.

Definition

A condition in the Matet forcing is a $p = (a, \bar{c})$, such that a is a finite subset of ω and $\bar{c}$ is an unmeshed sequence of finite subsets of $(\max(a), \omega)$. A stronger condition $q = (b, \bar{d})$ is gotten by taking as $b \setminus a$ some union of finitely many elements of $\bar{c}$, and dropping elements from the sequence $\bar{c}$ such that infinitely members stay and merge finite blocks of adjacent members of the intermediate sequence to get $\bar{d}$. $\bar{d}$ is called a condensation of $\bar{c}$.

Stable ordered-union ultrafilters

Let $\mathrm{FU}(\bar{c})$ be the set all condensations of $\bar{c}$.

Definition

A filter $\mathscr{F}$ on $\mathbb{F}$ is said to be an ordered-union filter if it has a basis of sets of the form $\mathrm{FU}(\bar{d})$ for $\bar{d} \in (\mathbb{F})^{\omega}$. An ordered-union filter is said to be stable if, whenever it contains $\mathrm{FU}(\bar{d}_n)$ for $\bar{d}_n \in (\mathbb{F})^{\omega}$, $n < \omega$, then it also contains some $\mathrm{FU}(\bar{e})$ for some $\bar{e}$ that is almost a condensation of each $\bar{d}_n$.

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Let $\mathscr{U}$ be a stable ordered-union ultrafilter.

Definition

$\mathbb{M}(\mathscr{U})$ contains the $(a, \bar{c})$ from $\mathbb{M}$ such that $\bar{c} \in \mathscr{U}$. The forcing partial order is inherited from $\mathbb{M}$.

From $\mathscr{P}_{<\omega}(\omega)$ to ω

Definition

Let $\mathbb{F} = \mathscr{P}_{<\omega}(\omega)$. Let $\mathscr{U}$ be an ordered-union ultrafilter on $\mathbb{F}$. The core of $\mathscr{U}$ is the filter $\Phi(\mathscr{U})$ such that

$$X \in \Phi(\mathscr{U}) \text{ iff } (\exists \operatorname{FU}(\bar{c}) \in \mathscr{U})(\bigcup_{n\in\omega} c_n \subseteq X).$$

If $\mathscr{U}$ is ultra, then $\Phi(\mathscr{U})$ is not meager.

$\mathscr{U}$ that do not harm $\mathscr{V}$

Theorem

(Eisworth "$\rightarrow$" Theorem 4, "$\leftarrow$" Cor. 2.5, this direction works also with non-P ultrafilters.)
Let $\mathscr{U}$ be a stable ordered-union ultrafilter on $\mathbb{F}$ and let $\mathscr{V}$ be a P-point. Iff $\mathscr{V} \not\geq_{RB} \Phi(\mathscr{U})$, then $\mathscr{V}$ continues to generate an ultrafilter after we force with $\mathbb{M}(\mathscr{U})$ or with $\mathbb{Q}(\mathscr{U})$.

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Locally Fréchet filters are not nearly ultra. The reverse does not hold (as we shall see in the steps of cofinality ω_1).

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$\mathscr{A}$ is not almost ultra

Also $\mathscr{A}$ will be very far from being ultra, because at any time it contains a tree on $2^{\aleph_1}$ mutually non-nearly coherent core filters $\Phi(\mathscr{U})$ as supersets and at stages $\alpha \in \aleph_2 \setminus S^2_1$ the filter $\mathscr{A}_\alpha$ is even locally Fréchet.

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We strengthen the latter properties of $\mathscr{A}_\alpha$ to a property of every two stages $\beta < \gamma$, $\beta, \gamma \in \aleph_2 \setminus S^2_1$ that is preserved in the iteration and that will allow us to work with stable ordered-union ultrafilters $\mathscr{U}$ on $\mathbb{F} = [\omega]^{<\aleph_0}$ such that $\Phi(\mathscr{U}) \not\leq_{RB} \mathscr{E}$.

Getting NCF nevertheless

Third: We get NCF with help of a diamond and special iterands: We let $S^2_1 = \{\alpha \in \aleph_2 \, : \, \mathrm{cf}(\alpha) = \aleph_1\}$. A diamond sequence on S^2_1 is a sequence $\langle S_\alpha \, : \, \alpha \in S^2_1 \rangle$ such that for all $X \subseteq \aleph_2$ the set $\{\alpha \in S^2_1 \, : \, X \cap \alpha = S_\alpha\}$ is stationary. $\diamondsuit(S^2_1)$ says that there is a diamond sequence for S^2_1.

