Minicourse on The Axioms of Zermelo and Fraenkel

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Abstract

These are the notes for a minicourse of four hours in the DAAD-summerschool in mathematical logic 2020. To be held virtually in Göttingen, October 10-16, 2020.

1 The Zermelo Fraenkel Axioms

Axiom: a plausible postulate, seen as evident truth that needs not be proved.

Logical axioms, e.g. $A \wedge B \rightarrow A$ for any statements A, B. We assume the Hilbert axioms, see below.

Now we focus on the non-logical axioms, the ones about the underlying set theoretic structure. Most of nowadays mathematics can be justified as built within the set theoretic universe or one of its extensions by classes or even a hierarchy of classes.

In my lecture I will report on the Zermelo Fraenkel Axioms. If no axioms are mentioned, usually the proof is based on these axioms. However, be careful, for example nowaday's known and accepted proof of Fermat's last theorem by Wiles uses "ZFC and there are infinitely many strongly inaccessible cardinals". It is conjectured that ZFC or even the Peano Arithmetic will do.

The language of the Zermelo Fraenkel axioms is the first order logic $\mathscr{L}(\tau)$ with signature $\tau = \{\in\}$. The symbol $\in$ stands for a binary relation.

However, we will work in English and thus expand the language by many defined notions. If you are worried about how this is compatible with the statement "the language of set theory is $\mathcal{L}(\{\in\})$" you can study the theory of language expansions by defined symbols. E.g., you may read page 85ff in [15]. Such expansions do not increase the expressive power. We expand the axioms just by the definitions of the intruduced additional symbols. This procedure does not increase the amount of what can be proved in the old language in comparison to before the expansion.

Recall $\mathscr{L}(\{\in\})$

In a minimal setup, there are the propositional conjunctions $\wedge$, $\neg$; and the only quantifier $\exists$. There are two relation symbols: $\in$ and $=$. There are infinitely many variables v_i, $i \in \mathbb{N}$.

Atomic formulae are: $v_i = v_j$, $v_i \in v_j$.

If φ, $\psi \in \mathscr{L}(\tau)$ then $(\varphi \wedge \psi)$, $\neg\varphi$, $\exists v_i \varphi$ are formulae as well. Each formula is built up in finitely many steps.

This minimal language is good in oder to carry out inductions over formulae: we have only two atomic steps and three types of induction steps.

However, we freely use $\forall$, $\vee$, $\rightarrow$ in formulae in their obvious meanings.

Definition 1.1. The function fr $:: \mathcal{L}(\{\in\}) \rightarrow \mathcal{P}_{<\mathbb{N}}(\{v_n \ : \ n \in \mathbb{N}\})$ denotes the assigment of the set of free variables in φ:

(1) For $i, j \in N$, $\mathrm{fr}(v_i = / \in v_j) = \{v_i, v_j\}$,

(2) $\mathrm{fr}((\varphi * \psi)) = \mathrm{fr}(\varphi) \cup \mathrm{fr}(\psi)$ each logical connective $*$,

(3) $\mathrm{fr}(\exists v_i \varphi) = \mathrm{fr}(\varphi) \smallsetminus \{v_i\}$.

Convention 1.2. The formula $\forall x(x \in y \rightarrow \varphi)$ is abbreviated by $\forall x \in y\, \varphi$. The formula $\exists x(x \in y \wedge \varphi)$ in abbreviated by $\exists x \in y\, \varphi$.

$\exists! x\varphi$ is an abbreviation of the formula $\exists x\varphi \wedge \forall y(\varphi(\frac{y}{x}) \rightarrow y = x)$. Here replacing x by y is denoted by $\frac{y}{x}$, and x and y are names for variables, i.e., members of the set $\{v_n \ : \ n \in \mathbb{N}\}$.

Definition 1.3. The axioms of Zermelo and Fraenkel ZFC are the following two schemes and the following seven single axioms:

1. Extensionality

"Two sets who have the same elements are equal."

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow y = x).$$

2. Regularity/Foundation.

The relation $\in$ is well-founded.

$$\forall x(\exists y \in x \rightarrow \exists y \in x(\neg\exists z(z \in y \wedge z \in x))).$$

3. Comprehension Scheme/Separation Scheme.

"Any definable subclass of a set is a set."

