Analytic equivalence relations and bi-embeddability

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A subset of a Polish space $X$ is said Borel if it belongs to the $\sigma$-algebra generated by the topology of $X$, and is said analytic if it is the projection of a closed (equiv. Borel) subset of $X \times \omega^\omega$, where $\omega^\omega$ is the Baire space.

A structure $(X, E)$ is called analytic equivalence relation (resp. Borel equivalence relation) if $X$ is a Polish space and $E$ is an equivalence relation on $X$ which is analytic (resp. Borel) as subset of $X \times X$. 

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Classification and invariants

The analysis of analytic equivalence relation arises from the idea of classifying mathematical objects:

Example: Let $X$ be the space of complex $n \times n$ matrices (i.e. $\mathbb{C}^{n^2}$), $E$ be the similarity relation, and $\phi$ be the function which maps each matrix into its Jordan's canonical form.
The analysis of analytic equivalence relation arises from the idea of classifying mathematical objects: If we view $X$ as a set of mathematical objects, which we are interested in up to $E$-equivalence, the classification problem for $(X, E)$ consists of finding some (concrete or nicely definable) set $I$ of invariants, together with some (concrete or nicely definable) assignment $\varphi : X \to I$ which satisfies $x E y \iff \varphi(x) = \varphi(y)$ for every $x, y \in X$.

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Borel reducibility

To compare the complexity of two analytic equivalence relations \((X, E)\) and \((Y, F)\) we use the notion of Borel-reducibility:

\[(X, E) \leq_B (Y, F) \text{ iff exists some Borel function } f: X \to Y \text{ s.t. } \forall x_1, x_2 \in X (x_1 E x_2 \iff f(x_1) F f(x_2)).\]

\[(X, E) \text{ and } (Y, F) \text{ are Borel-equivalent, } (X, E) \sim_B (Y, F) \text{ in symbols, iff } (X, E) \leq_B (Y, F) \text{ and } (Y, F) \leq_B (X, E).\]

Intuitive meaning of \(\leq_B\): "\((X, E) \leq_B (Y, F)\) = "\((X, E)\) is simpler than \((Y, F)\)." In fact, any solution to the classification problem for \((Y, F)\) can be converted via \(f\) into a solution to the classification problem for \((X, E)\)."
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In fact, any solution to the classification problem for \((Y, F)\) can be converted via \(f\) into a solution to the classification problem for \((X, E)\).
Analytic completeness

All the notions involved depend only on the Borel structure of the domain of the relation considered, hence it suffices that such domains are standard Borel spaces (rather than Polish spaces).

Definition

E is said analytic complete if for every analytic equivalence relation F one has $F \leq B_E$. 

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Analytic equivalence relations and bi-embeddability
The structure of analytic equivalence relations

The structure given by analytic equivalence relations under $\leq_B$ is extremely complicated!

There are relations which are neither analytic complete nor Borel (e.g. $x \mathrel{E} y \iff \omega^1 x = \omega^1 y$); there are incompatible relations (e.g. $E$ and $F$); there are infinite antichains of equivalence relations (even among very “simple” equivalence relations, such as the countable ones); there are infinite descending chains, and so on.

Positive structural results: dichotomy theorems. Silver’s Theorem; Harrington-Kechris-Louveau Theorem; Ulm-classifiability, and so on.
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**Example:** $\mathcal{L} = \{R\}$, with $R$ binary. Then each $x \in \omega^2$ code the structure $A_x = (\omega, R^{A_x})$, where $n R^{A_x} m$ if and only if $x(\langle n, m \rangle) = 1$. 

Therefore each isomorphism between countable $\mathcal{L}$-structures is simply a permutation of $\omega$. The isomorphism relation on a Borel class of countable structures is an analytic equivalence relation.
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Isomorphism relations

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$S_\infty = \text{Polish group of all permutations of } \omega$. 

**Definition**

A set $X \subseteq \text{Mod}_\mathcal{L}$ is said to be invariant if it is closed under isomorphism (i.e. closed under the action of $S_\infty$ on $\text{Mod}_\mathcal{L}$).

**Fact:** $X \subseteq \text{Mod}_\mathcal{L}$ is Borel and invariant if and only if there is some $\mathcal{L}_{\omega_1 \omega}$-sentence $\phi$ such that $X = \text{Mod}_{\phi}$. 

