Filters and scales

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Outline

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Filters on ω

Definition

A filter is a subset $\mathscr{F} \subseteq \mathscr{P}(\omega)$ that is closed

– under finite intersections

– and supersets

– and does not contain the empty set.

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Using topology and measure of 2^ω

$Y \subseteq \omega$ has a characteristic function $\chi_Y \in 2^\omega$, $\chi_Y(n) = 0$ iff $n \notin Y$.

We identify a filter $\mathscr{F}$ on ω with the set of characteristic functions $\{\chi_Y : Y \in \mathscr{F}\}$.

2^ω carries the usual topology, and there is the usual measure.

Then we may speak about meager filters, measurable filters, filters with the Baire property.

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   - Filters
   - Reduced powers of $(\omega, <)$
   - Mappings of filters
   - The Rudin-Blass ordering
2. Consistency results
3. Consequences of the principles
   - Rapid filters

Partial (pre) orders

Let $f, g\colon \omega \to \omega$, and let $\mathscr{F}$ be a filter on ω.

We write $f \leq_{\mathscr{F}} g$ iff $\{n : f(n) \leq g(n)\} \in \mathscr{F}$.

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Let $\mathfrak{d}$, the dominating number, be the smallest cardinal of an $\leq^*$-dominating subset of ω^ω.

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Scales

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A scale is a sequence $\langle f_\alpha : \alpha < \mathfrak{d} \rangle$ that is $\leq^*$-increasing and dominating.

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Talagrand's characterization of meager filters

For $X \in [\omega]^\omega$ let $\mathrm{en}_X : \omega \to \omega$ enumerate X increasingly: $\mathrm{en}_X(n) =$ the $n+1$-st element of X.

Theorem. Talagrand, 1984

The following are equivalent for any non-principal (semi)filter:

- $\mathscr{F}$ is meager.
- $\{\mathrm{en}_X \ : \ X \in \mathscr{F}\}$ is $\leq^*$-bounded.
- $(\exists g \in \omega^{\uparrow\omega})(\forall X \in \mathscr{F})(\forall^\infty i)(X \cap [g(i), g(i+1)) \neq \emptyset)$.

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Mappings between filters

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Let $g\colon \omega \to \omega$ be any function or be finite-to-one. We set

$$g(\mathscr{F}) = \{X \,:\, g^{-1}X \in \mathscr{F}\}.$$

$g(\mathscr{F})$ contains less information than $\mathscr{F}$:

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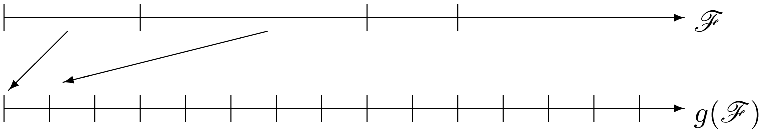

Talagrand's characterization revisited

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- $(\exists g \in \omega^{\uparrow\omega})(\forall X \in \mathscr{F})(\forall^\infty i)(X \cap [g(i), g(i+1)) \neq \emptyset)$
  The latter means: There is a finite-to-one function mapping $\mathscr{F}$ to the Fréchet filter.

Estimates for $\mathfrak{b}(\mathscr{F})$

$\mathfrak{b}(\mathscr{F}) = \mathfrak{b}(g(\mathscr{F}))$ for every finite-to-one function g.

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Groupwise density

Definition

$\mathscr{G} \subseteq [\omega]^\omega$ is called groupwise dense if $\forall \langle \pi_i : i < \omega \rangle \in \omega^{\uparrow\omega} \exists A \in [\omega]^\omega \bigcup_{i \in A} [\pi_i, \pi_{i+1}) \in \mathscr{G}$ and if $\mathscr{G}$ is closed under almost subsets.

Definition

The groupwise density number, $\mathfrak{g}$, is the minimum number of groupwise dense sets whose intersection is empty.

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More estimates for $\mathfrak{b}(\mathscr{F})$

Theorem, Blass and M., 1999

$\mathfrak{b}(\mathscr{F}) \geq \mathfrak{g}_f$ if $\mathscr{F}$ is not meager.

$\nu_X(n) = \min([n, \infty) \cap X)$ is the next-function.

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Nearly maximal filters

However, $h(\mathscr{U})$ can be an ultrafilter even if $\mathscr{U}$ is not ultra.

Example: Take an ultrafilter $\mathscr{U}$ on the even numbers and look at $\mathscr{U}' = \{X \cup \text{ odd numbers } : X \in \mathscr{U}\}$.

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The filter dichotomy principle

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Can $\mathfrak{g}$ be large?

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The character of an ultrafilter is the smallest size of a basis. The ultrafilter number $\mathfrak{u}$ is the smallest character of a non-principal ultrafilter.

$\mathfrak{b} \leq \mathfrak{u}$ in ZFC.