Assume that $\varphi \in \mathscr{L}(\in)$ and $\mathrm{fr}(\varphi) \subseteq \{x, z, w_1, w_2, \dots, w_n\}$. Then

$$\forall z \forall w_1 \dots \forall w_n \exists y \forall x (x \in y \leftrightarrow x \in z \wedge \varphi).$$

The variables $w_1, \dots, w_n$ are called parameters φ.

We write $y = \{x \in z : \varphi(x, z, \bar{w})\}$.

4. Pairing

"For any two sets, there is a set containing both of them as elements."

$$\forall x \forall y \exists z (x \in z \wedge y \in z).$$

Now $\{x\}$ and $\{x, y\}$ are defined.

5. Union

"For any set (of sets), the union of its elements is a subset of a set."

$$\forall F \exists A \forall Y \forall x (x \in Y \wedge Y \in F \rightarrow x \in A).$$

Now $\bigcup F = \{x : \exists Y (Y \in F \wedge x \in y)\}$ is defined.

We define the operation $\cup\colon V \times V \rightarrow V$ by $x \cup y = \bigcup\{x, y\}$.

6. Replacement Scheme

"Given a set and a definable operation, the image is a subset of a set."

We assume that $\varphi \in \mathscr{L}(\in)$ and $\mathrm{fr}(\varphi) \subseteq \{x, y, A, w_1, w_2, \dots, w_n\}$.

$$\forall A \forall w_1 \dots \forall w_n \Big(\forall x \in A \exists ! y \varphi(x, y) \rightarrow \exists Y (\forall z) (\exists x \in A \varphi(x, z) \rightarrow z \in Y) \Big)$$

7. Infinity.

"There is a infinite set."

$$\exists x (\emptyset \in x \wedge \forall y \in x (y \cup \{y\} \in x)).$$

8. Powerset

"Each set has a power set."

$$\forall x \exists y \forall z (z \subseteq x \rightarrow z \in y).$$

$x \subseteq y$ abbreviates $\forall z \in x (z \in y)$.

$\mathcal{P}(x)$ stands for $\{y : y \subseteq x\}$.

This ends the list ZF. In summary extensionality, foundation, comprehension, pairing, union, replacement, power, infinity.

The *Hilbert logic rules* say that
Existence

$$\exists x(x = x).$$

Some expositions count this as a ZFC-axiom. However, since the Hilbert proof rules are correct only in non-empty structures, we have that any model $(V, \in)$ of ZFC is not empty.

Recall the Hilbert proof rules:

(1) B1 Premise rule: $T \vdash \varphi$ if $\varphi \in T$.
(2) B2 Equality axioms.
(3) B3 Propositions tautologies (excluded middle belongs to these).
(4) B4 existence axiom: $\varphi(\frac{t}{x}) \to \exists x\varphi(x)$, if t can be substituted for x in φ.
(4) B5 Modus ponens.
(6) B6 Introduction of $\exists$ in premise. If $\varphi \to \psi$ is proved and $x \notin \mathrm{fr}(\psi)$, then $\exists x\varphi \to \psi$ is proved.

These are the classical proof rules.

Subsystems of ZF and one Extension

ZFC Axioms 1–8 and the Axiom of Choice AC.

ZF Axioms 1–8

ZF^- Axioms 1,3–8.

$\mathsf{ZF}-\mathsf{P}$ Axioms 1–7.

Exercise 1.4. *Suppose we have* ZF *in the background. Does* $\mathsf{ZF}-\mathsf{P}$ *has a set sized model?*

Exercise 1.5. *Suppose we renounce replacement. Are there small models of the remaining axioms?*

Exercise 1.6. *How often can we iterate the operation* $x \mapsto x \cup \mathcal{P}(x)$*? What do you do if the step counter comes to a limit ordinal?*

Exercise 1.7. *Read about the von Neuman hierarchy* V_α*,* $\alpha \in \mathrm{On}$ *(an ordinal) and* H_κ*. For the latter you have to know the notion of a cardinal and the notion of the transitive closure.*

Exercise 1.8. *We write* $\leftrightarrow$ *instead of second* $\rightarrow$ *in the replacement scheme. Then we can renounce the comprehension scheme. (So is the exposition of ZFC on the Israeli wikipedia)*

Definition 1.9. A collection of sets, e.g. $\{x : \varphi(x, y)\}$ is called *class*. A class can be a set or not, in the latter case we say it is a *proper class*.

