On Fraïssé's order type conjecture

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Introduction

Our results will be about the relation of embeddability between order types (isomorphism types of linearly ordered sets). If φ and ψ are order types, write $\varphi \leqq \psi$ to mean that φ is embeddable in ψ (i.e., if L and M are linearly ordered sets having type φ and ψ, respectively, then there is a 1–1 order preserving function from L into M), and let $\varphi < \psi$ mean that $\varphi \leqq \psi$ but $\psi \not\leqq \varphi$. If α is an ordinal and $\{\varphi_\gamma : \gamma < \alpha\}$ is a set of order types, then $\{\varphi_\gamma : \gamma < \alpha\}$ is called a descending sequence if $\gamma < \delta \rightarrow \varphi_\gamma > \varphi_\delta$. A set $\{\psi_x : x \in X\}$ of order types is called an antichain if $x \neq y \rightarrow \psi_x \not\leqq \psi_y$. The Dushnik-Miller-Sierpinski construction [1], [11] provides an example of an infinite descending sequence of order types, as well as an infinite antichain of order types. The order types which they construct to get these examples are all subtypes of the real line of power $2^{\aleph_0}$.

In [2], Fraïssé made conjectures to the effect that the embeddability relation is more well behaved in the case of countable order types. His main conjecture was

every descending sequence of countable order types is finite, and every antichain of countable order types is finite.

A few years later he extended this conjecture to apply to scattered order types of any cardinality (an order type φ is said to be scattered if the order type of the rationals is not embeddable in φ).

Let $\mathfrak{M}$ be the class of all order types φ which satisfy the following condition: if L is a linearly ordered set of type φ, then L can be written as the union of countably many subsets $L_1, L_2, \cdots, L_n, \cdots$ such that each L_i has scattered order type. Fraïssé's conjecture will be a consequence of the following theorem, which is the main result of this paper.

THEOREM. *Every descending sequence of members of $\mathfrak{M}$ is finite, and every antichain of members of $\mathfrak{M}$ is finite.*

We will explain the proof of this theorem by giving a summary of the sections. We would like to acknowledge at this point the mathematical debt

[1] This paper is a slightly expanded version of the author's Ph.D. thesis, written at the University of California, Berkeley, in 1969.

which we owe to the deep methods and results of Nash-Williams' paper [8]. His theory of better-quasi-orderings will play a basic role in this paper, as will be seen.

In § 1, the theory of well-quasi-orderings (arbitrary sets and classes on which a transitive, reflexive relation is defined which has the "no infinite descending sequence, no infinite antichain" property) as well as the theory of better-quasi-orderings (well-quasi-orderings of a particularly well-behaved type) will be reviewed. It turns out that better-quasi-orderings are more natural than well-quasi-orderings in almost every respect. However, their definition might appear unintuitive at first, so we will spend some time trying to motivate it.

In § 2, Nash-Williams' infinite tree theorem will be stated, and we will indicate how to generalize it to allow the trees to have their nodes labelled by members of a better-quasi-ordered set. The proof of this generalized tree theorem will be the only place in this paper where familiarity with [8] will be assumed; the rest of the paper is self contained.

In § 3, we will state a well known theorem of Hausdorff which characterizes the class of scattered order types, and will then present work of Galvin characterizing the larger class $\mathfrak{M}$. Galvin's results involve defining order types $\eta_{\alpha\beta}$ which act like η at higher cardinalities (not to be confused with Hausdorff's η_α's, which are generalizations of η is another direction).

In § 4 the main theorem will be proved. Originally the author proved Fraïssé's conjecture by induction on Hausdorff's hierarchy. We are indebted to Fred Galvin for then suggesting the generalization to $\mathfrak{M}$ and showing us his characterization of $\mathfrak{M}$ (we would also like to thank him for many other interesting conversations). The present theorem will be done in that more general setting. To make the induction hypothesis go through it will be strengthened in two ways. First, we will show that $\mathfrak{M}$ is better-quasi-ordered under $\leqq$ rather than just well-quasi-ordered. For the other strengthening, define a Q-type to be (intuitively) an order type whose points are labelled by members of Q, and let $Q^{\mathfrak{M}}$ be the collection of Q-types whose base types are in $\mathfrak{M}$. An ordering on Q induces an ordering on $Q^{\mathfrak{M}}$ in a natural way. Our strongest result is

THEOREM. *If Q is better-quasi-ordered, then $Q^{\mathfrak{M}}$ is better-quasi-ordered.*

In the remainder of § 4 we will show there are (up to equivalence) k^+ order types in $\mathfrak{M}$ of power $\leqq k$.

In § 5 we will mention some applications and problems.

Notation will be that of standard set theory (with classes). α, β, γ, δ, k, λ range over ordinals, where an ordinal is defined so that it is the set of all smaller ordinals; the class of ordinals will be denoted by On. Card is the class of cardinals (initial ordinals), Card X is the cardinality of X. If k is a cardinal then k^+ is the least cardinal greater than k; a successor cardinal is one of the form k^+. Cf(α), the cofinality of α, is the least β such that α is the limit of a β-sequence of ordinals. A cardinal k is regular if cf$(\kappa) = k$; RC is the class of infinite regular cardinals. X^α ($X^{<\alpha}$) is the collection of X-sequences of length α ($<\alpha$) (this will be slightly modified in § 4); members of X^α will be written in the form $\langle x_\gamma \rangle_{\gamma<\alpha}$. $\mathcal{P}(X)$ is the collection of subsets of X, more generally $\mathcal{P}^0(X) = X$, $\mathcal{P}^{\alpha+1}(X) = \mathcal{P}(\mathcal{P}^\alpha(X))$ and $\mathcal{P}^\lambda(X) = \bigcup_{\beta<\lambda} \mathcal{P}^\beta(X)$ for γ a limit ordinal.

1. wqo theory, basic bqo theory

Abstracting from the embeddability relation between order types, define a *quasi-order* to be a reflexive, transitive relation. Throughout this paper, the letters Q and R will range over quasi-ordered sets and classes. Various quasi-ordered spaces will be defined; in each case we will use the symbol $\leqq$ (perhaps with subscripts) to denote the quasi-order under consideration. If $q_1, q_2 \in Q$, write $q_1 < q_2$ to mean $q_1 \leqq q_2$ but $q_2 \nleqq q_1$, and write $q_1 \equiv q_2$ to mean $q_1 \leqq q_2$ and $q_2 \leqq q_1$. (All results could be done in terms of partial orderings (quasi-orderings where $\equiv \to =$) instead of quasi-orderings; we elect not to do this since it would mean continually taking equivalence classes.) Whenever a subset Q_1 of Q is defined, we assume that Q_1 is quasi-ordered as a subordering of Q.

We turn now to the definition of well-quasi-ordering, giving two equivalent formulations.

Q is *well-quasi-ordered* (wqo) $\leftrightarrow_{\text{df}}$

(i) for any sequence $\langle q_i \rangle_{i<\omega}$ of members of Q, $\exists i,\ j < \omega$: $i < j$ and $q_i \leqq q_j$, equivalently,

(ii) every descending sequence of members of Q is finite, and every antichain of members of Q is finite.

Thus, in these terms, the first of the two theorems listed in the introduction reads: *the class* $\mathfrak{M}$ *is* wqo *under the embeddability relation.*

Well-quasi-orderings were first studied by Higman in [5], where the equivalence of the two definitions (immediate from Ramsey's theorem) was observed. If $q \in Q$, let $Q_q = \{r \in Q : q \nleqq r\}$. From part (i) in the definition of wqo (which will be the version of wqo used from now on) we have immediately the following

Induction principle for well-quasi-orderings: If a proposition $\Phi(Q)$ is true of Q whenever it is true of each Q_q, $q \in Q$, and Φ is true of the empty (quasi-ordered) set, then $\Phi(Q)$ holds for all wqo Q.

Given arbitrary quasi-orderings Q, Q_1, Q_2, we now define quasi-orders on various spaces built up from them: Quasi-order the Cartesian product $Q_1 \times Q_2$ by the usual rule $\langle q_1, q_2\rangle \leqq \langle r_1, r_2\rangle \leftrightarrow_{\mathrm{df}} q_1 \leqq r_1$ and $q_2 \leqq r_2$. If $Q_1 \cap Q_2 = \varnothing$ then the *disjoint quasi-order on* $Q_1 \cup Q_2$ is defined by the rule $q \leqq r \leftrightarrow_{\mathrm{df}} \exists i$: $q, r \in Q_i$ and $q \leqq r$ as members of Q_i. Quasi-order $\bigcup_{\alpha \in \mathrm{On}} Q^\alpha$ by the rule

$\langle q_\beta\rangle_{\beta<\delta} \leqq \langle r_\alpha\rangle_{\alpha<\lambda} \leftrightarrow_{\mathrm{df}}$ there is a strictly increasing $f\colon \delta \to \lambda$ such that $q_\beta \leqq r_{f(\beta)}$ for each $\beta < \delta$.

Finally, $\mathcal{P}(Q)$ is quasi-ordered in two ways. For $X, Y \subseteq Q$, define $X \leqq_m Y$ $\leftrightarrow_{\mathrm{df}} \exists f\colon X \to Y$ with $x \leqq f(x)$, all $x \in X$, and $X \leqq_1 Y \leftrightarrow_{\mathrm{df}}$ in addition to the above, f is 1–1.

We now list for interest the preservation theorems for wqo's which are known. In his original paper [5], Higman showed that if Q is wqo, then so is $Q^{<\omega}$ (and hence so is the set of finite subsets of Q, under either $\leqq_m$ or $\leqq_1$). This theorem was strengthened in one direction by Kruskal [6], who showed that if Q is wqo then so is the collection of finite trees whose nodes are labelled by elements of Q (the quasi-order on this space is defined in § 2), and in another direction by Nash-Williams [7], who showed that if Q is wqo, then so is the subclass of $\bigcup_{\alpha \in \mathrm{On}} Q^\alpha$ consisting of those sequences which have finite range.

Note the finite character of these theorems. In the case of spaces built up from Q in an unrestricted infinite manner, the situation is different. We single out for this discussion what turns out to be the most natural such space, $\mathcal{P}(Q)$ under $\leqq_m$. In [10], Rado constructed a Q such that Q is wqo but $\mathcal{P}(Q)$ is not wqo. Working from Rado's counter-example (which will almost be given in the remarks below) Nash-Williams found a condition on quasi-ordered sets stronger than that of being well-quasi-ordered, namely that of being "better quasi-ordered" (bqo), which is preserved under passage from Q to $\mathcal{P}(Q)$. The condition given for Q to be bqo is combinatorial, but it amounts to saying that $\mathcal{P}^{\omega_1}(Q)$ is wqo, where $\mathcal{P}^{\omega_1}(Q)$ is quasi-ordered by the natural extension $\leqq_a$ of the $\leqq_m$ relation, defined as follows: for $X \in \mathcal{P}^\alpha(Q)$, $Y \in \mathcal{P}^\beta(Q)$ (the definition is by induction on $\alpha, \beta < \omega_1$), then $X \leqq_a Y \leftrightarrow_{\mathrm{df}}$

$\alpha = 0, \beta = 0$, and $X \leqq Y$ as elements of Q, or

$\alpha = 0, \beta > 0$, and $\exists Y' \in Y$, $X \leqq_a Y'$, or

$\alpha > 0, \beta > 0$, and $\forall X' \in X\ \exists Y' \in Y\ X' \leqq_a Y'$.

