Ludwig-Maximilians-Universität München

M.Sc. Theoretical and Mathematical Physics

Equivariant Cohomology and Localization

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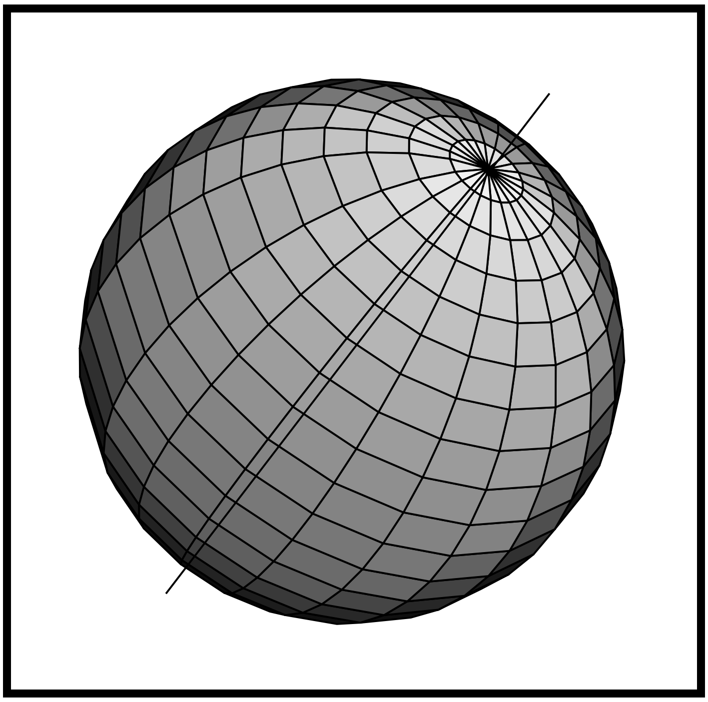

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Abstract

The fundamental theorem of calculus relates the definite integral of a function to the values of its antiderivative at the extrema of the interval of integration: we may regard this fact as a *localization* phenomenon, the reduction of information spread all over a set to a particular subset.

The thesis deals with localization on manifolds. The intuitive notion of symmetry of a space is formalized considering the action of a Lie group: integrals of forms which respect this symmetry are seen to be localizable, this time over the fixed point set of the action.

In Chapter 1 we review *equivariant cohomology*, the main tool needed to prove the localization theorem; Chapter 2 is dedicated to the proof of the theorem and the description of applications. We derive a formula for the volume of a class of homogeneous Kähler manifolds.

Contents

I. What is equivariant cohomology?

This chapter is devoted to the construction of a cohomology theory for $G-$spaces: we would like to introduce a theory taking in input not only the space itself, but also the group G it is acted upon, and its very action. We will present two equivalent ways of defining such a theory: the Borel construction and the Weil model.

I.1. The Borel construction

The basic idea of the Borel construction is to consider the cohomology the orbit space M/G of our $G-$space M; a direct approach in this sense is however unsuccessful, for without further hypotheses on the action M/G may lose many of the regularity conditions M has. We will want, typically, to compute the equivariant cohomology of some manifold, and we would like its orbit space M/G to stay a manifold: the slice theorem (see [6, p. 17]) tells us this is the case when the group G is a compact and connected Lie group, and its action free: we'll state it, and prove some corollaries, in the next paragraph.
The assumptions of the slice theorem can be loosened a bit remembering we are working up to homotopy. While it's not possible to do much with the group G, one would hope it is possible to find a substitute of M on which the action is free: observing that the (diagonal) action on a product of $G-$spaces is free as soon as it is on just one of the spaces, we want a space E such that:

- G acts freely on E;
- $M \times E$ has the same homotopy type as M.

In this section we'll discuss existence an uniqueness of such an E.

Remark I.1.0.1. All along the section, both $G-$spaces and their orbit space will be taken such that they have a CW structure; G is a compact, connected Lie group. The term *map* is reserved for continuous functions.