Project Proposal 1.10. The axiom system given by von Neumann, Bernays, and Gödel, short NBG. Explain their axioms. Discuss why they are conservative over ZF.

Present also MK, Morse–Kelley set theory. Here more classes are used.

Is ZF consistent?

Theorem 1.11. *Gödel's second incompleteness theorem for* ZF*: If* ZF *is consistent, then* ZF *does not prove its consistency.*

If you follow Reinhard Kahle's course on the incompleteness theorems for the Peano Arithmetic, then you can modify the (long) proof given there to a proof for ZF.

Theorem 1.12. *(Russell) The class of all sets is not a set.* $\neg\exists x \forall z (z \in x)$*.*

Proof. Assume the contrary. Then $\forall z(z \in x)$. By comprehension $\{z \in x : z \notin z\} = u$ is a set. Then

$$u \in u \leftrightarrow u \notin u.$$

□

Definition 1.13. The class $V = \{x : x \text{ set}\}$ is called the universe. We can also write $V = \{x : x = x\}$.

We think of $(V, \in)$ as of a model. Why is this legitimate?

An application of the Gödel completeness theorem gives

$$\mathsf{ZF} \vdash (\mathrm{Con}(\mathsf{ZF}) \rightarrow \exists M (M, \in) \models \mathsf{ZF})$$

1.1 Literature on Set Theory

General: Kunen's classic [8], the "new Kunen" [9][1], Jech [4] are the classical bibles on set theory. For large cardinals, Kanamori [5] is a first reference. For very basic beginnings, Enderton's book [1] is good and also Levy's book [10]. There are many, however, I do not have any pdf of these. You may want to learn from the wikipedia and wolfram mathematics.

My favorite book on the Axiom of Choice is [3]. There is only one small drawback for very advanced students: The book does not contain the full forcing proof that the prime ideal principle is strictly weaker than the axiom of choice. However, a proof for models with urelements in given.

Moore's book [12] is good on history, but contains some mathematical errors. I recommend to read its review on Mathscinet before using mathematical statements from this book.

A mainly 1963 book on many equivalents of AC is [16]. The second edition is from 1985. The book reflects the pre-forcing time. Of course, there could be another book on on equivalents to the Prime ideal Principle. I do not know of a book of this topic.

If you want to understand the non-implications, here are my favorite sources to forcing and independence proofs:

Jech again [4], and Kunen again [9].

Here is the proof about the prime ideal theorem's weakness: Halpern and Levy [2], Todorcevic Introduction to Ramsey Spaces [19].

To understand non-Choice and Determinacy Kechris [6], Moschovakis [13]

To understand constructibility Schindler [17]

2 The Axiom of Choice

Most mathematicians consider the use of the Axiom of Choice as a legitimate method of proof.

Axiom of Choice, AC

"Every set X of non-empty sets has a function $f\colon X \to \bigcup X$ such that

$$(\forall y \in X)(f(y) \in y).''$$

So we have $X = \{y : y \in X\}$, X a set, and for any $y \in X$, $y \neq \emptyset$, i.e., $(\exists z)(z \in y)$.

[1] I not not have a pdf file of this.

A partial formalisation in $\mathcal{L}(\{\in\})$ would be

$$\forall X\Big(\forall y \in X \exists z \in Y \rightarrow$$
$$\exists f(\text{“}f \text{ is a function”} \wedge \operatorname{dom}(f) = X \wedge \operatorname{rge}(f) \subseteq \bigcup X \wedge \forall y \in X f(y) \in y)\Big)$$

Now one would formalise ordered pairs (y, z) as $\{\{y\}, \{y, z\}\}$, identify the function f with its graph $\{(y, z) : (y, z) \in f\}$ and also express $z = f(y)$ as a formula $\varphi(f, y, z)$. Also the clauses “f is a function”, “$\operatorname{dom}(f) = X$” and “$\operatorname{rge}(f) \subseteq \bigcup X$” can be expressed in $\mathcal{L}(\{\in\})$.

Definition 2.1. A *choice function on a set* X is a function $f\colon X \to \bigcup X$ such that $(\forall y \in X)(y \neq \emptyset \rightarrow f(y) \in y)$.

So we can reformulate AC as: “Every set of non-empty sets has a choice function.”

In some cases we can prove the existence of a choice function without invoking AC. Here are two examples:

Example 2.2. Suppose X is a finite set of non-empty sets. By induction on the finite size, we can prove that there is a choice function.