Assume a Q is given such that $\mathcal{P}^{\omega_1}(Q)$ is not wqo (to avoid a trivial notational problem, assume $q \in Q \to q \notin \mathcal{P}^{\alpha+1}(Q)$, $0 \leqq \alpha < \omega_1$). Accordingly, there is a sequence $\langle X_i\rangle_{i<\omega}$ of members of $\mathcal{P}^{\omega_1}(Q)$ such that whenever $i < j$,

$$X_i \not\leq_a X_j \;.$$

Set $I_0 = \{i\colon X_i \notin Q\}$. Observe now that it is possible to choose, for each $m \in I_0$ and each $n > m$ a set $X_{mn} \in X_m$, such that whenever $j \in I_0$ and $i < j < k$,

$X_i \not\leq_a X_{jk}$ and, if $i \in I_0$, $X_{ij} \not\leq_a X_{jk}$.

Now let $I_1 = \{\langle i, j\rangle\colon X_{ij} \notin Q\}$, As above, there are sets $X_{ijk} \in X_{ij}$ for all $\langle i, j\rangle \in I_1$ and all $k > j$, such that whenever $i < j < k < l$ and $\langle j, k\rangle \in I_1$,

$X_i \not\leq_a X_{jkl}$, and, if $i \in I_0$, $X_{ij} \not\leq_a X_{jkl}$, and, if $\langle i, j\rangle \in I_1$, $X_{ijk} \not\leq_a X_{jkl}$.

By continuing this process, the original bad sequence on $\mathcal{P}^{\omega_1}(Q)$ is reduced to a bad array, of a certain kind, on Q.

The following definitions of Nash-Williams should now appear more natural to the reader. A set B of strictly increasing finite sequences of non-negative integers is called a *block* just in case for every strictly increasing infinite sequence $\langle s_i\rangle_{i<\omega}$ of non-negative integers there is an $n < \omega$ with $\langle s_i\rangle_{i\leqq n} \in B$. If $t, u \in B$ then write $t \lhd u$ to mean for some $s_0 < s_1 < \cdots < s_n$ and some r, $0 \leqq r < n$, $t = \langle s_i\rangle_{0\leqq i\leqq r}$ and $u = \langle s_j\rangle_{1\leqq j\leqq n}$.

Q is *better-quasi-ordered* $\leftrightarrow_{\mathrm{df}}$ for every block B and every $f\colon B \to Q$ there are $t, u \in B$ with $t \lhd u$ and $f(t) \leqq f(u)$.

In the application of bqo theory to order types we will need (besides the results on trees in § 2) the following basic facts about bqo's, taken from [8] and collected into one theorem.

THEOREM 1.1. [8].

(i) Q bqo $\to Q$ wqo

(ii) Q *well ordered* $\to Q$ bqo

(iii) $Q = Q_1 \cup Q_2$ *and* Q_1, Q_2 bqo $\to Q$ bqo

(iv) Q_1, Q_2 bqo $\to Q_1 \times Q_2$ bqo

(v) Q bqo $\to Q^{<\omega}$ bqo

(vi) Q bqo $\to \mathcal{P}(Q)$ bqo (*under both* $\leqq_m$ *and* $\leqq_1$).

The preservation properties (iii)—(v) remain true with "bqo" replaced by "wqo" (see [5], [6]); (vi) is the distinguishing feature of bqo's.

We list another fact about bqo's which will be used; this one is an immediate consequence of the definition of bqo.

Homomorphism property for bqo's: If Q is bqo, $Q' \subseteq Q$, and there is an order preserving function taking Q' onto R, then R is bqo.

2. A generalization of the infinite tree theorem

We turn now to the main theorem of [8]. A *tree* is a set T, partially ordered by a relation $\leqq_T$, such that for each $x \in T$, $\{y \in T\colon y \leqq_T x\}$ is well ordered by $\leqq_T$. If there is an $x \in T$ such that $x \leqq_T y$ for all $y \in T$, then T is said to be *rooted* and x is called the *root* of T. Let $\mathcal{T}$ be the class of all trees T such that

(i) T is rooted, and

(ii) there are no paths in T of length $>\omega$.

For $T \in \mathcal{T}$, $x, y \in T$, let $x \cap y$ be the glb of x and y in T. We quasi-order the class $\mathcal{T}$ by the following embeddability relation: for $T_1, T_2 \in \mathcal{T}$,

$T_1 \leqq T_2 \leftrightarrow_{\mathrm{df}}$ there is a 1-1 function $f\colon T_1 \to T_2$ such that for all $x, y \in T_1$, $f(x \cap y) = f(x) \cap f(y)$ (i.e. if y and z are distinct immediate successors of x in T_1, then $f(y)$ and $f(z)$ occur above distinct immediate successors of $f(x)$ in T_2). Nash-Williams' infinite tree theorem can now be stated.

THEOREM 2.1. [8]. *$\mathcal{T}$ is* bqo.

Actually, the condition that the trees be rooted is not demanded in [8]; the wider theorem (with respect to a suitable quasi-order) thus obtained is an immediate corollary of 2.1 and 1.1 (vi). We have only put the restriction in to simplify matters. As for condition (ii) in the definition of $\mathcal{T}$, it automatically holds for all the trees in [8], since "tree" in that paper is taken to mean "graph with no cycles."

We will need to generalize this theorem. Define a Q-tree to be a pair (T, l) where T is a tree and $l\colon T \to Q$. l is intuitively a function which labels the nodes of T. Let $\mathcal{T}_Q$ be the class of Q-trees (T, l) such that $T \in \mathcal{T}$. Quasi-order $\mathcal{T}_Q$ by the following rule: if $(T_1, l_1), (T_2, l_2) \in \mathcal{T}_Q$, then

$(T_1, l_1) \leqq (T_2, l_2) \leftrightarrow_{\mathrm{df}} T_1 \leqq T_2$ by a function f such that for all $x \in T_1$, $l_1(x) \leqq l_2(f(x))$.

For the proof of the next theorem, which generalizes 2.1, we assume familiarity with [8].

THEOREM 2.2. Q bqo $\to \mathcal{T}_Q$ bqo.

To avoid introducing a large amount of notation applicable to this theorem only, and because the basic idea of the generalization is straightforward, it will be proved in the following way. We will prove revised versions of [8, Lems. 29–32, 37]. (In addition, we will claim without proof that various other lemmas of that paper can be made to accommodate Q-trees without any difficulties.) It is left to the reader to verify that these revised lemmas, together with the rest of [8], do yield a proof of Theorem 2.2.

For $T \in \mathcal{T}$, let $\rho(T)$ be the root node of T. If $x \in T$, $S(x)$ is the set of immediate successor nodes of x. If $(T, l) \in \mathcal{T}_Q$ and $x \in T$, define $\mathrm{br}_{(T,l)}(x)$ (abbreviated $\mathrm{br}(x)$; similar abbreviations will be made below) to be the Q-tree obtained from (T, l) by restricting T to nodes $\geqq_T x$ ($\mathrm{br}(x)$ is the *branch* of (T, l) with root node x). A branch X of (T, l) is said to be *strict* if and only if $X < (T, l)$.

For $x \in T$, $(T, l) \in \mathcal{T}_Q$, we define
$J(x) = \{\mathrm{br}(y)\colon y \in S(x)$ and $\mathrm{br}(y)$ strict$\}$, and
$K(x) = \{y\colon y \in S(x)$ and $\mathrm{br}(y) \equiv (T, l)\}$.

Let

$$\Gamma_{(T,l)}(x) = \langle J(x), \mathrm{Card}\,(K(x)), l(x)\rangle\ .$$

$\Gamma_{(T,l)}(x)$ is a member of the space $(\mathcal{P}(\mathcal{T}_Q) \times \mathrm{Card} \times Q)$; we quasi-order this space by the ordering induced from the natural ordering on the class of cardinals and the $\leqq_1$ ordering on $\mathcal{P}(\mathcal{T}_Q)$.

We put

$$\Theta((T, l)) = \{\Gamma_{(T,l)}(x)\colon x \in T\}\ .$$

$\Theta((T, l))$ belongs to the space $\mathcal{P}(\mathcal{P}(\mathcal{T}_Q) \times \mathrm{Card} \times Q)$. We quasi-order this space by the ordering induced from the quasi-order on $(\mathcal{P}(T_Q) \times \mathrm{Card} \times Q)$ by the $\leqq_m$ relation.

The following replaces Lemma 29 of [8].

LEMMA 29′. $\Theta((T_1, l_1)) \leqq \Theta((T_2, l_2)) \rightarrow (T_1, l_1) \leqq (T_2, l_2)$.

PROOF. We will define a function f embedding (T_1, l_1) into (T_2, l_2) by induction on T_1. Suppose for each $u \leqq_T v$ in T_1, $f(u)$ has been defined such that there are sets $K_1(u)$ and $K_2(u)$ with $K_1(u) \cup K_2(u) = K(f(u))$ and $K_1(u) \cap K_2(u) = \varnothing$, and $J(u) \leqq_1 (J(f(u)) \cup \{\mathrm{br}(x)\colon x \in K_1(u)\})$, and $\mathrm{Card}(K(u)) \leqq \mathrm{Card}\,(K_2(u))$; want to define f above v. Extend f to an embedding of the members of $J(v)$ into distinct members of $J(f(v)) \cup \{\mathrm{br}(x)\colon x \in K_1(v)\}$. Since $\mathrm{Card}\,(K(v)) \leqq \mathrm{Card}\,(K_2(v))$, it remains to be shown that for $y \in K(v)$, $z \in K_2(v)$, we can pick an $f(y) \in \mathrm{br}(z)$ satisfying the induction hypothesis. Pick a function j embedding (T_2, l_2) into $\mathrm{br}\,(z)$. For some $w \in T_2$,

$$\Gamma_{(T_1,l_1)}(y) \leqq \Gamma_{(T_2,l_2)}(w)\ .$$

Let $f(y) = j(w)$. Note that $l_1(y) \leqq l_2(w) \leqq l_2(f(y))$. Define

$$V = \{v \in J(w)\colon j(v) \in \mathrm{br}\,(x) \text{ for some } x \in K(j(w))\}\ .$$

Now let $K_1(y)$ be

$$\{z \in S(f(y))\colon j(v) \geqq_{T_2} z \text{ for some } v \in V\}\ ,$$

and let $K_2(y) = K(f(y)) - K_1(y)$. We see from the nature of the embedding j that $K_1(y)$ and $K_2(y)$ are as desired. The induction step is now complete, and the f thus defined is an embedding of (T_1, l_1) into (T_2, l_2), completing the lemma.

$(T, l) \in \mathcal{T}_Q$ is said to be *descensionally infinite* if and only if there is an infinite sequence $x_1 <_T x_2 <_T x_3 <_T \cdots$ of nodes of T such that $\mathrm{br}\,(x_1) >$

$\mathrm{br}(x_2) > \mathrm{br}(x_3) > \cdots$; otherwise (T, l) is *descensionally finite*. Let $\mathcal{F}_Q$ be the class of all descensionally finite members of $\mathcal{T}_Q$ and let $F(T, l)$ be the set of descensionally finite branches of (T, l). For $x \in T$, define

$$L(x) = \{\mathrm{br}(y) \colon y \in S(x) \text{ and } \mathrm{br}(y) \in F(T, l)\} ,$$
$$M(x) = \{y \colon y \in S(x) \text{ and } \mathrm{br}(y) \notin F(T, l)\} .$$

Let

$$\Delta_{(T,l)}(x) = \langle L(x), \mathrm{Card}\,(M(x)), l(x)\rangle .$$

Finally, put

$$\Phi_{(T,l)}(x) = \{\Delta_{(T,l)}(y) \colon x \leqq_T y\} .$$

$\Phi_{(T,l)}(x) \in \mathcal{P}(\mathcal{P}(\mathcal{T}_Q) \times \text{Cardinals} \times Q)$, which is quasi-ordered as in Lemma 29′.

The next lemma replaces Lemmas 30–32 of [8]. We assume Q is bqo.