I.1.1. The slice theorem

Let's clarify the setting of the theorem. When we consider actions of compact groups on manifolds, the orbits are submanifolds. The inclusion map of the orbits induces a splitting of the tangent space at each point:

$$0 \longrightarrow T_x(G \cdot x) \longrightarrow T_xM \longrightarrow V_x \longrightarrow 0 \tag{I.1.1}$$

moreover, for each $g \in G_x$, the *stabilizer* of x, the differential of the associated action at x respects the splitting. We get a representation of G_x on V_x

$$G_x \to GL(V_x) : g \mapsto d_xg \tag{I.1.2}$$

We use this representation to define the diagonal action of G_x on $G \times V_x$. There is a fiber bundle $(G \times V_x)/G_x$ to G/G_x with fiber V_x, with projection map just given by

$$[g, v] \mapsto [g] \tag{I.1.3}$$

we can define a $G-$action on the bundle via $(G \times V_x)/G_x \to G/G_x : [g, v] \to g' \cdot [g, v] = [gg'^{-1}, v]$, and regard G/G_x as a submanifold, the zero section of the vector bundle. With this in mind, we can state the slice theorem:

Theorem I.1.1.1. *There exists an equivariant diffeomorphism ϕ from an equivariant open neighbourhood of G/G_x to an open neighbourhood $G \cdot x$ in M making the diagram commute:*

$$\begin{array}{ccc} G/G_x & \hookrightarrow & (G \times V_x)/G_x \\ \simeq \downarrow & & \downarrow \phi \\ G \cdot x & \hookrightarrow & M \end{array} \tag{I.1.4}$$

As an application of the theorem, we want to show the following property. This will be needed while discussing the localization theorem, in the second chapter:

Proposition I.1.1.2. *The set of fixed points of the $G-$action is a submanifold of M.*

To prove it, it is convenient to introduce the concept of *type of an orbit*:

Definition I.1.1.3. Let M be a manifold on which G acts smoothly, $x \in M$. The *type* of the orbit $G \cdot x$ is (G_x), the conjugacy class of G_x.

Recall that given $x, y \in G \cdot x$, G_x and G_y are conjugate, so that the definition is well-posed. Now I.1.1.2 is a corollary of the following:

Lemma I.1.1.4. *The union of all orbits of a given type is a submanifold of M.*

Proof. Fix a type (H), and a point $x \in M_{(H)} = \{y \in M | \ G_y \in (H)\}$. We will show that $M_{(H)}$ is a submanifold in a neighbourhood of $G \cdot x$: pick $y \in G \cdot x$ such that $G_y = H$, and consider orbits of type (H) in $(G \times V_y)/H$. Recall that a $G-$action on $(G \times V_y)/H$ was defined as

$$\phi_{g'} : [g, v] \mapsto [gg'^{-1}, v] \tag{I.1.5}$$

and look at the conjugacy class of the stabilizer of some $[g, v]$. We have

$$g' \cdot [g, v] = [g, v] \iff [gg'^{-1}, v] = [g, v] \iff \exists h \in H : \begin{cases} gg'^{-1} = hg \\ v = hv \end{cases} \tag{I.1.6}$$

so that $G_{[g,v]} = gH_v g^{-1}$, and all conjugates appear while g varies over G. Thus $G \cdot [g, v]$ is of type (H) if and only if $H_v = H$. As a consequence

$$(G \times V_y)/H \cap (G \times V_y)/H_{(H)} = \{[g, v] \in (G \times V_y)/H : \ hv = v \ \forall h \in H\} \tag{I.1.7}$$

is a subbundle of $(G \times V_y)/H$, with fiber the linear space $F = \{v \in V : \ hv = v \ \forall h \in H\}$. Now apply the slice theorem, and observe that a neighbourhood of $G \cdot x$ is diffeomorphic to a neighbourhood in this subbundle. The thesis follows. □

I.1.2. The universal bundle problem

We work in the setting of *principal bundles*:

Definition I.1.2.1. A *principal $G-$bundle* (E, B, π) is a fibration $\pi : E \to B$, where

(i) E is a $G-$space on which G acts freely;

(ii) B is its orbit space;

(iii) π is the related quotient map.

Example I.1.2.2. Easy instances of such bundles are given by the real, complex and quaternionic projective spaces:

1. $S^0 \simeq \mathbb{Z}_2 \curvearrowright S^n \rightsquigarrow \pi : S^n \to \mathbb{R}P^n$;
2. $S^1 \curvearrowright S^{2n+1} \rightsquigarrow \pi : S^{2n+1} \to \mathbb{C}P^n$;
3. $S^3 \curvearrowright S^{4n+3} \rightsquigarrow \pi : S^{4n+1} \to \mathbb{H}P^n$.

The general notion of (iso)morphism between vector bundles specializes in terms of equivariant mappings:

Definition I.1.2.3. Let (E, B, π), (E', B', π') be two principal $G-$bundles.

(i) A *morphism* between (E, B, π) and (E', B', π') is an equivariant map $\phi : E \to E'$;

(ii) An *isomorphism* between (E, B, π) and (E', B', π') is an equivariant homotopy equivalence $\phi : E \to E'$, i.e. such that
   - $\exists \gamma : E' \to E$ such that $\phi \cdot \gamma \simeq_F \text{id}$, $\gamma \cdot \phi \simeq_G \text{id}$;
   - $F(\cdot, t)$, $G(\cdot, t)$ are equivariant maps $\forall t$.