Example 2.3. Suppose that X is an arbitrary set of finite non-empty sets of real numbers. Then we have a choice function given by $y \mapsto \min(y)$. Every finite non-empty subset of the real numbers has a unique minimal element.

A small example in which our constructive methods break down, is the following:

Example 2.4. Suppose that X is an infinite (maybe countable) set of sets of the form $\{a, b\}$, and a and b are subsets of $\mathbb{R}$. Then the minimum $\min(a)$ is not defined and we cannot say that we pick a out of $\{a, b\}$ if e.g. $\min(a) < \min(b)$.

Paul Cohen proved in 1963 that there is a model of all the Zermelo Fraenkel axioms (ZF) in which there is a set X as in this example that does *not* have a choice function.

Also if we let X to be the power set of $\mathbb{R}$ without the empty set, so $X = \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$, then we cannot just define a choice function on X. We invoke AC to get one.

2.1 The Vitali set

Let μ denote the Lebesgue measure on $\mathbb{R}$. We let for $r, s \in \mathbb{R}$,

$$r \sim s \text{ if } s - r \in \mathbb{Q}.$$

The relation $\sim$ is an equivalence relation on $\mathbb{R}$. By AC we can pick a choice function on the set

$$\{r/\sim\colon r \in \mathbb{R}\} = \{r/\sim\colon r \in [0,1]\}$$

and the image of this choice function is a set of representatives $M \subseteq [0,1]$ for $\mathbb{R}/\sim$.

We have

$$\mathbb{R} = \bigcup\{M + q \,:\, q \in \mathbb{Q}\}$$

Here we write $M + r$ for the set $\{m + r \,:\, m \in M\}$ and call $M + r$ the shift of M by r of less precise a shift of M.

Now, for a contradiction, suppose that M were Lebesgue measurable.

Since $\mu([0,1]) = 1$ and since countably many shifts of M cover $[0,1]$, by σ-additivity, $\mu(M) > 0$.

So

$$\mu([0,2]) \geq \mu(\bigcup\{M_q \,:\, q \in \mathbb{Q} \cap [0,1]\}) = \sum_{q \in \mathbb{Q} \cap [0,1]} \mu(M) = \infty.$$

Contradiction.

Project Proposal 2.5. A paradoxial decomposition of the sphere. The Banach Tarksi paradoxon. Literature: E.g. Jech, The axiom of Choice.

Project Proposal 2.6. The Nielsen–Schreier Theorem. Every subgroup of a free group is free. See e.g. [18]

2.2 Three of the most useful equivalent forms of AC

Definition 2.7. Let A be a set. A linear order $<$ on A is called a well-order, if every non-empty subset of A has a minimal element.

Example 2.8. $(\mathbb{N}, <)$. Counterexamples $(\mathbb{R}, <)$, $(\mathbb{Z}, <)$.

Definition 2.9. The well-ordering principle says: Every set can be well-ordered. More formal, on every set A there is a well-order $< \subseteq A \times A$.

Definition 2.10. The following statement is called *Zorn's Lemma*: Let $(P, <)$ be a partially ordered set such that every chain in P has an upper bound. (So in particular P is not empty.) Then P has a *maximal element*, i.e., there is $m \in P$ such that $(\forall p \in P)(p \not> m)$.

End of second lecture on October 13 2020.

Definition 2.11. Let $\mathcal{F}$ be a set of sets. We say that $\mathcal{F}$ has *finite character* if for each X,

$$X \in \mathcal{F} \text{ if and only if every finite subset of } X \text{ is an element of } \mathcal{F}.$$

Definition 2.12. The following statement is called *Tukey's Lemma*: Let $\mathcal{F}$ be a non-empty set of sets. If $\mathcal{F}$ has finite character, then $(\mathcal{F}, \subseteq)$ has a maximal element.

2.3 Background: Ordinals and Transfinite Recursion

Definition 2.13. (in ZF minus foundation) A set α is called an *ordinal*, if

$$(\forall x \in \alpha)(x \subseteq \alpha)$$

(the latter ist called: α is transitive) and

$$(\alpha, \in) \text{ is a well-order.}$$

Remark 2.14. We define the notion of an ordinal in the background theory ZF. Then we can be a bit more thrifty and stipulate: A set α is called an ordinal, if

$$(\forall x \in \alpha)(x \subseteq \alpha)$$

(the latter ist called: α is transitive) and

$$(\alpha, \in) \text{ is a linear order.}$$

In ZF, both definitions are equivalent.