LEMMA 32′. $(T, l) \in \mathcal{T}_Q$, $F(T, l)$ bqo $\rightarrow (T, l) \in \mathcal{F}_Q$.

PROOF. It must be checked that the proof in [8] still holds in the presence of the new factor Q. Suppose $(T, l) \notin \mathcal{F}_Q$. Pick any $x \in T$, $\mathrm{br}(x) \notin \mathcal{F}_Q$. Will show $\Phi_{(T,l)}(x) > \Phi_{(T,l)}(x_1)$ for some $x_1 >_T x$, $\mathrm{br}(x_1) \notin \mathcal{F}_Q$ (but then by continuing the process a descending sequence $\Phi_{(T,l)}(x) > \Phi_{(T,l)}(x_1) > \Phi_{(T,l)}(x_2) > \cdots$ can be obtained; this contradicts the fact that, in view of Q bqo, the hypothesis, and 1.1 (ii), (iv), and (vi), $\mathcal{P}(\mathcal{P}(F(T, l)) \times \text{Cardinals} \times Q)$ is bqo and hence wqo). Suppose there is no such x_1. Clearly for all $z \geqq_T x$, $\Phi_{(T,l)}(z) \leqq \Phi_{(T,l)}(x)$, so by assumption $\Phi_{(T,l)}(x) \equiv \Phi_{(T,l)}(u)$ for all $u \geqq_T x$ such that $\mathrm{br}(u) \notin \mathcal{F}_Q$. Since $\mathrm{br}(x) \notin \mathcal{F}_Q$, we can pick a $y >_T x$ such that $\mathrm{br}(y) \notin \mathcal{F}_Q$ and $\mathrm{br}(y) < \mathrm{br}(x)$. But we now claim that $\mathrm{br}(x) \leqq \mathrm{br}(y)$, a contradiction. To establish this claim, construct an f embedding $\mathrm{br}(x)$ into $\mathrm{br}(y)$ in the following fashion. Pick a $z \geqq_T y$ such that $\Delta_{(T,l)}(x) \leqq \Delta_{(T,l)}(z)$, let $f(x) = z$ and note that $l(x) \leqq l(f(x))$. Extend f to embed the members of $L(x)$ into distinct members of $L(z)$. For each $v \in M(x)$, $w \in M(z)$, $\Phi_{(T,l)}(v) \equiv \Phi_{(T,l)}(w)$, so our initial assumption allows us to repeat this process indefinitely, yielding an embedding of $\mathrm{br}(x)$ into $\mathrm{br}(y)$. This contradiction gives the lemma.

We have now reduced the theorem to the problem of showing $\mathcal{F}_Q$ is bqo, since by Lemma 32′, if $\mathcal{F}_Q$ is bqo then $\mathcal{F}_Q = \mathcal{T}_Q$. If $\mathfrak{U} \subseteq \mathcal{F}_Q$, define $\mathrm{Br}(\mathfrak{U})$ to be

$$\{\mathrm{br}_{(T,l)}(x) \colon (T, l) \in \mathfrak{U} \text{ and } \mathrm{br}_{(T,l)}(x) < (T, l)\}.$$

$\mathfrak{U}$ is said to be *well branched* whenever $\mathrm{Br}(\mathfrak{U})$ is wqo, and $\mathfrak{U}$ is *closed* just in case $\mathrm{Br}(\mathfrak{U}) \subseteq \mathfrak{U}$. In [8], Lemmas 38–42 reduce the problem of showing $\mathcal{F}$ is bqo to that of showing that every closed, well branched subset of $\mathcal{F}$ is bqo. In fact, it can be checked that the same arguments yield the analogous reduction for $\mathcal{F}_Q$ here.

The last place where visible changes are needed is Lemma 37. We replace it with the following paragraph.

Suspose $\mathfrak{U}$ is a well branched, closed subset of $\mathcal{F}_Q$ which is not bqo. Accordingly there is a barrier B and a bad $f: B \longrightarrow \mathfrak{U}$. By Lemma 29′, Θf is also bad. Therefore, by Lemma 26, there is a bad $f': B^2 \longrightarrow (\mathcal{P}(\mathfrak{U}) \times \text{Card} \times Q)$ such that for all $(s \cdot t) \in B^2$, $f'(s \cdot t) \in \Theta(f(s))$. We now apply Lemma 22 twice (once to Q, once to the cardinals) to get a barrier $C \subseteq B^2$ and a bad $f'': C \longrightarrow \mathcal{P}(\mathfrak{U})$, such that for all $t \in C$, $f''(t) =$ the first coordinate of $f'(t)$. Since $\mathfrak{U}$ is well branched, $\bigcup_{t \in C} f''(t)$ is wqo, so, applying Lemma 28, a barrier $D \subseteq B^3$ and a bad $g: D \longrightarrow \mathfrak{U}$ are obtained such that for all $(s \cdot t \cdot u) \in D$, $g(s \cdot t \cdot u)$ is a branch of $f(s)$ and $g(s \cdot t \cdot u) < f(s)$.

By Lemma 36, now, there is an f_1 which "warily foreruns" f. Lemmas 33–35 show that this process can be repeated indefinitely, which is impossible, demonstrating the falsity of the assumption above that $\mathfrak{U}$ is not bqo. No changes are needed in Lemmas 33–36, as trees can be replaced by Q-trees without affecting the proofs. This concludes the proof of Theorem 2.2.

We introduce now another, more natural, quasi-order on $\mathcal{T}_Q$, with the idea of working with it, instead of the quasi-order we have been considering, in § 4. For $(T_1, l_1), (T_2, l_2) \in \mathcal{T}_Q$, we let

$(T_1, l_1) \leqq_m (T_2, l_2) \leftrightarrow_{\text{df}}$ there is a strictly increasing function $f: T_1 \longrightarrow T_2$ such that $l_1(x) \leqq l_2(f(x))$ for all $x \in T_1$.

This ordering is analogous to the $\leqq_m$ relation on $\mathcal{P}(Q)$; it differs from the previously defined quasi-order on $\mathcal{T}_Q$ in that the embedding function f need no longer be 1–1 and f may take distinct successors of x into nodes which occur above the same successor of $f(x)$. We have now

COROLLARY 2.3. Q bqo $\longrightarrow \mathcal{T}_Q$ bqo *under* $\leqq_m$.

PROOF. If $(T_1, l_1) \leqq (T_2, l_2)$, then clearly $(T_1, l_1) \leqq_m (T_2, l_2)$, and the corollary follows from 2.2 and the homomorphism property.

In concluding this section, we would like to mention the main theorem of Nash-Williams' more recent paper [9]: *If Q is* bqo, *then so is* $\bigcup_{\alpha \in \text{On}} Q^\alpha$.

This is of course a strengthening of 1.1 (v). This result will come out below as a corollary to our main Theorem 4.8, when the order types under consideration are restricted to be ordinals. (We remark that our methods totally differ from the methods of [9].)

3. Order types; preliminaries, characterizations of $\mathcal{S}$ and $\mathfrak{M}$

The letters L, M, N are reserved for sets which are linearly ordered by relations $\leqq_L$, $\leqq_M$, $\leqq_N$, respectively. Tp (L) is the order type (order isomorphism type) of L. Variables φ, χ, ψ, θ will range over order types. The following embeddability relation between order types, which quasi-orders the class of order types, is the main object of study of this paper.

For order types φ and ψ, $\varphi \leqq \psi \leftrightarrow_{\mathrm{df}}$ for tp $(L) = \varphi$, tp $(M) = \psi$, there is a strictly increasing function taking L into M.

We assume familiarity with the notion of the *sum* $\varphi + \psi$ of order types φ and ψ. More generally, if L is a linearly ordered set and for each $x \in L$, φ_x is an order type, then define the *ordered sum* $\sum_{x \in L} \varphi_x$ to be tp (M), where M is obtained from L by replacing each point x of L with an ordered set of type φ_x. Recall that the *product* $\varphi \cdot \psi$ of φ and ψ is $\sum_{x \in M} \varphi_x$, where each $\varphi_x = \varphi$ and tp $(M) = \psi$. If ψ is an order type (or $\mathcal{R}$ a collection of order types) then we will often express the fact that $\varphi = \sum_{x \in L} \varphi_x$, where tp $(L) = \psi$ (tp $(L) \in \mathcal{R}$) by saying that φ is a ψ sum ($\mathcal{R}$ sum) of the φ_x's. If a subset L' of a linearly ordered set L is specified, L' is assumed to be ordered as a subordering of L. If $\varphi =$ tp (L) then the *converse* of φ, written φ^*, is tp (M), where there is a bijection $f: L \to M$ such that for all $x, y \in L$, $x \leqq_L y \leftrightarrow f(x) \geqq_M f(y)$. If tp $(L) = \alpha^*$ for some $\alpha \in \mathrm{On}$ (ordinals will often be taken as order types), that L is *conversely well ordered*. If $x, y \in L$ and $x <_L y$, denote as usual by (x, y) and $[x, y]$ the open and closed intervals of L determined by x and y. The notation (L^1, L^2) will be used to denote a Dedekind cut of L.

Let η be the order type of the rationals. An order type φ is said to be *scattered* if and only if $\eta \not\leqq \varphi$. Let $\mathcal{S}$ denote the class of scattered order types. The first theorem of this section is Hausdorff's inductive classification of $\mathcal{S}$.

THEOREM 3.1. (Hausdorff [4]). $\mathcal{S} = \bigcup_{\alpha \in \mathrm{On}} \mathcal{S}_\alpha$, *where*
$\mathcal{S}_0 =$ *the set of order types* $\{0, 1\}$, *and for* $\beta > 0$,
$\mathcal{S}_\beta = \{\varphi$: φ *is a well ordered or conversely well ordered sum of members of* $\bigcup_{\gamma < \beta} \mathcal{S}_\gamma\}$.

All results about scattered types will be proved essentially by induction on this hierarchy. In the following lemma we collect some simple, well known facts about order types which we will need.

LEMMA 3.2. (i) *A scattered sum of scattered types is scattered.*

(ii) *If* $\kappa \in \mathrm{RC}$ *and* $\kappa \leqq \sum_{y \in M} \varphi_y$, *then either* $\kappa \leqq$ tp (M) *or for some* y, $\kappa \leqq \varphi_y$.

(iii) *If* $\kappa \in \mathrm{RC}$, $\lambda < \kappa$, $L = \bigcup_{\gamma < \lambda} L_\gamma$, *and* $\kappa \leqq$ tp (L), *then* $\kappa \leqq$ tp (L_γ) *for some* γ.

(iv) *If* $\kappa \in \text{RC}$, $\varphi \in \mathcal{S}$, $\operatorname{Card} \varphi \geqq \kappa$, *then* $\kappa \leqq \varphi$ *or* $\kappa^* \leqq \varphi$.

PROOF. (i) is well-known, (ii) and (iii) are easily verified, and (iv) is a well-known consequence of 3.1.

The larger class of order types $\mathfrak{M}$ mentioned in the introduction will now be considered. $\mathfrak{M}$, the class of types which are the countable unions of scattered types, is defined formally as follows:

$\varphi \in \mathfrak{M} \leftrightarrow_{\text{df}}$ whenever $\operatorname{tp}(L) = \varphi$, then there exist subsets $L_1, L_2, \cdots, L_n, \cdots, (n < \omega)$ of L such that $L = \bigcup_{n<\omega} L_n$ and for each n, $\operatorname{tp}(L_n) \in \mathcal{S}$.

Thus clearly $\mathcal{S} \subset \mathfrak{M}$, $\eta \in \mathfrak{M}$, in fact any $\mathfrak{M}$ sum of members in $\mathfrak{M}$ is itself in $\mathfrak{M}$, and for any cardinal κ there are "κ-dense" order types in $\mathfrak{M}$. In the remainder of this section Galvin's characterization of $\mathfrak{M}$ will be presented.