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- shall diagonalize $\mathscr{A}_\alpha$ by adding an infinite set X_α.

So $\mathscr{A}_\alpha$ becomes by this procedure again locally Fréchet, and thus in the whole extension $\mathscr{A}$ is not mapped by any finite-to-one function to an ultrafilter.

An iteration

We fix a diamond sequence $\langle S_\alpha : \alpha \in S^2_1 \rangle$. We also fix a P-point $\mathscr{E} \in \mathbf{V}$ that will be preserved throughout our iteration. Let $\underset{\sim}{f}_\alpha$, $\alpha \in \aleph_2 \setminus S^2_1$, be an enumeration of all $\mathbb{P}_{\aleph_2}$-names for finite-to-one functions, each appearing cofinally often. Let $\underset{\sim}{f}_\alpha$ be a $\mathbb{P}_\alpha$-name. Since all $\mathbb{Q}_\alpha$ have size $\aleph_1$ and are proper, such an enumeration exists.

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We construct (carefully) by induction on $\alpha < \aleph_2$ a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \aleph_2, \alpha \leq \aleph_2 \rangle$ and two sequences of names $\langle \underset{\sim}{A}_\alpha : \alpha \in \aleph_2 \setminus S^2_1 \rangle$ and $\langle \underset{\sim}{X}_\alpha : \alpha \in S^2_1 \rangle$ such that

The desired properties

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(P3) We write $A_\alpha = \underset{\sim}{A}_\alpha[G_{\alpha+1}]$. $\{A_\beta : \beta \in \alpha \setminus S^2_1\}$ has the finite intersection property and for each $\alpha \notin S^2_1$, $f_\alpha(A_\alpha) \neq^* \omega$. We let $\mathscr{A}_\alpha = \text{filter}(\{A_\beta : \beta \in \alpha \setminus S^2_1\})$. So A_α shows that $f_\alpha(\mathscr{A}_{\alpha+1})$ is not the Fréchet filter.

... more properties

(P4) Let $\underset{\sim}{\mathscr{A}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathscr{A}_\alpha$. If $\alpha \in S^2_1$ and the S_α is a $\mathbb{P}_\alpha$-name $\underset{\sim}{\mathscr{D}}$ for an ultrafilter in $V^{\mathbb{P}_\alpha}$, then $\Vdash_{\mathbb{P}_{\alpha+1}}$ "$\underset{\sim}{\mathscr{D}}$ and filter($\mathscr{E}$) are nearly coherent, filter($\mathscr{E}$) is ultra, and $\underset{\sim}{X}_\alpha$ diagonalises $\underset{\sim}{\mathscr{A}}_\alpha$".

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(P5) For $\beta < \gamma \notin S^2_1$ we have $(\exists X \in [\omega]^{\aleph_0})\mathscr{A}_\beta \restriction X = \mathrm{CFF}_X$ and if $G_\gamma \subseteq \mathbb{P}_\gamma$ is generic over $\mathbf{V}$ and $G_\beta = \mathbb{P}_\beta \cap G_\gamma$ then

if $\mathbf{V}_\beta \models$ "$(\bar{c}, \mathscr{R})$ is a witness over $\mathscr{A}_\beta$"

then $\mathbf{V}_\gamma \models$ "$(\exists \bar{d})(\bar{c} \leq^* \bar{d} \wedge (\bar{d}, \mathscr{R})$ is a witness over $\mathscr{A}_\gamma)$".

The properties in (P5)

Definition

We say $(\bar{c}, \mathscr{R})$ is a *witness over* $\mathscr{A}$ when:

(a) $\mathscr{A} \subseteq [\omega]^{\aleph_0}$,

(b) $\bar{c} = \langle c_n : n < \omega \rangle$ is a pure member of $\mathbb{Q}$ or of $\mathbb{M}$,

(c) $\mathscr{R}$ is a countable subset of
$\{R \subseteq \omega \times \omega : (\forall m)(\exists^{<\aleph_0} n)(mRn) \wedge (\forall n)(\exists^{<\aleph_0} m)(mRn)\}$,

(d) $\mathscr{R} \neq \emptyset$,

(e) if $R \in \mathscr{R}$ then $\mathscr{A} \restriction R(\mathrm{set}(\bar{c})) = \mathrm{CFF}(R(\mathrm{set}(\bar{c})))$.