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Sketch of proof

Fix a basis $\{U_\alpha : \alpha < \mathfrak{u}\}$ of an ultrafilter with character $\mathfrak{u}$. Let $\mathscr{F}$ be a non-meager filter. Then show that for $\alpha < \mathfrak{u}$

$$\mathscr{G}_\alpha = \{X \in [\omega]^\omega : \exists F \in \mathscr{F} (\forall^\infty x < y \in X)([x, y) \cap F \neq \emptyset \rightarrow [x, y) \cap U_\alpha \neq \emptyset)\}$$

is groupwise dense.

The reverse direction is open.

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More upper bounds on $\mathfrak{g}$

Theorem, Shelah 2008

$\mathfrak{g} \leq \mathfrak{b}^+$.

$\mathfrak{g}_f \leq \mathfrak{b}^+$.

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Consequences of $\mathfrak{u} < \mathfrak{g}$ onto cardinal characteristics

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FD implies $\mathfrak{u} = \mathfrak{b}$ and $\mathfrak{d} = \mathfrak{c}$.

Definition

$\mathfrak{s}$, the splitting number, is the smallest size of a set $\mathscr{S}$ such that for any $X \in [\omega]^\omega$ there is $S \in \mathscr{S}$ such that $X \cap S$ and $X \setminus S$ are both infinite.

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$\mathfrak{s}$, the splitting number, is the smallest size of a set $\mathscr{S}$ such that for any $X \in [\omega]^\omega$ there is $S \in \mathscr{S}$ such that $X \cap S$ and $X \setminus S$ are both infinite.

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$\mathfrak{s} \leq \mathfrak{d}(\mathscr{F}) \cdot \mathfrak{d}(\mathscr{G})$, $\mathscr{F}$ and $\mathscr{G}$ not nearly coherent.

Consequences of $\mathfrak{u} < \mathfrak{g}$ onto cardinal characteristics

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A new consequence of NCF

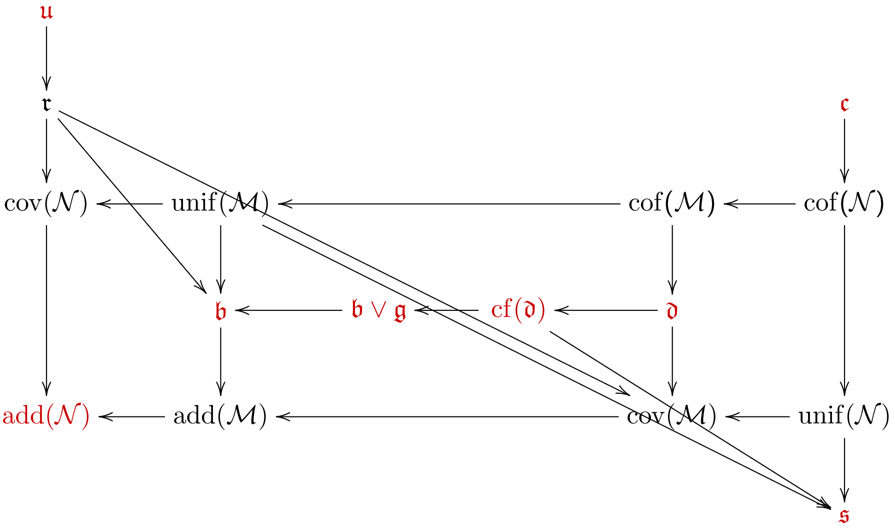
Definition

$\mathrm{add}(\mathcal{N})$, the additivity of the Lebesgue null sets, is the smallest number of Lebesgue null sets whose union is not a null set.

Theorem. M., new

NCF implies $\mathrm{add}(\mathcal{N}) = \aleph_1$.

An extension of Cichoń's diagram



Outline

Rapid filters

Definition. Mokobodzki

A filter $\mathscr{F}$ is called rapid if for every $f\colon \omega \to \omega$ there is a $X \in \mathscr{F}$ such that $\forall n |X \cap f(n)| \leq n$.

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Rapid filters

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If $\mathrm{add}(\mathcal{N}) > \aleph_1$ then there is a rapid filter.

Proposition

Under NCF there is no rapid filter.

Proof: NCF implies that $\mathfrak{u} < \mathfrak{d}$. Hence every ultrafilter is mapped by a suitable finite-to-one map to a finite-to-one image of the ultrafilter witnessing $\mathfrak{u}$. So every ultrafilter has a finite-to-one image that is not rapid. If a filter is rapid, then also all its finite-to-one images are rapid. There is a rapid filter iff there is a rapid ultrafilter.

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Question. Blass, 1989

Does the filter dichotomy principle imply $\mathfrak{u} < \mathfrak{g}$?

Theorem. M.,2001

$\mathfrak{u} < \mathfrak{g}_f$ is equivalent to the filter dichotomy principle. The filter dichotomy implies $\mathfrak{g}_f = \mathfrak{d}$.

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In all known models of $\mathfrak{u} < \mathfrak{d}$ we have $\mathfrak{u} = \mathfrak{b}$.

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