We now define the order types $\eta_{\alpha\beta}$ mentioned in the introduction. $\eta_{\alpha\beta}$ will be defined if and only if α and β are uncountable regular cardinals (considered in the construction below as order types) and $\max\{\alpha, \beta\}$ is a successor cardinal ($\alpha = \beta$ is allowed). Call $\langle\alpha, \beta\rangle$ *admissible* if the above conditions hold.

Given an admissible $\langle\alpha, \beta\rangle$, it is convenient to choose first an auxiliary type $\sigma_{\alpha\beta}$. Suppose $\alpha = \gamma^+$, $\beta = \delta^+$. Then $\sigma_{\alpha\beta} = \gamma^* \cdot \delta$. Now suppose $\alpha \neq \gamma^+$ for all γ (and hence $\alpha < \beta$, α is weakly inaccessible, and $\beta = \delta^+$ for some δ). Then $\sigma_{\alpha\beta} = \sum_{x \in M} \varphi_x$, where $\operatorname{tp}(M) = \delta$, each $\varphi_x < \alpha^*$, and for each $\lambda < \alpha$ there is some x with $\varphi_x \geqq \lambda^*$ (this last condition is possible to satisfy since $\alpha < \beta$). Finally, suppose $\beta \neq \delta^+$ for all δ. Then $\sigma_{\alpha\beta} = (\sigma_{\beta\alpha})^*$.

$\eta_{\alpha\beta}$, now, will be $\operatorname{tp}(L)$, where $L = \bigcup_{n<\omega} L_n$ and the sets $L_0 \subset L_1 \subset \cdots \subset L_n \subset \cdots$ are chosen as follows:

(i) L_0 is a linearly ordered set of type $\sigma_{\alpha\beta}$;

(ii) L_{n+1} is obtained from L_n by inserting into each empty interval (x, y) of L_n a copy of $\sigma_{\alpha\beta}$.

The following theorem isolates the properties of $\eta_{\alpha\beta}$ which will be needed, and shows that the non-uniqueness of the construction of $\eta_{\alpha\beta}$ in the case that $\min\{\alpha, \beta\}$ is weakly inaccessible is unimportant inasmuch as the possible choices of $\eta_{\alpha\beta}$ are all equivalent. Assume for this theorem that $\eta_{\alpha\beta} = \operatorname{tp}(L)$, where $L = \bigcup_{n<\omega} L_n$ as above.

THEOREM 3.3. [3]. (i) $\eta_{\alpha\beta} \in \mathfrak{M}$

(ii) $\alpha^* \not\leqq \eta_{\alpha\beta}$, $\beta \not\leqq \eta_{\alpha\beta}$

(iii) *If* (x, y) *is an interval of* L, *then* $\operatorname{tp}(x, y) \geqq \alpha_0^*$, *all* $\alpha_0 < \alpha$, *and* $\operatorname{tp}(x, y) \geqq \beta_0$, *all* $\beta_0 < \beta$.

Conversely, if $\varphi \neq 0, 1$, $\varphi = \operatorname{tp}(M)$ *and* φ *and* M *satisfy* (i)–(iii) *in place of* $\eta_{\alpha\beta}$ *and* L *(where* α *and* β *are taken to be arbitrary), then* $\langle\alpha, \beta\rangle$ *is admissible and* $\varphi \equiv \eta_{\alpha\beta}$.

PROOF. (i) Clearly $\operatorname{tp}(L_{n+1})$ can be written $\sum_{x \in L_n} \varphi_x$, where each φ_x is either 1 or $\sigma_{\alpha\beta}+1$. Since $\sigma_{\alpha\beta} \in \mathfrak{S}$, the result follows from 3.2 (i) by induction on n.

(ii) We will show $\beta \not\leqq \eta_{\alpha\beta}$, and the case $\alpha^* \not\leqq \eta_{\alpha\beta}$ will be symmetric. Clearly $\beta \not\leqq \sigma_{\alpha\beta}$, so, using the sums given in (i) above, we obtain from 3.2 (ii) that $\beta \not\leqq \operatorname{tp}(L_n)$ for each n, by induction on n. Since $\beta > \omega$, it follows from 3.2 (iii) that $\beta \not\leqq \operatorname{tp}(L) = \eta_{\alpha\beta}$.

(iii) Will show it for β, and again a symmetric argument will hold for α^*. Suppose $\gamma < \beta$ and (iii) holds for all $\delta < \gamma$, and let an interval (x, y) of L be given. $x, y \in L_m$, for some m, and lest $\eta \leqq \operatorname{tp}(L_m)$ it is clear that the interval $[x, y]$ of L_m contains an empty (in L_m) subinterval (u, v). The L interval (x, y) thus embeds $\sigma_{\alpha\beta}$ and hence all cardinals $<\beta$; hence $\operatorname{cf}(\gamma) \leqq \operatorname{tp}(x, y)$. So can assume $\operatorname{cf}(\gamma) < \gamma$; to see that $\gamma \leqq \operatorname{tp}(x, y)$, embed $\operatorname{cf}(\gamma)$ into (x, y), express γ as the $\operatorname{cf}(\gamma)$ sum of smaller ordinals, and use the induction hypothesis to embed the smaller ordinals into the appropriate intervals.

For the converse, we are given φ and M, where by (i) we may take $M = \bigcup_{n \in \omega} M_n$, each $\operatorname{tp}(M_n) \in S$. Will first show that $\langle \alpha, \beta \rangle$ is admissible. If β, say, were not regular, property (iii) could be used to map β (expressed as a $\operatorname{cf}(\beta)$ sum of smaller ordinals) into ψ, contradicting (ii). Likewise α is regular. Since Card $M > 1$, repeated applications of (iii) yield $\omega, \omega^* \leqq \varphi$, so α and β must be uncountable. Now suppose $\max\{\alpha, \beta\}$ $(=\beta$, say) is not a successor cardinal. By (iii) then, Card $M \geqq \beta$, but since $\beta \in \mathrm{RC}$ and $\beta > \omega$, Card $M_n \geqq \beta$ for some n. But now by 3.2 (iv) we must have $\operatorname{tp}(M_n) \geqq \beta$ or $\operatorname{tp}(M_n) \geqq \beta^* \geqq \alpha^*$, contradicting (ii). So $\langle \alpha, \beta \rangle$ is admissible.

To show $\varphi \equiv \eta_{\alpha\beta}$ it suffices by symmetry to show that if Card $N > 1$ and N satisfies (iii), then $\varphi \leqq \operatorname{tp}(N)$. To do this, we will show that if $\operatorname{tp}(N_1) \in \mathfrak{S}$ and $\operatorname{tp}(N_1)$ satisfies (ii), then N_1 is embeddable in N by a function f such that for any Dekekind cut (N_1^1, N_1^2) of N_1 there is an interval (x, y) of N such that

$$z \in (x, y),\ u \in N_1^1,\ v \in N_1^2 \rightarrow f(u) <_N z <_N f(v)\ .$$

The proof is by induction on $\mathfrak{S}$. Assume by 3.1 that $\operatorname{tp}(N_1)$ is the, say, δ sum of smaller types for which the claim holds, where we have $\delta < \operatorname{tp}(N)$. Clearly δ can be mapped into N satisfying the Dedekind cut condition, and then the induction hypothesis can be used (every interval of N satisfies (iii)) to embed the smaller types into N in an appropriate manner, establishing the claim. Now, to embed M into N, first embed M_0 into N so that the Dedekind cut condition holds, then extend this to a map $M_0 \cup M_1$ into N satisfying the Dedekind cut condition (recall every interval of N satisfies (iii)), and so on, getting $\varphi \leqq \operatorname{tp}(N)$ as desired. This completes the proof of Theorem 3.3.

Theorem 3.3, together with the final paragraph in its proof, gives

COROLLARY 3.4. $\varphi \leqq \eta_{\alpha\beta} \leftrightarrow \varphi \in \mathfrak{M}, \alpha^* \not\leqq \varphi, \beta \not\leqq \varphi$.

The sense in which Hausdorff's 3.1 can be generalized to $\mathfrak{M}$ is shown in part (ii) of the next theorem. Let $\mathfrak{D}_{\alpha\beta} = \{\varphi : \varphi < \eta_{\alpha\beta}\}$.

THEOREM 3.5. [3]. (i) *A $\mathfrak{D}_{\alpha\beta}$ sum of members of $\mathfrak{D}_{\alpha\beta}$ is itself in $\mathfrak{D}_{\alpha\beta}$;*

(ii) $\mathfrak{D}_{\alpha\beta} = \bigcup_{\gamma < \max\{\alpha,\beta\}} (\mathfrak{D}_{\alpha\beta})_\gamma$, *where* $(\mathfrak{D}_{\alpha\beta})_0 = \{0, 1\}$, *and for $\delta > 0$, $\varphi \in (\mathfrak{D}_{\alpha\beta})_\delta \leftrightarrow \varphi$ is an α_0^* sum, for some $\alpha_0 < \alpha$, or a β_0 sum, for some $\beta_0 < \beta$, or an $\eta_{\alpha_0\beta_0}$ sum, for some $\langle\alpha_0, \beta_0\rangle \in \mathrm{On} \times \mathrm{On}$ with $\langle\alpha_0, \beta_0\rangle < \langle\alpha, \beta\rangle$, of members of $\bigcup_{\gamma<\delta} (\mathfrak{D}_{\alpha\beta})_\gamma$.*

PROOF. (i) Given $\mathrm{tp}(M) \in \mathfrak{D}_{\alpha\beta}$ and $\varphi_x \in \mathfrak{D}_{\alpha\beta}$ for each $x \in M$, we want to show $\sum_{x \in M} \varphi_x \in \mathfrak{D}_{\alpha\beta}$. It can be quickly seen from the previous theorem that $(\eta_{\alpha\beta})^2 \equiv \eta_{\alpha\beta}$, i.e., $\sum_{x \in L} \varphi_x \equiv \eta_{\alpha\beta}$, where each $\varphi_x = \mathrm{tp}(L) = \eta_{\alpha\beta}$. Thus clearly $\sum_{x \in M} \varphi_x \leqq \eta_{\alpha\beta}$. If $\eta_{\alpha\beta} \leqq \sum_{x \in M} \varphi_x$, then clearly either $\eta_{\alpha\beta} \leqq \mathrm{tp}(M)$ or some interval of $\eta_{\alpha\beta}$ (and hence $\eta_{\alpha\beta}$ itself) is $\leqq$ some φ_x. Both cases are ruled out, so we have $\sum_{x \in L} \varphi_x < \eta_{\alpha\beta}$.