Lemma 2.15. *The empty set is an ordinal.*

Lemma 2.16. *If α is an ordinal, then also $s(\alpha) = \alpha \cup \{\alpha\}$ is an ordinal.*

Lemma 2.17. *If S is a transitive set of ordinals, then S an ordinal.*

Proposition 2.18. *(Burali-Forti) There class of all ordinals is not a set.*

Definition 2.19. We write On for the class of ordinals.

This is a definable class, so just an abbreviation for a formula.

Theorem 2.20. *Recursion over all ordinals, also called transfinite recursion.*

If $G: V \to V$ is a definable class sized function, then there is a unique definable class sized function $F: \mathrm{On} \to V$ such that

$$(\forall \alpha \in \mathrm{On})(F(\alpha) = G(F \upharpoonright \alpha).$$

Proof. Uniqueness is proved by induction.

Existence: Also construct be induction. We let $f_\emptyset = \emptyset$.

Successor step:
Suppose $f_\alpha \colon \alpha \to V$ is already constructed and $\forall \beta < \alpha)(f_\alpha(\beta) = G(f_\alpha \restriction \beta)$. We let

$$f_{s(\alpha)} = f_\alpha \cup \{(\alpha, G(f_\alpha))\}$$

and thus get that $f_{s(\alpha)}$ fulfils $\forall \beta < s(\alpha))(f_{s(\alpha)}(\beta) = G(f_{s(\alpha)} \restriction \beta)$. Here we use that we already know the uniqueness.

Limit step:
If f_β, $\beta < \alpha$, is defined and each f_β respects the recursion condition and α is a limit ordinal, then we let

$$f_\alpha = \bigcup \{f_\beta \,:\, \beta < \alpha\}.$$

Here for existence, the replacement scheme is invoked. Now for checking that this is well-defined, we use that we already know the uniqueness. It is also easily checked that f_α fulfils the recursion condition $(\forall \gamma < \beta)(f_\alpha(\gamma) = G(f_\alpha \restriction \gamma)$.

□

Remark 2.21. A much better exposition is given e.g. in [8, page 102 ff]

Now we resume the main thread of Section 2.2.

Theorem 2.22. *The following statement are equivalent (on the basis of* ZF*):*

(1) AC*.*

(2) The well-ordering principle.

(3) Zorn's lemma.

(4) Tukey's Lemma.

Proof. (1) implies (2).
Let A be a set and let c be a choice function on $\mathcal{P}(A) \setminus \{\emptyset\}$. By transfinite recursion on $\alpha \in \mathrm{On}$ we define

$$F(\alpha) = \begin{cases} c(A \setminus \{F(\beta) \,:\, \beta < \alpha\}), & \text{if } \{F(\beta) \,:\, \beta < \alpha\} \neq A; \\ A, & \text{else.} \end{cases}$$

By the replacement scheme, there is a δ such that

$$\{F(\beta) \,:\, \beta < \delta\} = A,$$

since F is an injective function before we exhaust A. There is no injective function from a proper class into the set A.

Then we have an enumeration

$$\{(\alpha, F(\alpha)) : \alpha < \delta\}$$

of A. Now we define a well-order by letting for $\alpha, \beta \in \delta$,

$$F(\alpha) < F(\beta) :\Leftrightarrow \alpha < \beta.$$

(2) implies (3).
Let $(P, <)$ be a partially ordered set in which each chain has an upper bound. By assumption P can be well ordered via $(p_\alpha : \alpha < |P|)$. We choose a maximal element of P by the following transfinite induction: $c_0 = p_0$, c_ξ is the p_α such that α is the minimal ordinal such that p_α is an upper bound of $C = \{c_\beta : \beta < \xi\}$ and $p_\alpha \notin C$. The construction comes to a halt at a chain $\{c_\beta : \beta < \xi\}$ and then $c_{\xi-1}$ is a maximal element of P.

(3) implies (4).
Let $\mathcal{F}$ be a non-empty set of sets and assume that $\mathcal{F}$ has finite character. Thanks to the finite character, every $\subseteq$-chain in $\mathcal{F}$ has an upper bound, namely the union of the chain. By Zorn's Lemma we get a maximal element of $(\mathcal{F}, \subseteq)$.