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Now consider an \( \mathcal{L}_{\omega_1\omega} \)-sentence \( \varphi \) and let \( \text{Mod}_\varphi \) be the collection of all \( x \in \text{Mod}_\mathcal{L} \) which are models of \( \varphi \).
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- in particular, no isomorphism relation is analytic complete!
All the definitions about analytic equivalence relations can be stated in the setup of quasi-orders (i.e. reflexive and transitive relations):
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3. \((X, R)\) is complete analytic if \(S \leq_B R\) for every analytic quasi-order \((Y, S)\), and so on.
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Each analytic quasi-order \(R\) canonically induce the analytic equivalence relation \(E_R = R \cap R^{-1}\), and if \(R\) is complete analytic then so is \(E_R\).
Let $\mathcal{L}$, $\text{Mod}_\mathcal{L}$, $\varphi$ and $\text{Mod}_\varphi$ be as before.

Remark: $\sqsubseteq$ restricted to $\text{Mod}_\varphi$ is an analytic quasi-order, and can canonically induce the analytic equivalence relation $\equiv$ of bi-embeddability.

Example 1: On well-founded linear orders of length $\leq \alpha$, some fixed countable ordinal, isomorphism and bi-embeddability coincide (Schröder-Bernstein Theorem).

Example 2: $\equiv_{\text{LO}}$ is $\mathcal{S}_\infty$-complete (H. Friedman-Stanley), while $\equiv_{\text{LO}}$ has only $\aleph_1$-many classes but does not Borel-reduce equality on $\omega_2$ (Laver: $\sqsubseteq_{\text{LO}}$ is a bqo).
Embeddings

Let $\mathcal{L}$, $\text{Mod}_\mathcal{L}$, $\varphi$ and $\text{Mod}_\varphi$ be as before.

Given two $\mathcal{L}$-structures $A$ and $B$, we say that $A$ embeds in $B$ ($A \sqsubseteq B$ in symbols) if there is an injection $f : A \to B$ which is an isomorphism between $A$ and its image under $f$.
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Main results

Let $\mathcal{L}$ be a language with two binary relational symbols.

Theorem (S.-D. Friedman-M.)
For each analytic quasi-order $R$ there is an $\mathcal{L}_{\omega_1^{\omega}}$-sentence $\varphi$ such that $R$ is Borel-equivalent to $\subseteq$ on $\text{Mod} \varphi$.

Corollary (S.-D. Friedman-M.)
$E$ is an analytic equivalence relation iff there is an $\mathcal{L}_{\omega_1^{\omega}}$-sentence $\varphi$ such that $E$ is Borel-equivalent to $\equiv$ on $\text{Mod} \varphi$.

The same results hold if we replace "embedding" with homomorphism; weak homomorphism; isometric embeddings on a Borel class of discrete Polish metric spaces closed under isometry.
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- isometric embeddings on a Borel class of discrete Polish metric spaces closed under isometry.
Normal trees

For $s, t \in {}^n\omega$, put $s \leq t$ if $s(i) \leq t(i)$ for each $i < n$, and define $s + t$ by setting $s + t(i) = s(i) + t(i)$ for each $i \leq n$. 
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A function \( f : {}^{<\omega}\omega \to {}^{<\omega}\omega \) is said to be Lipschitz if \( s \subseteq t \Rightarrow f(s) \subseteq f(t) \) and \( |s| = |f(s)| \) for every \( s, t \in {}^{<\omega}\omega \).
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A tree $T$ on $X \times \omega$ is said to be normal if $(u, s) \in T$ and $s \leq t$ implies $(u, t) \in T$. 

Remark: $\leq_{\max}$ is an analytic quasi-order.
Normal trees

For $s, t \in n\omega$, put $s \leq t$ if $s(i) \leq t(i)$ for each $i < n$, and define $s + t$ by setting $s + t(i) = s(i) + t(i)$ for each $i \leq n$.

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Definition

If $S, T$ are normal trees on $2 \times \omega$, we put $S \leq_{\text{max}} T$ iff exists a Lipschitz $f$ such that $S(s) = \{u \in <\omega, 2 | (u, s) \in S\} \subseteq T(f(s)) = \{u \in <\omega, 2 | (u, f(s)) \in T\}$ for every $s \in <\omega, \omega$.
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**Definition**

If \( S, T \) are normal trees on \( 2 \times \omega \), we put \( S \leq_{\text{max}} T \) iff exists a Lipschitz \( f \) such that \( S(s) = \{u \in {}^{<\omega}2 \mid (u, s) \in S\} \subseteq T(f(s)) = \{u \in {}^{<\omega}2 \mid (u, f(s)) \in T\} \) for every \( s \in {}^{<\omega}\omega \).