(ii) Let $\mathcal{C}_{\alpha\beta} = \bigcup_{\gamma < \max\{\alpha,\beta\}} (\mathfrak{D}_{\alpha\beta})_\gamma$. If $\mathrm{tp}(L) \leqq \eta_{\alpha\beta}$, then $\mathrm{Card}\, L < \max\{\alpha, \beta\}$; consequently a $\mathcal{C}_{\alpha\beta}$ sum of members of $\mathcal{C}_{\alpha\beta}$ is itself in $\mathcal{C}_{\alpha\beta}$. Since $\alpha_0 < \alpha$, $\beta_0 < \beta$ and $\langle\alpha_0, \beta_0\rangle < \langle\alpha, \beta\rangle$ respectively imply $\alpha_0^* \in \mathfrak{D}_{\alpha\beta}$, $\beta_0 \in \mathfrak{D}_{\alpha\beta}$, and $\eta_{\alpha_0\beta_0} \in \mathfrak{D}_{\alpha\beta}$ by 3.3, we have $\mathcal{C}_{\alpha\beta} \subseteq \mathfrak{D}_{\alpha\beta}$ by part (i) of this theorem. Now suppose there is an L with $\mathrm{tp}(L) \in \mathfrak{D}_{\alpha\beta} - \mathcal{C}_{\alpha\beta}$. For $x, y \in L$, $x \leqq_L y$, we define

$$x \sim y \leftrightarrow \mathrm{tp}\big((x, y)\big) \in \mathcal{C}_{\alpha\beta} \, .$$

We also let $x \sim x$, and $x \sim y \leftrightarrow y \sim x$. Clearly $\sim$ is an equivalence relation which partitions L into intervals. Suppose $X \subseteq L$ is an equivalence class. By picking sequences co-initial and cofinal with X, $\mathrm{tp}(X)$ can be written as the $\alpha_0^* + \beta_0$ sum of types in $\mathcal{C}_{\alpha\beta}$, for some $\alpha_0 < \alpha$, $\beta_0 < \beta$. Since $\alpha_0^* + \beta_0 \in \mathcal{C}_{\alpha\beta}$, we obtain $\mathrm{tp}(X) \in \mathcal{C}_{\alpha\beta}$. Let L' be a subset of L obtained by picking one member out of each equivalence class. Claim each interval (u, v) of L' has type in $\mathfrak{D}_{\alpha\beta} - \mathcal{C}_{\alpha\beta}$; otherwise, the interval (u, v) of L would be a $\mathcal{C}_{\alpha\beta}$ sum of members of $\mathcal{C}_{\alpha\beta}$ and would thus itself have type in $\mathcal{C}_{\alpha\beta}$, contradicting the fact that $u \not\sim v$. Since $\mathrm{tp}(L') \in \mathfrak{D}_{\alpha\beta}$, there must be an interval (x_0, y_0) of L' which fails to embed some α_0^* for $\alpha_0 < \alpha$, or some β_0, for $\beta_0 < \beta$, by Theorem 3.3. Assume the latter case is true (the case $\alpha_0 < \alpha$ will be symmetric) and that every interval (u, v) of L' embeds all $\beta_1 < \beta_0$. Now pick $\alpha_0 \leqq \alpha$ to be the least ordinal δ such that some subinterval (x_1, y_1) of (x_0, y_0) fails to embed δ^*. Recall $\mathrm{tp}(x_1, y_1) \in \mathfrak{D}_{\alpha\beta} - \mathcal{C}_{\alpha\beta}$. Now $\mathrm{tp}(x_1, y_1)$, α_0, and β_0 clearly satisfy the hypothesis of the second half of 3.3 (in place of φ, α, β) so $\langle\alpha_0, \beta_0\rangle$ is admissible and $\eta_{\alpha_0\beta_0} \equiv \mathrm{tp}(x_1, y_1)$. But now, since $\langle\alpha_0, \beta_0\rangle < \langle\alpha, \beta\rangle$, $\eta_{\alpha_0\beta_0} \in \mathcal{C}_{\alpha\beta}$; hence

$\mathrm{tp}(x_1, y_1) \in \mathcal{C}_{\alpha\beta}$. This contradiction gives the theorem.

4. The main theorem

We now define the Q-labelled order types mentioned in the introduction. For Q quasi-ordered, a *Q-linear ordering* is a pair (L, l) where L is a linearly ordered set and l is a function from L into Q. Two Q-linear orderings (L_1, l_1) and (L_2, l_2) are called *Q-isomorphic* if and only if there is an order isomorphism h taking L_1 onto L_2 such that for all $x \in L_1$, $l_1(x) = l_2(h(x))$. A *Q-type* is the Q-isomorphism type of a Q-linear ordering. As in the case of order types, $\mathrm{tp}(L, l)$ is the Q-type of (L, l). We will let variables Φ, χ, Ψ, Θ range over Q-types.

We quasi-order the class of Q-types by the following embeddability relation, which is a natural extension of the one for order types: if $\Phi = \mathrm{tp}(L_1, l_1)$, $\Psi = \mathrm{tp}(L_2, l_2)$, then

$\Phi \leqq \Psi \leftrightarrow_{\mathrm{df}}$ there is a strictly increasing $f\colon L_1 \to L_2$ with $l_1(x) \leqq l_2(f(x))$ for all $x \in L_1$.

If a Q-type $\Phi = \mathrm{tp}(L, l)$ then $\mathrm{tp}(L)$ is called the *base* of Φ and is written $\mathrm{bs}(\Phi)$. The ordered sum $\sum_{x \in L} \Phi_x$ of Q-types is the naturally obtained Q-type whose base is $\sum_{x \in L} \mathrm{bs}(\Phi_x)$. The Q-type with base 0 will itself be written 0; for $q \in Q$, let $1_q = \mathrm{tp}(\{x\}, l)$, where $\{x\}$ is a one element linearly ordered set and $l(x) = q$.

If Q is a quasi-ordered set or class and φ is an order type, let $Q^{\varphi}(Q^{\leq\varphi}, Q^{\equiv\varphi})$ be the collection of all Q types Φ such that $\mathrm{bs}(\Phi) = \varphi$ $(\mathrm{bs}(\Phi) \leqq \varphi, \equiv \varphi)$. If $\mathcal{R}$ is a collection of order types let $Q^{\mathcal{R}} = \{\Phi\colon \Phi \in Q^{\varphi}$ for some $\varphi \in \mathcal{R}\}$.

To prove the main theorem (Q bqo $\to Q^{\mathfrak{M}}$ bqo), a class $\mathcal{H}(Q) \subset Q^{\mathfrak{M}}$ will first be defined, where $\mathcal{H}(Q)$ is singled out because its members will turn out to be representable in a certain way by Q^+-trees, for a certain $Q^+ \supset Q$.

If Φ is a Q-type, $\mathcal{U}$ is a set of Q-types, and κ is an infinite cardinal, then Φ is called a *$(\mathcal{U}, \kappa)$-unbounded sum* if and only if Φ can be written in the form $\sum_{x \in L} \Phi_x$, where $\mathrm{tp}(L) = \kappa$, $\{\Phi_x\colon x \in L\} = \mathcal{U}$, and

$$\forall x \in L\ \exists Y \subseteq L(\mathrm{Card}\ Y = \kappa \text{ and } \forall y \in Y\ \Phi_x \leqq \Phi_y)\ .$$

Similarly, if the same conditions hold except that $\mathrm{tp}(L) = \kappa^*$, then Φ is called a *$(\mathcal{U}, \kappa^*)$-unbounded sum.*

We make a simple observation about these sums.

LEMMA 4.1. *Suppose $\delta \in \mathrm{RC}$, $\kappa \leqq \delta$, Φ is a $(\mathcal{U}, \kappa)$-unbounded sum, Ψ is a $(\mathcal{V}, \delta)$ unbounded sum (or Φ is a $(\mathcal{U}, \kappa^*)$-unbounded sum, and Ψ is a $(\mathcal{V}, \delta^*)$-unbounded sum) and $\forall \Theta \in \mathcal{U}\ \exists \chi \in \mathcal{V}\ \Theta \leqq \chi$. Then $\Phi \leqq \Psi$.*

PROOF. Let the unbounded sums be given by $\Phi = \sum_{x \in L} \Phi_x$, $\Psi = \sum_{y \in M} \Psi_y$;

we want to find a function h embedding Φ into Ψ. Suppose h has been defined on an initial segment $\sum_{x<x_0} \Phi_x$ of Φ and embeds that segment into an initial segment $\sum_{y<y_0} \Psi_y$ of Ψ. By assumption $\exists y_1 \in M\ \Phi_{x_0} \leqq \Psi_{y_1}$. By unboundedness $\exists y_2 \geqq_M y_0\ \Psi_{y_1} \leqq \Psi_{y_2}$. Let h embed Φ_{x_0} into Ψ_{y_2}. Since $\delta \in \mathrm{RC}$ and $\kappa \leqq \delta$, for each $\gamma < \kappa$, a γ limit of initial segments of Ψ is itself an initial segment of Ψ, so this process can be continued for each $\gamma < \kappa$ to obtain an embedding of Φ into Ψ. The argument is symmetric in the case of κ^*, δ^* sums.

Suppose $\mathcal{R}$ is a set of Q-types and $\Psi \in \mathcal{R}^\varphi$ for some φ, i.e., $\Psi = \mathrm{tp}(L, l)$ where $l: L \rightarrow \mathcal{R}$ and $\mathrm{tp}(L) = \varphi$. We associate to Ψ a Q-type $\bar{\Psi}$ in the natural way, i.e.,

$$\bar{\Psi} = \sum_{x \in L} l(x) .$$

For Q an arbitrary quasi-ordered set, Φ is (Q, α, β)-*universal* if and only if $\Phi \in Q^{\equiv \eta_{\alpha\beta}}$, and whenever $\Psi \in Q^{\leqq \eta_{\alpha\beta}}$, then $\Psi \leqq \Phi$. If $\mathcal{R} \subset Q^{\mathfrak{M}}$ and Φ is $(\mathcal{R}, \alpha, \beta)$-universal for some α, β, then $\bar{\Phi}$ will be called an $(\mathcal{R}, \alpha, \beta)$-*shuffle*.

LEMMA 4.2. *If Φ is a $(\mathcal{U}, \alpha, \beta)$-shuffle, Ψ a $(\mathcal{V}, \gamma, \delta)$-shuffle, $\langle\alpha, \beta\rangle \leqq \langle\gamma, \delta\rangle$, and $\forall \chi \in \mathcal{U}\ \exists \Theta \in \mathcal{V}\ \chi \leqq \Theta$, then $\Phi \leqq \Psi$.*

PROOF. We have $\eta_{\alpha\beta} \leqq \eta_{\gamma\delta}$. The lemma follows quickly from the definition of $(\mathcal{V}, \gamma, \delta)$-universal and the fact that for $\Phi_1, \Psi_1 \in \mathcal{V}^{\mathfrak{M}}$, $\Phi_1 \leqq \Psi_1 \rightarrow \bar{\Phi}_1 \leqq \bar{\Psi}_1$.

We will now define a class $\mathcal{H}(Q) \subset Q^{\mathfrak{M}}$. $\mathcal{H}(Q) = \bigcup_{\alpha \in \mathrm{On}} \mathcal{H}_\alpha(Q)$, where $\mathcal{H}_0(Q) = \{0\} \cup \{1_q: q \in Q\}$, and if $\alpha > 0$, $\Phi \in \mathcal{H}_\alpha(Q) \leftrightarrow_{\mathrm{df}}$ for some $\mathcal{U} \subseteq \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q)$ either

(i) Φ is a $(\mathcal{U}, k)$ or a $(\mathcal{U}, k^*)$-unbounded sum for some $k \in \mathrm{RC}$, or

(ii) Φ is a $(\mathcal{U}, \alpha, \beta)$-shuffle for some (admissible) $\langle\alpha, \beta\rangle$.

If Q is a quasi-ordered set or class, we will choose a quasi-ordered class $Q^+ \supset Q$ in the following way. Add to Q new elements a_k, b_k, for each $k \in \mathrm{RC}$, and also elements $c_{\alpha\beta}$ for each admissible pair $\langle\alpha, \beta\rangle$ (passing first to an isomorphic copy of Q if there are not enough new objects to choose from; it will be assumed below that this has not been necessary). Quasi-order Q^+ as the disjoint union of the quasi-ordered sets Q, $\{a_k: k \in \mathrm{RC}\}$, $\{b_k: k \in \mathrm{RC}\}$, and $\{c_{\alpha\beta}: \langle\alpha, \beta\rangle \text{ admissible}\}$, where

$$a_k \leqq a_\lambda \longleftrightarrow b_k \leqq b_k \longleftrightarrow k \leqq \lambda, \text{ and } c_{\alpha\beta} \leqq c_{\gamma\delta} \longleftrightarrow \langle\alpha, \beta\rangle \leqq \langle\gamma, \delta\rangle .$$

For $T \in \mathcal{T}$, recall that $\rho(T)$ is the root of $\mathcal{T}$ and that for $x \in T$, $S(x)$ is the set of immediate successors of x. If Q is a quasi-ordered set, $q \in Q$, and $\mathcal{B} \subseteq \mathcal{T}_Q$, let $[q: \mathcal{B}]$ be a pair $(T, l) \in \mathcal{T}_Q$ such that $l(\rho(T)) = q$ and $\{\mathrm{br}\,(x): x \in S(\rho(T))\} = \mathcal{B}$ (assume a convention whereby the trees in $\mathcal{B}$ are disjoint in such a construction). For $q \in Q$, let 1^q be some one element tree, labelled by q.

