The descriptive set theory of the Lebesgue Density Theorem

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A point x has density r in a measurable set $A \subseteq [0;1]$ if

$$\lim_{\varepsilon \to 0} \mu(A \cap (x-\varepsilon; x+\varepsilon))/2\varepsilon = r$$

and let $\Phi(A)$ be the set of all points of density 1 in A. The Lebesgue Density Theorem says that almost every point in $[0;1]$ has density 0 or 1 in A, hence the function Φ picks a representative in each equivalence class of the measure algebra. It is known that $\hat{\Phi}([A]) \in \mathbf{\Pi}^0_3$, hence a map $\hat{\Phi} \colon \textsc{Malg} \to \mathbf{\Pi}^0_3$ is defined. (These result hold for the Cantor space *verbatim*.)

We prove several results on the the Lebesgue density function on the Cantor space:

- $\Phi(A)$ can attain any topological complexity inside $\mathbf{\Pi}^0_3$, that is for every $B \in \mathbf{\Pi}^0_3$ there is an A such that $\Phi(A)$ is Wadge equivalent to B. Hence $\mathscr{W}_{\boldsymbol{d}} = \{[A] \mid \hat{\Phi}([A]) \in \boldsymbol{d}\}$ is nonempty, for every Wadge degree $\boldsymbol{d} \subseteq \mathbf{\Pi}^0_3$.
- $\mathscr{W}_{\boldsymbol{d}}$ is dense in the topological space $\textsc{Malg}$ for $\boldsymbol{d} \neq [\emptyset]_{\mathrm{W}}, [{}^{\omega}2]_{\mathrm{W}}$.
- If $\boldsymbol{d} = \mathbf{\Pi}^0_3 \setminus \mathbf{\Delta}^0_3$, then $\boldsymbol{d}$ is the unique Wadge degre such that $\mathscr{W}_{\boldsymbol{d}}$ is dense in the boolean algebra $\textsc{Malg}$.
- If $\boldsymbol{d} = \mathbf{\Pi}^0_3 \setminus \mathbf{\Delta}^0_3$, then $\boldsymbol{d}$ is the unique Wadge degre such that $\mathscr{W}_{\boldsymbol{d}}$ is comeager in the topological space $\textsc{Malg}$.
- The set of all $\left\{[A] \in \textsc{Malg} \mid [A] \cap \mathbf{\Delta}^0_2 = \emptyset\right\}$ is comeger, hence most sets are equal up to a measure zero set to a $\mathbf{\Sigma}^0_2$ (and to a $\mathbf{\Pi}^0_2$) but are not equivalent to anything simpler.