**Remark:** \( \leq_{\text{max}} \) is an analytic quasi-order.
Normal form for quasi-orders

Each analytic subset of $\omega^2 \times \omega^2$ is the projection of a tree on $2 \times 2 \times \omega$. 

Theorem (Louveau-Rosendal)

Let $R$ be a quasi-order on $\omega^2$. Then there is a normal tree $S$ on $2 \times 2 \times \omega$ such that:

- $R$ is the projection of $S$;
- $(u, u, s) \in S$ for every $u \in <\omega^2$ and $s \in <\omega^\omega$ of the same length;
- $(u, v, s) \in S$ and $(v, w, t) \in S$ implies $(u, w, s + t) \in T$.

Modification:

Require also that $(u, v, 0 | u) \in S$ implies $u = v$.

(Simply drop all sequences of the form $(u, v, 0 | u)$ form the $S$ constructed above and check that all the other properties still hold.)
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Lucas Motto Ros

Analytic equivalence relations and bi-embeddability
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Completeness of $\leq_{\text{max}}$

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Idea of the proof: Given an analytic quasi-order $R$, construct $S$ as before and define the Borel-reduction

$$f(x) = S^x = \{(u, s) \in <\omega (2 \times \omega) \mid (u, x \upharpoonright |u|, s) \in S\}.$$
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**Remark:** Our modification gives that such $f$ is injective. In fact $(u, u, 0^{\|u\|}) \in S^x$ iff $u \subseteq x$: hence if $x \neq y$ and $n$ is such that $u = x \upharpoonright n \not\subseteq y$ then $(u, u, 0^n) \in S^x \setminus S^y$. 

Luca Motto Ros

Analytic equivalence relations and bi-embeddability
Completeness of $\sqsubseteq_{CT}$

A combinatorial tree is a connected and acyclic graph.
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**Theorem (Louveau-Rosendal)**

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**Idea of the proof:** Borel-reduce $\leq_{\max}$ to $\sqsubseteq_{CT}$ via a function $T \mapsto G_T$, where $T$ is a normal tree on $2^{<\omega} \times \omega$ and $G_T \in CT$. 

**Construction of $G_T$:**

First define $G_0$: nodes are of the type $s \in \omega^\omega$ or $s^*$ (for $s \in \omega^\omega \setminus \{\emptyset\}$), and put edges between $s^*$ and $s$ and between $s^*$ and the predecessor $s^-$ of $s$.

Choose an injective enumeration $\theta$ of $\omega^2$ such that $|u| \leq |v| \Rightarrow \theta(u) \leq \theta(v)$ (so that $\theta(\emptyset) = 0$).

For each $(u, s) \in T$ add vertices $(u, s, x)$ where $x \subseteq \vec{0}$ or $x \subseteq 0^2 \theta(u) + 2^{\vec{1} \vec{0}}$. Link $(u, s, x)$ to $(u, s, x^-)$ and $(u, s, \emptyset)$ to $s$.
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Sketch of the proof

Given a witness $f$ of $S \leq_{\text{max}} T$, construct an embedding between $G_S$ and $G_T$ by sending $s$ to $f(s)$, $s^*$ to $f(s)^*$, and $(u, s, x)$ to $(u, f(s), x)$.
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For the other direction, note that embeddings must preserve distances and (at most) increase valence of vertices. Then use the fact that:

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- elements of the form $(u, s, 0^{2\theta(u)+2})$ (for $(u, s) \in T$) have valence 3 and distance $2\theta(u) + 3$ from $s$;
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and prove that an embedding from $G_S$ to $G_T$ restricted to $<\omega \omega$ is a (well-defined) witness of $S \leq_{\text{max}} T$. 
An ordered combinatorial tree is a combinatorial tree with a new transitive relation defined on (some of) its vertices.
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Ordered combinatorial trees

An ordered combinatorial tree is a combinatorial tree with a new transitive relation defined on (some of) its vertices. $OCT$ is the collection of all countable ordered combinatorial trees.