To each $\Phi \in \mathcal{H}(Q)$ we assign, by induction on $\mathcal{H}(Q)$, a $T(\Phi) \in \mathcal{T}_{Q^+}$, as follows:

(i) $T(0)$ = the empty Q^+-tree, and for $q \in Q$, $T(1_q) = 1^q$.

(ii) Suppose $T(\Psi)$ has been defined for each $\Psi \in \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q)$, and $\Phi \in \mathcal{H}_\alpha(Q) - \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q)$. There are then by definition sets $\mathcal{U} \subseteq \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q)$ such that either

(1) For some $\lambda \in \mathrm{RC}$, Φ is a $(\mathcal{U}, \lambda)$-unbounded sum; or

(2) For some $\lambda \in \mathrm{RC}$, Φ is a $(\mathcal{U}, \lambda^*)$-unbounded sum; or

(3) For some admissible (γ, δ), Φ is a $(\mathcal{U}, \gamma, \delta)$-shuffle.

To define $T(\Phi)$, pick some $\mathcal{U}$ as above. If $\mathcal{U}$ satisfies (1), let $T(\Phi) = [a_\lambda; \{T(\Theta): \Theta \in \mathcal{U}\}]$, if $\mathcal{U}$ satisfies (2), let $T(\Phi) = [b_\lambda; \{T(\Theta): \Theta \in \mathcal{U}\}]$, and if $\mathcal{U}$ satisfies (3), let $T(\Phi) = [c_{\gamma\delta}; \{T(\Theta): \Theta \in \mathcal{U}\}]$.

The next theorem reduces the bqo question for $\mathcal{H}(Q)$ to that for T_{Q^+}.

THEOREM 4.3. *If $\Phi \in \mathcal{H}_\alpha(Q)$, $\Psi \in \mathcal{H}_\beta(Q)$, and $T(\Phi) \leqq_m T(\Psi)$, then $\Phi \leqq \Psi$.*

PROOF. Assume the theorem holds for all $\langle\alpha_0, \beta_0\rangle \in \mathrm{On} \times \mathrm{On}$ such that $\langle\alpha_0, \beta_0\rangle < \langle\alpha, \beta\rangle$. Let $T(\Phi) = (T_1, l_1)$ and $T(\Psi) = (T_2, l_2)$ and let $f: T_1 \to T_2$ be a $\leqq_m$ embedding of $T(\Phi)$ into $T(\Psi)$. It can be assumed that $f(\rho(T_1)) = \rho(T_2)$ since otherwise the theorem holds by induction and the observation that if $x \in S(\rho(T_2))$, then $\mathrm{br}_{(T_2, l_2)}(x) = T(\chi)$ for some $\chi \in \bigcup_{\gamma<\beta} \mathcal{H}_\gamma(Q)$, $\chi \leqq \Psi$.

Case 1. $T(\Phi)$ is the empty tree, or for some $q \in Q$, $l_1(\rho(T_1)) = q$. The theorem is clear in the first case. If $l_1(\rho(T_1)) = q$, then $l_2(\rho(T_2)) = r$, where $r \in Q$ and $r \geqq q$, so $\Phi = 1_q \leqq 1_r = \Psi$.

Case 2a. $l_1(\rho(T_1)) = a_\delta$, some $\delta \in \mathrm{RC}$. Then $l_2(\rho(T_2)) = a_k$ for some $k \in \mathrm{RC}$, $k \geqq \delta$. Also, Φ is a $(\mathcal{U}, \delta)$-unbounded sum and Ψ is a $(\mathcal{V}, k)$-unbounded sum, where

$\mathcal{U} = \{\Theta$: for some $x \in S(\rho(T_1))$, $T(\Theta) = \mathrm{br}_{(T_1, l_1)}(x)\}$, and

$\mathcal{V} = \{\chi$: for some $y \in S(\rho(T_2))$, $T(\chi) = \mathrm{br}_{(T_2, l_2)}(x)\}$.

Clearly the function f yields a $\leqq_m$ embedding of each $\mathrm{br}(x)$, $x \in S(\rho(T_1))$, into some $\mathrm{br}(y)$, $y \in S(\rho(T_2))$. By the induction hypothesis, then,

$$\forall \Theta \in \mathcal{U}\ \exists \chi \in \mathcal{V}\ \Theta \leqq \chi\,.$$

By 4.1, then, we have $\Phi \leqq \Psi$.

Case 2b. $l_1(\rho(T_1)) = b_\delta$, some $\delta \in \mathrm{RC}$. Then $l_2(\rho(T_2)) = b_k$, some $\kappa \in \mathrm{RC}$, $k \geqq \delta$, and the argument is symmetric to 2a.

Case 3. $l_1(\rho(T_1)) = c_{\alpha\beta}$, some admissible $\langle\alpha, \beta\rangle$. Then $l_2(\rho(T_2)) = c_{\gamma\delta}$, where $\langle\alpha, \beta\rangle \leqq \langle\gamma, \delta\rangle$. Thus, Φ is a $(\mathcal{U}, \alpha, \beta)$-shuffle and Ψ is a $(\mathcal{V}, \gamma, \delta)$-shuffle, where $\mathcal{U}$ and $\mathcal{V}$ are as in case 2a. As before, the induction hypothesis gives $\forall \Theta \in \mathcal{U}\ \exists \chi \in \mathcal{V}\ \Theta \leqq \chi$. By 4.2, then, $\Phi \leqq \Psi$.

This completes the proof.

COROLLARY 4.4. Q bqo $\longrightarrow \mathcal{H}(Q)$ bqo.

PROOF. We need to show that Q^+ is bqo. $\{a_k: k \in \text{RC}\}$ and $\{b_k: k \in \text{RC}\}$ are bqo by 1.1 (ii), and $\{c_{\alpha\beta}: \langle\alpha, \beta\rangle \text{ admissible}\}$ is bqo by 1.1 (ii) and (iv), so Q^+ is bqo by 1.1 (iii). Hence, by 2.3, T_{Q^+} is bqo under $\leqq_m$. From 4.3 and the homomorphism property, $\mathcal{H}(Q)$ is bqo.

The next theorem, which enables one to jump over the $\eta_{\alpha\beta}$'s in the hierarchy 3.5 (ii), was first proved in the case of η (i.e., $\eta_{\omega_1\omega_1}$) by Galvin and the author, independently. The generalization here to types $\eta_{\alpha\beta}$ is due to Galvin.

THEOREM 4.5. *Suppose Q is* wqo, *and $\Phi \in Q^{\leqq\eta_{\alpha\beta}}$. Then Φ is a $\mathfrak{D}_{\alpha\beta}$ sum of types 1_q, where $q \in Q$, and (R, α_0, β_0)-universal types, where $R \subseteq Q$, $\langle\alpha_0, \beta_0\rangle \leqq \langle\alpha, \beta\rangle$.*

PROOF. By the induction principle for wqo sets we can assume the theorem is true whenever $q \in Q$ and Q is replaced by Q_q $\left(=\{r \in Q: q \not\leqq r\}\right)$. We are given $\Phi \in Q^{\leqq\eta_{\alpha\beta}}$, $\Phi = \text{tp}\,(L, l)$. For $y, z \in L$, $y <_L z$, write $y \sim z$ to mean that for every subinterval (u, v) of (y, z), $\text{tp}\left((u, v), l\right)$ is a $\mathfrak{D}_{\alpha\beta}$ sum of types 1_q, $q \in Q$ and (R, α_0, β_0)-universal types, $R \subseteq Q$, $\langle\alpha_0, \beta_0\rangle \leqq \langle\alpha, \beta\rangle$. Putting also $x \sim x$ and $x \sim y \leftrightarrow y \sim x$, it is clear that $\sim$ is an equivalence relation which partitions L into intervals. Let $|x|$ be the equivalence class of x. Picking sequence co-initial and cofinal with $|x|$ yields $|x|$ as a $\gamma^* + \delta$ sum of 1_q's and subintervals (u, v) of $|x|$, where $\gamma^* < \alpha^*$, $\delta < \beta$ by 3.3 (ii) (and thus $\gamma^* + \delta \in \mathfrak{D}_{\alpha\beta}$). Since $u \sim v$ for each such subinterval (u, v), $\text{tp}\,(|x|, l)$ is itself a $\mathfrak{D}_{\alpha\beta}$ sum of 1_q's and (R, α_0, β_0)-universal types, by 3.5 (i). Therefore, if L itself is a single equivalence class, Φ is as desired. So suppose there are $x, y \in L$, $x \not\sim y$. Let $|L| = \{|x|: x \in L\}$, where $|L|$ is linearly ordered by the rule $|x| \leqq_{|L|} |y| \leftrightarrow x \leqq_L y$. Claim if $\left(|x|, |y|\right)$ is an interval of $|L|$, then

(i) $\text{tp}\left(|x|, |y|\right) \equiv \eta_{\alpha\beta}$, and

(ii) $\forall q \in Q \ \exists z \in L \ \ |z| \in \left(|x|, |y|\right)$ and $l(z) \geqq q$.

Proof of (i). Otherwise $\text{tp}\left(|x|, |y|\right) \in \mathfrak{D}_{\alpha\beta}$, which would give $x \sim y$ by 3.5 (ii).

Proof of (ii). Otherwise for some $r \in Q$,

$$\{l(z): |z| \in (|x|, |y|)\} \subseteq Q_r\ ,$$

but then by the induction hypothesis, $\{z: |z| \in (|x|, |y|)\}$ is part of one equivalence class, contradicting part (i) of the claim. Now we claim that Φ itself is (Q, α, β)-universal (which will give the theorem). Given $\text{tp}(M, l') \in Q^{\leqq\eta_{\alpha\beta}}$, since $(\eta_{\alpha\beta})^2 \equiv \eta_{\alpha\beta}$ there clearly is an embedding $f: M \longrightarrow |L|$ such that for each $x \in M$ there is an interval $\left(|y|, |z|\right) = |L|_x$ of $|L|$ with

$$\{x_0: f(x_0) \in |L|_x\} = \{x\}\ ,$$

where in addition the $|L|_x$'s are disjoint. By part (ii) of the previous claim, it is clear that f gives rise to an embedding $f'\colon M \to L$ such that $l'(y) \leqq l(f'(y))$, all $y \in M$. Thus $\mathrm{tp}\,(M, l') \leqq \Phi$, and Φ is (Q, α, β)-universal, completing 4.5.

The next theorem, which reduces the bqo problem for $Q^{\mathfrak{M}}$ to that for $\mathcal{H}(Q)$, is the point in the proof where the necessity for using Q-types rather than just order types becomes apparent. We first prove a lemma.