(4) implies (1).
Let X be a set of non-empty sets. The set $\mathcal{F}$ of partial choice functions $f \colon Y \to \bigcup Y$, $Y \subseteq X$ has finite character. By Tukey's Principle, $\mathcal{F}$ has a $\subseteq$-maximal element. By maximality, any maximal element is the a total choice function. □

Third lecture until here

3 The Axiom of Determinacy

In 1962 Mycielski and Steinhaus proposed the Axiom of Determinacy AD. [14]. This axiom contradicts the Axiom of Choice.

This axiom, AD says: "Every infinite two person game with perfect information that is played over the natural numbers is determined."

Now I explain the notions. Suppose X is a nonempty set, e.g., the set $\mathbb{N}$. Let ${}^{\mathbb{N}}X = \{f : f \colon \mathbb{N} \to X\} = \{x \in \mathcal{PPP}(X \cup \mathbb{N}) \mid x \colon \mathbb{N} \to X\}$ (the set of functions from $\mathbb{N}$ to X, and the latter is really a set).

Definition 3.1. Let $A \subseteq {}^{\mathbb{N}}X$. The following game $\mathcal{G}_X(A)$ is called *the two person game over X with payoff A*: Players I and II draw alternately in $\mathbb{N}$ many rounds.

I	$x(0)$		$x(2)$		$x(4)$		$\dots$
II		$x(1)$		$x(3)$		$x(5)$	

The rules are: $x(n) \in X$. Player I wins if

$$x = \{(n, x(n)) : n \in \mathbb{N}\} = (x(n))_{n \in N} = \langle x_n : n < \omega \rangle \in A.$$

Otherwise II wins. The function x is called a play, initial segments of x are called partial plays, and the set A is called the payoff.

Definition 3.2. (1) A *strategy* σ for I is a function $\sigma \colon \bigcup_n {}^{2n}X \supset\!\!\to X$ tells the player I how to move, given the previous moves. Its domain must at least contain all the positions that can be reached by initial plays. (This could be made exact by induction on the length of the initial play. I show how this goes in detail in the proof of the theorem on open games below.)

I	$\sigma(\emptyset)$		$\sigma(\langle \sigma(\emptyset), y(0) \rangle)$		$\sigma(\dots)$		$\dots$
II		$x(1) = y(0)$		$x(3) = y(1)$		$x(5) = y(2)$	

This play is also written as $\sigma * y$, for $y = \{(n, y(n)) : n < \mathbb{N}\}$.

(2) A strategy τ for player II is a function $\tau \colon \bigcup_n {}^{2n+1}X \supset\!\!\to X$ tells the player II how to move, given the previous moves.

I	$z(0) = x(0)$		$x(2) = z(1)$		$x(4) = z(2)$		
II		$\tau(\langle z(0) \rangle)$		$\tau(\langle z(0), \tau(\langle z(0) \rangle), z(1) \rangle)$			$\dots$

This play is also written as $z * \tau$, for $z = \{(n, z(n)) : n < \mathbb{N}\}$.

(3) A strategy for player I/II is called a *winning strategy for player I/II* if the playing according to the strategy always results in winning.

(4) The game $G_X(A)$ is called *determined*, if either player has a winning strategy.

(5) The axiom AD says: For every $A \subseteq {}^{\mathbb{N}}\mathbb{N}$, the game $G_{\mathbb{N}}(A)$ is determined.

The following theorem is beyond the scope of this lecture and maybe of any lecture.

Theorem 3.3. *(Woodin) Given* ZFC *and infinitely many Woodin cardinals,* ZF *and* AD *holds in* $L(\mathbb{R})$ *(the latter is a variation of the Gödels constructible universe).*

You can consult the Kanamori book [5] on the definition of Woodin cardinals. However, Woodin's proof is far beyond the Kanamori book and can be found in [20].