For $s, t \in <\omega \omega$ put $s \preceq t$ if either $|s| < |t|$ or $|s| = |t|$ and $s \leq_{\text{lex}} t$. 
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For each normal tree $T$, construct an ordered combinatorial tree $G_T$ using the same construction as before but adjoining the following order $\leq_T$: 

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Analytic equivalence relations and bi-embeddability
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- \( s^* \leq_T t^* \) iff \( s \leq t \)
- \( (u, s, x) \leq_T (v, t, y) \) iff \( s \prec_T t \), or \( s = t \) and \( u \prec_T v \), or \( s = t \), \( u = v \) and \( x \leq y \).
Completeness of $\sqsubseteq_{OCT}$

Proposition (S.-D. Friedman-M.)

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The proof is identical to the one of the Louveau-Rosendal Theorem:
Completeness of $\sqsubseteq_{OCT}$

Proposition (S.-D. Friedman-M.)

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The proof is identical to the one of the Louveau-Rosendal Theorem: just check that the orders defined doesn’t obstruct the construction of the embedding from $G_S$ and $G_T$ arising from a witness $f$ of $S \leq_{\max} T$, since $f$ can be taken such that $s \preceq t \iff f(s) \preceq f(t)$ (this is because we consider only normal trees).
Lemma

If $S, T$ are two distinct normal trees then $G_S$ and $G_T$ are not isomorphic.
Lemma

If $S$, $T$ are two distinct normal trees then $G_S$ and $G_T$ are not isomorphic.

Proof: We prove that if $G_S$ and $G_T$ are isomorphic then $S = T$. An isomorphism $i$ between $G_S$ and $G_T$ must respect the orders $\leq_S$ and $\leq_T$. Since they coincide on their initial segment $<\omega \omega$, $i$ must be the identity on $<\omega \omega$. Therefore

$$
(u, s) \in S \iff (u, s, 0^{2\theta(u)+2}) \in G_S \\
\iff (u, i(s), 0^{2\theta(u)+2}) = (u, s, 0^{2\theta(u)+2}) \in G_T \\
\iff (u, s) \in T,
$$

i.e. $S \subseteq T$. Similarly, using $i^{-1}$ instead of $i$ we get $T \subseteq S$. 

Luca Motto Ros
Analytic equivalence relations and bi-embeddability
Injective action of $S_{\infty}$

Each $G_S$ can be coded Borel-in-$S$ into an ordered combinatorial tree $\hat{G}_S$ on $\omega$: we identify $G_S$ with $\hat{G}_S$.

Lemma
For every normal tree $S$ and every distinct permutations $p, q \in S_{\infty}$, $j_L(p, G_S) \neq j_L(q, G_S)$.

Proof: $S$ is a well-founded linear order, therefore distinct permutations must at least "exchange" two elements with respect to $\leq_S$.

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The main theorem

Theorem (S.-D. Friedman-M.)

If $R$ is an analytic quasi-order then there is an $\mathcal{L}_{\omega_1\omega}$-sentence $\varphi$ such that $R$ is Borel-equivalent to $\sqsubseteq$ on $\text{Mod}_\varphi$. 
The main theorem

**Theorem (S.-D. Friedman-M.)**

*If R is an analytic quasi-order then there is an $\mathcal{L}_{\omega_1\omega}$-sentence $\varphi$ such that R is Borel-equivalent to $\sqsubseteq$ on Mod$_{\varphi}$.*

**Proof:** Let $R'$ be the analytic quasi-order on $\omega^2 \times S_\infty$ defined by

$$(x, p) R' (y, q) \iff x R y.$$

Obviously $R \sim_B R'$, so it is enough to prove the result for $R'$. 

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Analytic equivalence relations and bi-embeddability
The main theorem

Theorem (S.-D. Friedman-M.)

If $R$ is an analytic quasi-order then there is an $\mathcal{L}_{\omega_1\omega}$-sentence $\varphi$ such that $R$ is Borel-equivalent to $\sqsubseteq$ on $\text{Mod}_\varphi$.

**Proof:** Let $R'$ be the analytic quasi-order on $\omega_2 \times S_\infty$ defined by

$$(x, p) R' (y, q) \iff x R y.$$

Obviously $R \sim_B R'$, so it is enough to prove the result for $R'$. Consider the Borel map $f$ which sends $(x, p)$ to $j_L(p, G_{S^x})$, where $S^x$ is defined as before. $f$ reduces $R'$ to $\sqsubseteq_{OCT}$ because