LEMMA 4.6. *If $\chi \in \mathcal{H}_\gamma(\mathcal{H}(Q))$ (considering $\mathcal{H}(Q)$ as quasi-ordered under the embeddability relation of $Q^{\mathfrak{M}}$) then $\bar{\chi} \in \mathcal{H}(Q)$.*

PROOF. By induction on γ. If $\gamma = 0$ the result is clear. Suppose $k \in \mathrm{RC}$ and χ is a $(\mathfrak{U}, k)$-unbounded sum, where

$$\mathfrak{U} \subseteq \bigcup_{\beta<\gamma} \mathcal{H}_\beta(\mathcal{H}(Q)) \,.$$

Then $\bar{\chi}$ is a corresponding $(\{\bar{\chi}_i\colon \chi_i \in \mathfrak{U}\}, k)$ sum, which is clearly unbounded since $\chi_1 \leqq \chi_2 \to \bar{\chi}_1 \leqq \bar{\chi}_2$. Each $\bar{\chi}_i \in \mathcal{H}(Q)$ by the induction hypothesis and hence $\bar{\chi} \in \mathcal{H}(Q)$. A symmetric argument applies if χ is a $(\mathfrak{U}, k^*)$-unbounded sum. Finally, if χ is a $(\mathfrak{U}, \delta, \lambda)$-shuffle for some $\mathfrak{U} \subseteq \bigcup_{\beta<\gamma} \mathcal{H}_\beta(\mathcal{H}(Q))$ and some δ, λ, it is not hard to see from the definition of $(\mathfrak{U}, \delta, \lambda)$-universal that $\bar{\chi}$ is a $(\{\bar{\chi}_i\colon \chi_i \in \mathfrak{U}\}, \delta, \lambda)$-shuffle. Again, each $\bar{\chi}_i \in \mathcal{H}(Q)$ by induction, so $\bar{\chi} \in \mathcal{H}(Q)$, and the lemma is shown.

THEOREM 4.7. *If Q is* bqo, *$\Phi \in Q^{\leqq \eta_{\alpha\beta}}$, then Φ is a finite sum of members of $\mathcal{H}(Q)$.*

PROOF. Assume the theorem holds for all $\langle \alpha_0, \beta_0 \rangle \in \mathrm{On} \times \mathrm{On}$ such that $\langle \alpha_0, \beta \rangle < \langle \alpha, \beta \rangle$. We will first show the theorem under the assumption that $\Phi \in Q^{\mathfrak{D}_{\alpha\beta}}$. We are clearly done if $\Phi \in Q^{(\mathfrak{D}_{\alpha\beta})_0}$; assume that $\gamma \geqq 1$, $\Phi \in Q^{(\mathfrak{D}_{\alpha\beta})_\gamma}$, and the theorem holds for all $\delta < \lambda$. By 3.5 (ii), $\mathrm{bs}\,(\Phi)$ is either a β_0 sum (some $\beta_0 < \beta$) or an α_0^* sum (some $\alpha_0 < \alpha$) or an $\eta_{\alpha_0\beta_0}$ sum (some $\langle \alpha_0, \beta_0 \rangle < \langle \alpha, \beta \rangle$) of members of $\bigcup_{\delta<\gamma} (\mathfrak{D}_{\alpha\beta})_\delta$.

Case 1. $\mathrm{bs}\,(\Phi)$ is a β_0 sum. Suppose the theorem fails for Φ. There exists, then, a least ordinal λ such that for some $\Theta \in Q^{(\mathfrak{D}_{\alpha\beta})_\gamma}$, Θ is not a finite sum of $\mathcal{H}(Q)$'s but Θ can be written $\sum_{x\in L} \Theta_x$, where each Θ_x is a finite sum of $\mathcal{H}(Q)$'s and $\mathrm{tp}\,(L) = \lambda$. Clearly λ must be infinite. We claim now that $\lambda \in \mathrm{RC}$. To see this, write Θ in the form $\sum_{y\in M} \Theta^y$, where $\mathrm{tp}\,(M) = \mathrm{cf}\,(\lambda)$ and each Θ^y is a $<\lambda$ sum of Θ_x's. By minimality of λ, each Θ^y must be a finite sum of $\mathcal{H}(Q)$'s; thus, by minimality of λ again, $\mathrm{cf}\,(\lambda) = \lambda$, establishing the claim. Now Θ is a λ sum of finite sums of $\mathcal{H}(Q)$'s, so Θ can be written $\sum_{x\in L} \Theta'_x$, where each $\Theta'_x \in \mathcal{H}(Q)$ and $\mathrm{tp}\,(L) = \lambda$. We claim now that

$$(*) \qquad \exists x_0 \in L\, \forall y, z \in L(x_0 \leqq_L y \leqq_L z \longrightarrow \exists u \in L(z \leqq_L u \text{ and } \Theta'_y \leqq \Theta'_u)) \,.$$

Suppose there is no such x_0, i.e., for arbitrarily large $y \in L$,

$$\exists z \in L(y <_L z \text{ and } \forall u \in L(z \leqq_L u \longrightarrow \Theta'_y \nleqq \Theta'_u)) \,.$$

It follows then that we can choose an increasing sequence $y_1, y_2, \cdots, y_n, \cdots$ $(n < \omega)$ of members of L such that

$$m < n \longrightarrow \Theta'_{y_m} \nleqq \Theta'_{y_n} \,.$$

This, however, contradicts the fact (from the hypothesis and 4.4) that $\mathcal{H}(Q)$ is bqo and hence wqo. Thus $(*)$ holds, and so, since $\sum_{x \geqq x_0} \Theta'_x$ is a $(\{\Theta'_x : x \geqq x_0\}, \lambda)$-unbounded sum, and since each $\Theta'_x \in \mathcal{H}(Q)$, we have that $\sum_{x \geqq x_0} \Theta'_x \in \mathcal{H}(Q)$. Now $\sum_{z < x_0} \Theta'_z$ is a finite sum of $\mathcal{H}(Q)$'s by minimality of λ, and thus Θ itself is a finite sum of $\mathcal{H}(Q)$'s, contradicting the initial assumption and giving Case 1.

Case 2. bs (Φ) is an α_0^* sum. The argument is symmetric to that of Case 1.

Case 3. bs (Φ) is an $\eta_{\alpha_0\beta_0}$ sum. Φ is then an $\eta_{\alpha_0\beta_0}$ sum of finite sums of $\mathcal{H}(Q)$'s, by the induction hypothesis for $\mathcal{D}_{\alpha\beta}$. Upon collecting these sums, Φ becomes expressed as a certain φ sum of $\mathcal{H}(Q)$'s, where, in view of $(\eta_{\alpha_0\beta_0})^2 \equiv \eta_{\alpha_0\beta_0}$, we have $\varphi \equiv \eta_{\alpha_0\beta_0}$. Via this expression for Φ we obtain a χ such that

$$\chi \in (\mathcal{H}(Q))^\varphi \text{ and } \bar{\chi} = \Phi \,.$$

Now $\mathcal{H}(Q)$ is bqo by 4.4, and bs $(\chi) \equiv \eta_{\alpha_0\beta_0}$, $\langle \alpha_0, \beta_0 \rangle < \langle \alpha, \beta \rangle$, so by the initial induction hypothesis,

$$\chi = \chi_0 + \chi_1 + \cdots + \chi_n \,,$$

where each $\chi_n \in \mathcal{H}(\mathcal{H}(Q))$. By 4.6, each $\bar{\chi}_n \in \mathcal{H}(Q)$, and thus

$$\Phi = \bar{\chi} = \bar{\chi}_0 + \bar{\chi}_1 + \cdots + \bar{\chi}_n$$

is as desired, completing Case 3, and giving the theorem in the case that $\Phi \in Q^{\mathcal{D}_{\alpha\beta}}$.

Considering now the general case where $\Phi \in Q^{\leqq \eta_{\alpha\beta}}$, we have by 4.5 that Φ is a φ sum, for some $\varphi \in \mathcal{D}_{\alpha\beta}$, of 1_q's, $q \in Q$, and (R, γ, δ)-universal types, $R \subseteq Q$, $\langle \gamma, \delta \rangle \leqq \langle \alpha, \beta \rangle$. Since every (R, γ, δ)-universal type is clearly an $\mathcal{H}(Q)$, Φ is a φ sum of $\mathcal{H}(Q)$'s. As in Case 3 above, this sum gives rise to a $\chi \in (\mathcal{H}(Q))^\varphi$ such that $\bar{\chi} = \Phi$. Since $\mathcal{H}(Q)$ is bqo and bs $(\chi) = \varphi \in \mathcal{D}_{\alpha\beta}$, we have by the first part of this theorem that χ is a finite sum of $\mathcal{H}(\mathcal{H}(Q))$'s. Applying 4.6 as in Case 3, we obtain that $\Phi = \bar{\chi}$ is a finite sum of $\mathcal{H}(Q)$'s. This completes the proof of 4.7.

THEOREM 4.8. Q bqo $\rightarrow Q^{\mathfrak{M}}$ bqo.

PROOF. If $\Phi \in Q^{\mathfrak{M}}$ then bs $(\Phi) \leqq \eta_{\alpha\beta}$, some α, β, by 3.4, so by 4.7 each $\Phi \in Q^{\mathfrak{M}}$ is a finite sum of members of $\mathcal{H}(Q)$. Letting $f\colon (\mathcal{H}(Q))^{<\omega} \rightarrow Q^{\mathfrak{M}}$ be defined by

$$f(\langle \Phi_i \rangle_{i<n}) = \textstyle\sum_{i<n} \Phi_i \,,$$

f is a homomorphism whose range is thus $Q^{\mathfrak{M}}$. By 4.4 and 1.1 (v), $(\mathcal{H}(Q))^{<\omega}$ is bqo, so it follows from the homomorphism property that $Q^{\mathfrak{M}}$ is bqo.

Noting that $\mathfrak{M} \cong A^{\mathfrak{M}}$, where A is some one element quasi-ordered (and hence better-quasi-ordered) set, we obtain

COROLLARY 4.9. *$\mathfrak{M}$ is* bqo.

The class $\mathcal{H}(Q)$ was defined structurally for purposes of representation by Q^+-trees. It turns out from 4.7 that, assuming Q is bqo, $\mathcal{H}(Q)$ has a more familiar description. Call a Q-type Φ *additively indecomposable* (AI) if $\Phi = \Phi_1 + \Phi_2$ implies $\Phi \leqq \Phi_1$ or $\Phi \leqq \Phi_2$.

COROLLARY 4.10. *If Q is* bqo *then $\mathcal{H}(Q)$ is the class of additively indecomposable members of $Q^{\mathfrak{M}}$.*

PROOF. It is easy to verify that $(\mathcal{U}, k)$ and $(\mathcal{U}, k^*)$-unbounded sums, as well as $(\mathcal{R}, \alpha, \beta)$-shuffles, are always AI; thus every $\Phi \in \mathcal{H}(Q)$ is AI (where in fact the hypothesis that Q is bqo is not needed). We will omit the straightforward proof of the converse, other than to say that it involves showing from 4.7, by induction on γ, that if $\Phi \in \mathcal{H}_\gamma(Q)$ is a $(\mathcal{U}, k)$-unbounded sum and $\exists \chi \in \mathcal{U}\ \chi \equiv \Phi$, then Φ is an $(\mathcal{R}, \alpha, \beta)$-shuffle for some $\mathcal{R} \subset \mathcal{H}(Q)$, α, β.

In the remainder of this section we will determine the number (up to equivalence) of types in $\mathfrak{M}$ of power k. For Q an arbitrary quasi-ordered set, $q \in Q$, let $|q| = \{r \in Q : r \equiv q\}$ and let $|Q| = \{|q| : q \in Q\}$. k will always be an infinite cardinal below.

THEOREM 4.11. *If Q is* bqo, *$\operatorname{Card}|Q| \leqq k$, and $0 \leqq \alpha < k^+$, then $\operatorname{Card}|Q^\alpha| \leqq k$.*

PROOF. Suppose the theorem holds for all $\alpha_0 < \alpha$. Also assume in the case of α that the theorem holds for all Q_q, $q \in Q$, by the wqo induction principle. Since $\operatorname{Card}|Q^\beta| \leqq k$ for each $\beta < \alpha$, $\operatorname{Card}|\bigcup_{\beta<\alpha} Q^\beta| \leqq k$. α can clearly be assumed to be infinite, and we distinguish two cases.