Remark I.1.2.4. Every morphism ϕ induces by equivariance a map between the orbits:

$$\begin{array}{ccc} E & \xrightarrow{\ \phi\ } & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow[{[x] \to [\phi(x)]}]{} & B' \end{array}$$

so that equivariance is just the $G-$bundles version of fiber preservation, and an equivariant homotopy equivalence is just an homotopy equivalence for which there exists a fiber preserving homotopy, in the sense specified above.

New principal bundles can be obtained via pullback:

Proposition I.1.2.5. *Let (E, B, π) be a principal $G-$bundle, B' a space, f a map from B' to B. Then $f^*\pi : f^*E \to B'$ is a principal $G-$bundle.*

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

Proof. The pullback of a fibration is a fibration (see [2, p. 91]), so we just need to check that the induced action of G on f^*E is free, and that $f^*\pi$ is the related quotient map. Recall

$$f^*E = \{(e, b') \in E \times B' : \pi(e) = f(b')\} \tag{I.1.8}$$

then the induced action is given by $g \cdot (e, b') = (g \cdot e, b')$, and it is well defined since $\pi(g \cdot e) = \pi(e)$; by freeness of the original action this action is free, and the orbits have the form

$$\begin{aligned} G \cdot (e, b') &= \{(f, c') \in f^*E : g \cdot (f, c') = (e, b') \text{ for some } g \in G\} \\ &= \{(f, b') \in f^*E : g \cdot f = e \text{ for some } g \in G\} = \{(f, b') \in f^*E\} \end{aligned} \tag{I.1.9}$$

then $f^*\pi$, given pointwise by $f^*\pi(e, b') = b'$, is the orbit map of this action. □

The universal bundle problem consists in finding a principal bundle generating all the other bundles via pullback. More precisely:

Definition I.1.2.6. A principal $G-$bundle $(\mathcal{E}, \mathcal{B}, p)$ is called *universal* if any principal $G-$bundle (E, B, π) can be obtained by pulling back some map $f : B \to \mathcal{B}$.

Remark I.1.2.7. Uniqueness (up to homotopy) of universal $G-$bundles follows from the definition: given two such bundles (E_1, B_1, p_1), (E_2, B_2, p_2) we obtain by definition maps f, g such that

$$\begin{array}{ccccc} \mathcal{E}_1 & \xrightarrow{\gamma} & \mathcal{E}_2 & \xrightarrow{\phi} & \mathcal{E}_1 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} \\ \mathcal{B}_1 & \xrightarrow{g} & \mathcal{B}_2 & \xrightarrow{f} & \mathcal{B}_1 \end{array}$$

where both squares are pullbacks; but pullbacks are unique up to homotopy, so that $f \cdot_H g \simeq \mathrm{id}$. We can then pullback along H to obtain an equivariant homotopy between $\phi \cdot \gamma$ and id:

$$\begin{array}{ccc} H^*\mathcal{E}_1 & \longrightarrow & \mathcal{E}_1 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} \\ \mathcal{B}_1 & \xrightarrow[H]{} & \mathcal{B}_1 \end{array}$$

the converse equivariant homotopy can be obtained analogously.

I.1.3. Universal bundles and contractibility

We want to construct explicitly an universal $G-$bundle. As a first step, we reduce our problem to finding a contractible space on which G acts freely: to do so, we need a preliminary lemma.

Lemma I.1.3.1. *Consider two principal $G-$bundles $\pi : M \to M/G$ and $p : E \to E/G$, with E contractible, and let G act on $M \times E$ via the diagonal action $g \cdot (x, e) = (g \cdot x, g \cdot e)$ $\forall (x, e) \in M \times E$. Then*

(i) $\Pi : (M \times E)/G \to M/G : [(x, e)] \mapsto [x]$ *is a fibration;*

(ii) $\Pi : (M \times E)/G \to M/G$ *has a section.*

Proof. Ad (i). First, check that this assignment is well defined, i.e. $[(x, e)] = [(x', e')]$ implies $[x] = [x']$: this follows by definition of the diagonal action, since the first equality implies in particular $x = gx'$ for some $g \in G$.
To show that this map is a fibration, we exhibit a local trivialization.

$$\begin{array}{ccc} (M \times E)/G & \xleftarrow{\pi_\times} & M \times E \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle p_1} \\ M/G & \xleftarrow[\pi]{} & M \end{array}$$

