# Many countable support iterations of proper forcings preserve Souslin trees

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**Definition**

The club principle $\clubsuit$, also called Ostaszewski's club, says: There is a sequence $\langle A_\alpha \mid \alpha \in \omega, \lim(\alpha)\rangle$, with the following properties: $A_\alpha$ is a cofinal subset of $\alpha$, and for every uncountable $X \subseteq \omega_1$, there are stationarily many $\alpha$ with $A_\alpha \subseteq X$.

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**Theorem, Miyamoto**

If the club principle holds and $\mathrm{cov}(\mathcal{M}) = \aleph_2$ then there is a Souslin tree.

# A Souslin tree from few meager sets

**Theorem, Brendle 2006**

If the club principle holds and $\mathrm{cof}(\mathcal{M}) = \aleph_1$ then there is a Souslin tree.

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Examples are the Miller model and the Blass-Shelah model.

# Are there Souslin trees in these models?

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Proof: The reason is the preservation of Souslin trees. The diamond principle holds after $\omega_1$ iteration steps. So there are Souslin trees after $\omega_1$ steps of the iteration.

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Third one: For nep forcings. Preserving $\omega$-many Cohen generics over many (not necessarily elementary) models is a sufficient criterion. This is weaker than the game-theoretic criterion. However, it is applicable only to forcings that are nep in a strong sense. However, it does not cover all the forcings from the second case.

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Iteration issues. Preserving a Souslin tree is iterable.

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For convenience we write is such that $[\omega\alpha, \omega\alpha + \omega) \setminus \{0\}$ being a superset of the $(1 + \alpha)$-th level and $\{0\}$ being the zero-th level. We require that the trees are pruned, i.e., for every node $t$ on level $\alpha$ for every $\beta > \alpha$ there is $t' >_T t$ on level $\beta$. Moreover, every node shall be a splitting node.

# Forcing with a Souslin tree

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Let for $\delta \in \omega_1$, we let $Y(\delta) = \{t^\delta_n \mid n < \gamma_\delta\} \subseteq T_\delta$, $\gamma_\delta \leq \omega$. Let $Y = \bigcup\{Y(\delta) \mid \delta \in \omega_1\}$. Let $\mathscr{S} \subseteq [\omega_1]^\omega$ be stationary.

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## Definition

We say $T$ is $(Y, \mathscr{S})$-proper iff $Y \subseteq T$ and $\mathscr{S} \subseteq [\omega_1]^\omega$ and for every sufficiently large $\chi$ for every countable $N \prec \mathscr{H}(\chi)$ with $\{T, \mathscr{S}\} \subset N$ and $N \cap \omega_1 \in \mathscr{S}$, $\delta = \sup(N \cap \omega_1)$ for every $t \in Y(\delta)$, $T_{<_T t} := \{s \mid s <_T t\}$ is $(N, T)$ generic.

# An old characterisation of Souslinness

If every node of $T$ is splitting and $T$ has an uncountable chain then $T$ has an uncountable antichain. So an everywhere splitting tree $T$ is Souslin iff it has no uncountable antichain. The latter is used in the proof of the following:

**Lemma**

*The following are equivalent:*

*(1)* $T$ *is Souslin.*

*(2)* $T$ *is* $(Y, \mathscr{S})$*-proper for every stationary* $\mathscr{S} \subseteq [\omega_1]^\omega$ *and for every* $Y$ *of the form* $\bigcup_{\delta \in W} T_\delta$*, such that* $W \subseteq \{\sup(a) \mid a \in \mathscr{S}\}$ *stationary.*

*(3)* $T$ *is* $(Y, \mathscr{S})$*-proper for some stationary* $\mathscr{S} \subseteq [\omega_1]^\omega$ *and for some* $Y$ *of the form* $\bigcup_{\delta \in W} T_\delta$*, such that* $W \subseteq \{\sup(a) \mid a \in \mathscr{S}\}$ *stationary.*

## Preserving generic branches and hence Souslinness

### Definition

We say $\mathbb{P}$ is $(T, Y, \mathscr{S})$-preserving iff the following holds: Let $\mathscr{S} \subseteq \omega_1$ be stationary. There is $x \in H(\chi)$, for every $N \prec \mathscr{H}(\chi)$ with $\{x, Y, T, \mathbb{P}, \mathscr{S}\} \subseteq N$ and $p \in \mathbb{P} \cap N$: if $\sup(N \cap \omega_1) = \delta$, $N \cap \omega_1 \in \mathscr{S}$, and for every $n < \gamma_\delta$, $\{s \mid s <_T t_n^\delta\}$ is $(N, \mathbb{P}, p)$-generic, then there is $q \geq_{\mathbb{P}} p$ such that $q$ is $(N, \mathbb{P})$-generic and

$$q \Vdash_{\mathbb{P}} (\forall n < \gamma_\delta)(\{s \mid s <_T t_n^\delta\} \text{ is } (N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], T)\text{-generic}).$$

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Remark: If $\mathscr{S}$ and $Y$ are large enough, this implies properness. We use that $\mathbb{P}$ is $(T, Y, \mathscr{S})$-preserving for a stationary $\mathscr{S}$ and a set $Y$ containing all points of a stationary set of levels of the Souslin tree $T$ and thus we get that $\mathbb{P}$ preserves that $T$ is a Souslin tree.