$$(x, p) R' (y, q) \iff x R y$$

$$\iff S^x \leq_{\text{max}} S^y$$

$$\iff G_{S^x} \sqsubseteq G_{S^y}$$

$$\iff f(x, p) \sqsubseteq f(y, q).$$
Now check that \( f \) is injective. Take \((x, p) \neq (y, q)\): if \( x \neq y \) then \( f(x, p) \) is not isomorphic to \( f(y, q) \) by strong injectivity of the map \( T \mapsto G_T \) (in particular, \( f(x, p) \neq f(y, q) \)), while if \( x = y \) but \( p \neq q \) then \( f(x, p) \neq f(y, q) \) by the second lemma above.
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Note also that the range of $f$ is invariant by definition of $f$. Since $f$ is an injective Borel map defined on a Borel set we have that:
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- the range of $f$ is also Borel: being Borel and invariant, it coincides with $\text{Mod}_\varphi$ for some $\mathcal{L}_{\omega_1\omega}$-sentence $\varphi$;
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- the range of $f$ is also Borel: being Borel and invariant, it coincides with $\text{Mod}_\varphi$ for some $\mathcal{L}_{\omega_1\omega}$-sentence $\varphi$;
- $f^{-1}$ is a Borel function and therefore also a Borel reduction of $\sqsubseteq$ on $\text{Mod}_\varphi$ to $R'$. 

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- \( f^{-1} \) is a Borel function and therefore also a Borel reduction of \( \sqsubseteq \) on \( Mod_\varphi \) to \( R' \).

This concludes our proof.
Homomorphism and weak homomorphism

Given two \( L \)-structures \( A \) and \( B \) (where \( L = \{ P, Q \} \) contains two binary relational symbols as before), an homomorphism between them is a function \( f : A \rightarrow B \) such that for all \( x, y \in A \)

\[ x P^A y \iff f(x) P^B f(y) \quad \text{and} \quad x Q^A y \iff f(x) Q^B f(y). \]
Homomorphism and weak homomorphism

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$f$ is said \textbf{weak homomorphism} if for all $x, y \in A$

\[ x \, P^A \, y \Rightarrow f(x) \, P^B \, f(y) \text{ and } x \, Q^A \, y \Rightarrow f(x) \, Q^B \, f(y). \]
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**Theorem (S.-D. Friedman-M.)**

*If $R$ is an analytic quasi-order then there is an $\mathcal{L}_{\omega_1 \omega}$-sentence $\varphi$ such that $R$ is Borel-equivalent to the relation of homomorphism (resp. weak homomorphism) on $\text{Mod}_\varphi$.***
Sketch of the proof

We have just to modify our main construction:

given a normal tree $T$ on $2 \times \omega$ construct the ordered combinatorial tree $G_T$ as before but replacing $\leq_T$ with its strict part $< T$. We claim that on (isomorphic copies of) ordered combinatorial trees of this kind, weak homomorphism, homomorphism and embedding coincide.

Let $f$ be a weak homomorphism between $G_S$ and $G_T$. First note that $f$ must be injective: if $x, y$ are distinct elements of $G_S$, either $x < S y$ or $y < S x$. But if $f(x) = f(y)$ then neither $f(x) < T f(y)$ nor $f(y) < T f(x)$.

Now assume $f(x) < T f(y)$: if $x \not\preceq_S y$ then $y < S x$ and hence $f(y) < T f(x)$ (since $f$ is a weak homomorphism), a contradiction!

Finally, let $f(x)$ and $f(y)$ be two linked vertices of $G_T$: if $x$ and $y$ are not linked in $G_S$, then the proper (and unique) chain connecting $x$ and $y$ should be mapped to a proper chain connecting $f(x)$ and $f(y)$, contradiction!
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Our main theorem remains true (in the case of embeddings) if we replace $\mathcal{L}$ with the language $\mathcal{L}'$ containing just one binary relational symbol (in this case we use combinatorial trees rather than ordered combinatorial trees). However the construction is a little bit more difficult.
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**Disadvantages:** One can still prove the analogous statement for homomorphism (in a rather different way), but **NOT** the one for weak homomorphism.
Getting rid of the order

Our main theorem remains true (in the case of embeddings) if we replace $\mathcal{L}$ with the language $\mathcal{L}'$ containing just one binary relational symbol (in this case we use combinatorial trees rather than ordered combinatorial trees). However the construction is a little bit more difficult.

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**Advantages:** One gets as a corollary that each analytic quasi-order is Borel-equivalent to the relation of isometric embedding (injective metric-preserving maps) on a Borel class of discrete Polish metric spaces closed under isomorphism (each discrete space is viewed here as a space on $\omega$).