Lemma 3.4. *(ZFC) Given any strategy σ for player I, the set $B = \{\sigma * y \,:\, y \in {}^{\mathbb{N}}\mathbb{N}\}$ is of size $2^{\mathbb{N}}$ and also ${}^{\mathbb{N}}\mathbb{N} \smallsetminus B$ is of size $2^{\mathbb{N}}$.*

Theorem 3.5. *(ZFC) 1953 Gale and Stewart. There is a payoff set A such that $G_{\mathbb{N}}(A)$ is not determined.*

Proof. We enumerate the possible total stategies for I and for II as $\{(\alpha, \sigma_\alpha, \tau_\alpha) \,:\, \alpha < \mathrm{card}({}^{\mathbb{N}}\mathbb{N})\}$. Note that $\mathrm{card}({}^{\mathbb{N}}\mathbb{N}) = \mathrm{card}({}^{({}^{<\mathbb{N}}\mathbb{N})}\mathbb{N})$ since ${}^{<\mathbb{N}}\mathbb{N}$ is countable and infinite. By induction on $\alpha < |{}^{\mathbb{N}}\mathbb{N}|$ we choose a_α such that

$$\text{for some } z\ (a_\alpha = z * \tau_\alpha \in {}^{\mathbb{N}}\mathbb{N} \text{ and } a_\alpha \in \mathbb{N}^{\mathbb{N}} \smallsetminus \{b_\beta \,:\, \beta < \alpha\})$$

and b_α such that

$$\text{for some } y\ (b_\alpha = \sigma_\alpha * y \text{ and } b_\alpha \in {}^{\mathbb{N}}\mathbb{N} \smallsetminus \{a_\beta \,:\, \beta \leq \alpha\}).$$

Such elements a_α, b_α exist, since $z \mapsto z * \tau$ is injective and hence

$$|\{z * \tau \,:\, z \in {}^{\mathbb{N}}\mathbb{N}\}| = |{}^{\mathbb{N}}\mathbb{N}|$$

and

$$|\{b_\beta \,:\, \beta < \alpha\}| \leq |\alpha| < |{}^{\mathbb{N}}\mathbb{N}|$$

and similar for the $\sigma * y$. Then let $A = \{a_\alpha \,:\, \alpha < |{}^{\mathbb{N}}\mathbb{N}|\}$. □

By the way, $|{}^{\mathbb{N}}\mathbb{N}| = 2^{\aleph_0} = |\mathbb{R}|$.

We give X the discrete topology and endow the space ${}^{\mathbb{N}}X$ with the product topology.

Definition 3.6. DC (the Axiom of Dependent Choice) says: If Y is a set and R is a binary relation over Y such that

$$(\forall x \in Y)(\exists y \in Y)(x, y) \in R$$

then there is a function $f \colon \mathbb{N} \to X$ such that

$$(\forall n \in \mathbb{N})((f(n), f(n+1)) \in R).$$

Exercise 3.7. $\mathsf{ZFC} \vdash \mathsf{DC}$

Theorem 3.8. *Kechris 1984 [7]. If* AD *holds in* $L(\mathbb{R})$*, then also* DC.

So under the assumption of infinitely many Woodin cardinals and ZFC, the theory ZF + AD + DCis consistent.

Theorem 3.9. *(*ZF + AC*) (even* ZF + DC*) 1953, Gale and Stewart. For every open or closed payoff set A the game $G_X(A)$ is determined.*

Proof. For any $B \subseteq {}^{\mathbb{N}}X$ and $s \in {}^{<\mathbb{N}}X$ let

$$B/s = \{x | s\hat{\ }x \in B\}.$$

If I has no winning strategy in the game after s, i.e. in $G_X(B/s)$, then there is a $j \in X$ such that I has no winning strategy in the game $G_X(B/(s\hat{\ }i\hat{\ }j)$. Otherwise, I would have a winning strategy in the game $G_X(B/s)$ after all: Initially make the move i, and after any reply j by II, play according to σ.

Suppose now that $A \subseteq {}^{\mathbb{N}}X$ is open, and assume that I has no winning strategy in $G_X(A)$. Then by the above remark, a strategy τ for II can be defined recursively over the length of the initial segment (that is also played according to this very strategy that we are about to define).

The start of the induction is $\tau_0 = \emptyset$.

The induction hypothesis is: I has no winning stategy in $G_X(B/t)$ and t is of even length $2n$ and played according to the part τ_n that is defined on sequences of length $2n - 1$ (and which we already know).

Player I plays something, say i. Then $t\hat{\ }i$ of odd length $2n + 1$, and

$$\begin{array}{l}\text{player II shall play such a } j \text{ that}\\ \text{player I has no winning strategy in } G_X(A/t\hat{\ }i\hat{\ }j).\end{array} \qquad (*_n)$$

This gives a possible τ_{n+1}.