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"$\mathbb{Q}$ is $(T, Y, \mathscr{S})$-preserving" is an iterable property.

**Definition**

Let $\mathbb{P}$ be a notion of forcing and $p \in \mathbb{P}$.
The game $\Game^2(\mathbb{P}, p)$ is played in $\omega$ rounds. In round $n$, player COM chooses an $\ell_n \in \omega \setminus \{0\}$ and a sequence $\langle p_{n,\ell} \mid \ell < \ell_n \rangle$ of conditions $p_{n,\ell} \in \mathbb{P}$ and then player INC plays $\langle q_{n,\ell} \mid n < \omega \rangle$ such that $p_{n,\ell} \le q_{n,\ell}$. After $\omega$ rounds, COM wins the game iff for every infinite $u \subseteq \omega$ there is $q_u \ge p$ such that

$$q_u \Vdash (\exists^\infty n \in u)(\exists \ell < \ell_n)(q_{n,\ell} \in \underset{\sim}{\mathbf{G}}).$$

# Using the game

**Theorem**

*Assume $\alpha(*) = \omega_1$ and $\mathscr{S} \subseteq \omega_1$ is stationary. Let $T$ be an $\omega_1$-tree and $Y \subseteq T$. If COM has a winning strategy in the game $\eth^2(\mathbb{P})$, then $\mathbb{P}$ is $(T, Y, \mathscr{S})$-preserving.*

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**Example**

In the Laver forcing COM does not have a winning strategy in $\Game^2(\mathbb{P})$. The existence of such a winning strategy implies that $\mathbb{P}$ is almost $\omega^\omega$-bounding. Nevertheless Laver forcing preserves Souslin trees.

# Proof of the previous lemma: A strategy for COM

Let $v \subseteq {}^{\omega>}\omega$. We let $\mathrm{dcl}(v) = \{\eta \restriction k \mid \eta \in v, k < \lg(\eta)\}$ be the descending closure of $v$.

# Proof of the previous lemma: A strategy for COM

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Let $v \subseteq {}^{\omega>}\omega$. We let $\mathrm{dcl}(v) = \{\eta \restriction k \mid \eta \in v, k < \lg(\eta)\}$ be the descending closure of $v$.
We describe a strategy $\mathbf{st}$ for COM in $\eth^2(\mathbb{P}, p)$.
On the side after the in the $n$-th move COM chooses a finite set of nodes $v_n$ such the sequence $\langle \bar{p}_n, \bar{q}_n, v_n \mid n \in \omega\rangle$ has the following properties

# A strategy for COM, II

(0) $\ell_0 = 1$, $p_{0,0} = p$, $q_{0,0} \geq p_{0,0}$, $v_0 = \{\mathrm{tr}(q_{0,0})\}$.

(1) For $n \geq 1$, given $v_{n-1}$, COM chooses $\ell_n = |v_{n-1}|$ and for $\eta \in v_{n-1}$, $\eta = \mathrm{tr}(q_{n',\ell})$ for some $n' < n$, $\ell < \ell_{n'}$ he lets

$$m(\eta, n) = \min\{k \mid \eta\hat{\ }k \in q_{n',\ell} \setminus \mathrm{dcl}(v_{n-1})\}.$$

Let $\{\eta^n_\ell \mid \ell < \ell_n\}$ enumerate $v_{n-1}$ and let $\eta^n_\ell = \mathrm{tr}(q_{n',\ell'})$. Now COM chooses $p_{n,\ell} = q_{n',\ell'}^{[\eta^n_\ell\hat{\ }m(\eta_\ell,n)]}$.

(2) INC plays $q_{n,\ell} \geq p_{n,\ell}$.

(3) Now COM chooses his new helper: $v_n = v_{n-1} \cup \{\mathrm{tr}(q_{n,\ell}) \mid \ell < \ell_n\}$, and the round is finished. Indeed $\ell_{n+1} = 2\ell_n$ and $\ell_0 = 1$, but this is not important.

# This is a winning strategy

The strategy $\mathbf{st}$ is a winning strategy for COM: Let $u \subseteq \omega$ be infinite. By induction on $n \in u$ we choose $s_n \subseteq v_n \smallsetminus v_{n-1}$ . If $n = \min(u)$, then $s_n \subseteq v_n \smallsetminus v_{n-1}$ is a singleton. For $n > \min(u)$, let

$$s_n = s_{\max(u \cap n)} \cup \{\eta \in v_n \smallsetminus v_{n-1} \mid \nu = \max_{\triangleleft}\{\varrho \in v_n \mid \varrho \trianglelefteq \eta\} \in s_{\max(u \cap n)}\}.$$

Lastly we let

$$q_u = \{\varrho \mid (\exists n \in u)(\exists \eta \in s_n)(\varrho \trianglelefteq \eta)\}.$$

# Preserving Souslin trees for another reason

We compute with the norms in the linear creature case.

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**Lemma**

*Blass-Shelah forcing is* $(T, Y, \mathscr{S})$*-preserving.*