Case 1. $\alpha \notin$ RC. Since $\bigcup_{\beta<\alpha} Q^\beta$ is bqo by 4.8, the induction hypothesis gives

$$\operatorname{Card}|(\textstyle\bigcup_{\beta<\alpha} Q^\beta)^{\operatorname{cf}(\alpha)}| \leqq k .$$

There is a natural homomorphism from $(\bigcup_{\beta<\alpha} Q^\beta)^{\operatorname{cf}(\alpha)}$ onto $\bigcup_{\beta\leqq\alpha} Q^\beta$, and it follows that $\operatorname{Card}|Q^\alpha| \leqq k$.

Case 2. $\alpha \in$ RC. Given $\Phi \in Q^\alpha$, by 4.7 Φ is of the form $\Phi_1 + \Phi_2$, where $\Phi_1 \in \bigcup_{\beta<\alpha} Q^\beta$ and $\Phi_2 \in \mathcal{H}(Q)$. Let $Z_\alpha = Q^\alpha \cap \mathcal{H}(Q)$; the theorem reduces to showing $\operatorname{Card}|Z_\alpha| \leqq k$. Now suppose $\operatorname{Card}|Z_\alpha| > k$. We claim there is a $\Phi \in Z_\alpha$ such that

$$(*) \qquad \operatorname{Card}\{|\Psi| : \Psi \in Z_\alpha \text{ and } \Phi \not\leqq \Psi\} > k .$$

Suppose there is no such Φ. There will then be a strictly increasing sequence $\langle\Phi_\gamma\rangle_{\gamma<k^+}$ of members of Z_α, since if we are given $\langle\Phi_\gamma\rangle_{\gamma<\delta}$ for $\delta < k$, where $\gamma_1 < \gamma_2 \to \Phi_{\gamma_1} < \Phi_{\gamma_2}$, by a cardinality argument there will be a Φ_δ such that $\gamma < \delta \to \Phi_\gamma < \Phi_\delta$. By regularity of k^+, now, there is a $\lambda < k^+$ with

$$\{q\colon \exists\gamma < k^+ \, 1_q \leqq \Phi_\gamma\} = \{q\colon \exists\gamma < \lambda \, 1_q \leqq \Phi_\gamma\} .$$

Since each $\Phi_\gamma \in Z_\alpha$, however, it follows from 4.1 that $\Phi_{\lambda+1} \leqq \Phi_\lambda$, a contradiction, and the existence of a $\Phi \in Z_\alpha$ satisfying $(*)$ is established. Write $\Phi = \operatorname{tp}(L, l)$, where $\operatorname{tp}(L) = \alpha$ and $l\colon L \to Q$. If $\Phi \not\leqq \Psi$ for $\Psi \in Z_\alpha$, then by 4.1, $\exists x \in L \, 1_{l(x)} \not\leqq \Psi$. Since $k^+ \in \mathrm{RC}$ there will be an $x_0 \in L$ and $H \subseteq \{|\Psi|\colon \Psi \in Z_\alpha$ and $\Phi \not\leqq \Psi\}$ such that Card $H > k$ and

$$|\Psi| \in H \longrightarrow 1_{l(x_0)} \not\leqq \Psi .$$

But now

$$|\Psi| \in H \longrightarrow \Psi \in (Q_{l(x_0)})^\alpha ,$$

and, applying the wqo induction hypothesis, we get Card $H \leqq$ Card $|(Q_{l(x_0)})^\alpha| \leqq k$, a contradiction. This completes the proof of Theorem 4.11.

From 4.11 it is immediate that if Q is bqo and Card $|Q| \leqq k$, then Card $|\mathcal{P}(Q)| \leqq k$, where $\mathcal{P}(Q)$ can be taken as quasi-ordered either by $\leqq_m$ or by $\leqq_1$.

THEOREM 4.12. *If $\langle\alpha, \beta\rangle$ is admissible,* Q bqo, Card $|Q| \leqq k$ *and* $\max\{\alpha, \beta\} \leqq k$, *then* Card $|Q^{\leqq\eta_{\alpha\beta}}| \leqq k$.

PROOF. Assume the theorem holds for all $\langle\alpha_0, \beta_0\rangle < \langle\alpha, \beta\rangle$. We will first show that for each $\gamma < \max\{\alpha, \beta\}$, Card $|Q^{(\mathfrak{D}_{\alpha\beta})_\gamma}| \leqq k$, by induction on γ. If $\gamma = 0$ it is clear. Letting

$$\mathfrak{A} = \{\alpha_0^*\colon \alpha_0 < \alpha\} \cup \{\beta_0\colon \beta_0 < \beta\} \cup \{\eta_{\alpha_0\beta_0}\colon \langle\alpha_0, \beta_0\rangle < \langle\alpha, \beta\rangle\} ,$$

we have by 3.5 that if $\Phi \in Q^{(\mathfrak{D}_{\alpha\beta})_\gamma}$, then Φ can be written in the form $\bar{\Psi}$, where

$$\Psi \in (Q^{\bigcup_{\delta<\gamma}(\mathfrak{D}_{\alpha\beta})_\delta})^{\mathfrak{A}} .$$

Now $Q^{\bigcup_{\delta<\gamma}(\mathfrak{D}_{\alpha\beta})_\delta}$ is bqo by 4.8 and has power $\leqq k$ (up to equivalence) by the $\mathfrak{D}_{\alpha\beta}$ induction hypothesis and the fact that $\delta < k$, so for each $\varphi \in \mathfrak{A}$,

$$\text{Card}\,|(Q^{\bigcup_{\delta<\gamma}(\mathfrak{D}_{\alpha\beta})_\delta})^\varphi| \leqq k ,$$

by 4.11 if φ is an α_0^* or a β_0, and by the main induction hypothesis if φ is an $\eta_{\alpha_0\beta_0}$. Hence clearly, since Card $\mathfrak{A} \leqq k$,

$$\text{Card}\,|(Q^{\bigcup_{\delta<\gamma}(\mathfrak{D}_{\alpha\beta})_\delta})^{\mathfrak{A}}| \leqq k ,$$

and it follows that Card $|Q^{(\mathfrak{D}_{\alpha\beta})_\gamma}| \leqq k$ for each γ, implying in turn that Card $|Q^{\mathfrak{D}_{\alpha\beta}}| \leqq k$.

Turning now to the general case, if $\Phi \in Q^{\leq \eta_{\alpha\beta}}$ then by 4.5, Φ is a $\mathcal{D}_{\alpha\beta}$ sum of 1_q's, $q \in Q$, and (R, α_0, β_0)-universal types, $R \subseteq Q$, $\langle \alpha_0, \beta_0 \rangle \leqq \langle \alpha, \beta \rangle$. Let

$$\mathcal{B} = \big(Q \cup (\mathcal{P}(Q) \times \alpha + 1 \times \beta + 1)\big) .$$

$\mathcal{B}$ is quasi-ordered as a disjoint union of Q and $\big(\mathcal{P}(Q) \times \alpha + 1 \times \beta + 1\big)$, where the latter space is quasi-ordered by the ordering induced from the usual orders on $\alpha + 1$, $\beta + 1$, and the $\leqq_m$ order on $\mathcal{P}(Q)$. From 4.1 and the above, it follows that $Q^{\leq \eta_{\alpha\beta}}$ is a certain homomorphic image of $\mathcal{B}^{\mathcal{D}_{\alpha\beta}}$. $\mathcal{B}$ is bqo by various parts of 1.1. Card $|\mathcal{P}(Q)| \leqq k$ by the remark following 4.11, so it follows that Card $|\mathcal{B}| \leqq k$. Hence, by the $\mathcal{D}_{\alpha\beta}$ part of this theorem Card $|\mathcal{B}^{\mathcal{D}_{\alpha\beta}}| \leqq k$. It follows then that Card $|Q^{\leq \eta_{\alpha\beta}}| \leqq k$, completing the proof.

Letting $\mathfrak{M}_{(k)}$, $(\mathcal{S}_{(k)})$ be the set of order types in $\mathfrak{M}(\mathcal{S})$ which are of power k, we obtain

COROLLARY 4.13. *If Q is* bqo, Card $|Q| \leqq k^+$, *then*

$$\text{Card}\,|\mathcal{S}_{(k)}| = \text{Card}\,|\mathfrak{M}_{(k)}| = \text{Card}\,|Q^{\mathcal{S}_{(k)}}| = \text{Card}\,|Q^{\mathfrak{M}_{(k)}}| = k^+ .$$

PROOF. Card $|\mathcal{S}_{(k)}| \geqq k^+$ in view of the ordinals in $\mathcal{S}_{(k)}$. It is not hard to see that $\varphi \in \mathfrak{M}_{(k)} \leftrightarrow \varphi \leqq \eta_{k^+k^+}$, and the theorem then follows from 4.12.

We would like to state without proof one further result on the structure of $\mathcal{S}$ and $\mathfrak{M}$ under embeddability. The *dimension* of a partially ordered set P (written dim P) is the least cardinal λ such that P can be isomorphically embedded in the direct product of λ linear orderings. In [12] it was mentioned as an open question whether there exist partially ordered, well-quasi-ordered sets having uncountable dimension. The following theorem shows there are such sets of arbitrary dimension.

THEOREM 4.14. dim $|\mathcal{S}_{(k)}|$ = dim $|\mathfrak{M}_{(k)}| = k^+$.

5. Conclusion

The results in this paper are expected to be of help in getting a better understanding of the structure of the members of $\mathfrak{M}$ (especially the countable order types). Results of this nature have been proved recently; we will state two of them here (proofs will be given elsewhere).

(1) If tp $(L) = \varphi \in \mathfrak{M}$, then there is an $n < \omega$ such that if $L = \bigcup_{i<r} L_i$ for some $r < \omega$, then for some $\leqq n$-element set $I \subseteq \{0, 1, \cdots, r\}$, $\varphi \leqq$ tp $(\bigcup_{i \in I} L_i)$.

(2) Let $\mathcal{C}_0$ = the set of order types $\{0, 1\}$

$$\mathcal{C}_\alpha = \{\varphi \colon \varphi = \textstyle\sum_{i<\omega} \varphi_i \text{ or } \varphi = (\sum_{i<\omega} \varphi_i)^* ,$$
$$\text{where each } \varphi_i \in \textstyle\bigcup_{\beta<\alpha} \mathcal{C}_\beta \text{ and } i < j \rightarrow \varphi_i \leqq \varphi_j\} .$$

Then for each countable scattered type φ, φ is strongly indecomposable if and only if $\varphi \equiv \psi$ for some $\psi \in \bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha$.[2] (There is an analogous theorem characterizing the strongly indecomposable members of $\mathfrak{M}$.)

In addition to the problem of obtaining more detailed information about the order types in $\mathfrak{M}$, there is of course the general problem of determining which classes of mathematical objects are wqo (or bqo) under naturally defined quasi-orders. Some examples of spaces for which the answer is not known are the class of all graphs, quasi-ordered by immersability (Nash-Williams [8], a generalized form of Vázsonyi's trivalent graph conjecture), the class of all graphs, quasi-ordered by homomorphic contraction (Mader), and the class of all trees which have countable limit levels, quasi-ordered by a natural extension of the ordering on $\mathcal{T}$.

Finally, the question arises as to how the order types outside of $\mathfrak{M}$ behave under embeddability. For instance, it is known that there are order types not in $\mathfrak{M}$ which do not embed any uncountable set of reals, and to which the Dushnik-Miller-Sierpinski construction, thus, does not apply. In some specific cases the answers are known to be independent of ZFC.

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[2] Immediate from this theorem is the closely related Conjecture IV of [2].