Note that for each τ_n there is in general a large set of possible τ_{n+1}, so here is the point were the axiom DC enters. If X has a well-order, then we do not need to invoke DC.

Using dependent choice DC, applied to

$$Y = \bigcup\{\tau' : (\exists m \in \mathbb{N})(\tau' \colon {}^{\leq 2m-1}X \supset\!\!\!\rightarrow X \text{ according to the rules } (*_n), n < m)\})$$

and the relation R of extending for some n a strategy τ_n that is defined on a subset of ${}^{\leq 2n-1}X$ to a possible τ_{n+1} that is defined on a subset of ${}^{\leq 2n+1}X$ according to the rules for the relevant lengths, we get an increasing sequence

$$\{(n, \tau_n) : n \in \mathbb{N}\}.$$

Now we let $\tau = \bigcup\{\tau_n : n \in \mathbb{N}\}$.

We show that τ is actually a winning strategy for II. If x were a play according to τ yet $x \in A$, then by openness there would be a $2n \in \mathbb{N}$ such that

$$O(x \restriction 2n) = \{f \in {}^{\mathbb{N}}X : f \restriction 2n = x \restriction 2n\} \subseteq A.$$

But then, any strategy for I in $G_x(A/x \restriction 2n)$ would be a winning one, reaching a contradiction. Hence, τ is a winning strategy for II in $G_X(A)$.

The argument for closed A is analogous, with the roles of I and II interchanged.

$\square$

Project Proposal 3.10. Games and regularity properties. Kanamori [5, pp 373–377].

Corollary 3.11. *If we count the draw as say a win for player I (or for player II, also this is possible), then Chess is an open game and hence determined.*

Theorem 3.12. *(*ZF + AC*) (* ZF + DC*suffices) 1953, Donald Antony Martin [11]. For every Borel payoff set A the game $G_X(A)$ is determined.*

A very nice and readable proof of the Borel determinacy can be found in Kechris's book [6].

4 Models of Set Theory

We assume that ZF is consistent and investigate models of ZF and large fragments of it.

Definition 4.1. The von Neumann hiercharchy. By recursion over the ordinals we define

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\delta &= \bigcup\{V_\alpha : \alpha < \delta\} \text{ for limit ordinals } \delta. \end{aligned}$$

We have for $\alpha < \beta$, that $V_\alpha \subseteq V_\beta$. Any V_α is *transitive*, that means each element of V_α ist also a subset of V_α.

Exercise 4.2. *Show that the powerset of a transitive set is again transitive. Show that any union of transitive sets is transitive.*

Give a small non-transitive set.

Under ZF (minus the axiom of regularity), this hierarchy is well-defined.

Theorem 4.3. *(In ZF minus regularity)*

$$V = \bigcup\{V_\alpha : \alpha \textit{ is an ordinal}\} \Leftrightarrow \textit{ regularity.}$$

Proof. The proof of the forward implication is by easy induction on the ordinals. The proof of the backwards implication uses transfinite $\in$-induction, one of the strongest kinds of set-theoretic recursion. For a thorough coverage of this technique see [9]. □

Are there within the universe some set-sized models of ZFC? Note that an affirmative answer does not contradict the fact, the each model sees that its own universe is a proper class.

If α is a limit cardinal, then the set sized model $(V_\alpha, \in)$ fulfils all the ZFC axioms but the replacement scheme.

If κ is an uncountable regular strong limit cardinal, then $(V_\kappa, \in) \models \mathsf{ZFC}$. So

$$\mathsf{ZFC} + \text{ there is an unc. reg. strong limit cardinal } \rightarrow \mathrm{Con}(\mathsf{ZFC}).$$

Such a cardinal is called a strongly inaccessible cardinal and belongs to the hierarchy of large cardinals. Much stronger assumptions have been investigated, up to the "verge of inconsistency".

Project Proposal 4.4. Measurable cardinals and Scott's ultrapower construction.

Now we possibly study Gödel's constructible universe.

Project Proposal 4.5. Present some basic knowledge on cardinals. Show Hessenberg equality $\kappa \cdot \kappa = \kappa$, learn about cofinality. Maybe Julius König's sum versus product theorem.

Project Proposal 4.6. Counting and cardinals in connection with the real numbers. Which is the smallest ordinal that cannot order-embedded into the real line? How high is the Borel hierarchy?

Maybe Cantor Bendixsson derivative.

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