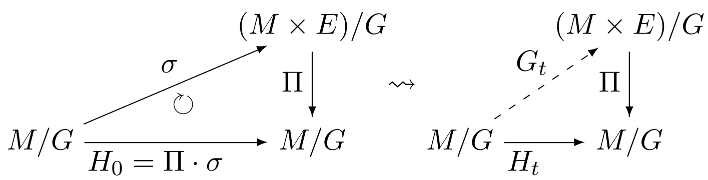
We can find an open set $U \subset M/G$ such that its preimage in M is homeomorphic to $U \times G$ using the local triviality hypothesis on $\pi : M \to M/G$; the preimage along p_1 gives us a set of the form $(U \times G) \times E$. Now, the action of G on $U \times G$ is given by $g \cdot ([x], g') = ([x], gg')$, so that two elements $([x], g, e), ([x]', g', e')$ are in the same orbit of the diagonal action if and only if $[x] = [x]'$, and $e = (gg'^{-1}) \cdot e'$.
This means, in turn, that representatives of each equivalence classes are given by elements of the form $([x], 1, e)$, and lastly that we can identify $\pi_\times((U \times G) \times E)$ with $U \times E$ via the assignment

$$[([x], g, e)] \mapsto ([x], g^{-1} \cdot e) \tag{I.1.10}$$

having as inverse $([x], e) \mapsto ([x], 1, e)$.
Ad (ii). Observe that the fiber of Π is E:

$$\Pi^{-1}([x]) = \{[(x, e)] \in (M \times E)/G : e \in E\} \simeq E \tag{I.1.11}$$

in fact the assignment $e \mapsto [x, e]$ is clearly surjective, and injectivity follows by freeness of the action.
Since E is contractible, Π is a weak homotopy equivalence, hence a homotopy equivalence. In particular $\exists \sigma : (M \times E)/G \to M/G$: $\Pi \cdot \sigma \simeq \mathrm{id}_{M/G}$, that is, we have a "homotopy section". To turn this into a proper section, let H_t be an homotopy between $\Pi \cdot \sigma$ and the identity, and consider the following homotopy extension problem:



then $s = G_1$ is a section of Π.

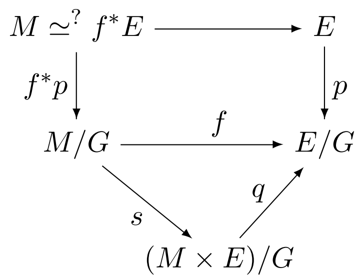
□

Remark I.1.3.2. Aside: notice that part (i) of the proof carries over verbatim to the following, more general result:

Proposition I.1.3.3. *Let G act smoothly on the spaces N, M, and let $\pi : N \to M$ be an equivariant fibration. Then $\pi_G : (N \times E)/G \to (M \times E)/G$ is a fibration.*

Where we call *equivariant fibration* a fibration of $G-$spaces whose projection map is equivariant.

Let's proceed with our discussion. The section s together with the assignment $q : (M \times E)/G \to E/G : [(x, e)] \to [e]$ produces $f = q \cdot s : M/G \to E/G$,: it's the map along which we would like to pull back $(E, E/G, p)$, our candidate universal bundle, to obtain $(M, M/G, \pi)$. It remains to show that $f^*E \simeq M$ as $G-$bundles: this is the content of the next theorem.



Theorem I.1.3.4. *Consider two principal $G-$bundles $\pi : M \to M/G$ and $p : E \to E/G$, with E contractible. Then there exists a map $f : M/G \to E/G$ such that $f^*E \simeq M$ as $G-$bundles.*

Proof. As remarked in the discussion above, we set $f = q \cdot s$. Observe that each equivalence class $[(x, e)] \in (M \times E)/G$ defines an equivariant map $h_{[x]} : \Pi^{-1}[x] \to E$ just by setting $h(x) = e$ and extending equivariantly. We can easily check that $h_{[x]}$ is the same map regardless of the chosen representatives: if $(x', e') = (g \cdot x, g \cdot e)$ and we set $h'_{[x]}(x') = e'$, we get $h'_{[x]}(x) = g^{-1} \cdot h'_{[x]}(x') = g^{-1} \cdot e' = e$, so that $h = h'$.

We can then think of the section s as a map yielding one such function for every equivalence class of M/G, i.e $s([x]) = [(x, h_{[x]}(x))]$. We then get an assignment h defined on the whole space M via

$$h(x) = h_{[x]}(x) \tag{I.1.12}$$

This assignment is continuous: pick an open set $V \subset E$, we want to show that $\forall x \in h^{-1}(V)$ there exists an open neighbourhood W of x contained in V. Set $y = h(x)$ and construct it as follows:

- Observe that there exists a neighbourhood V_1 of y inside V and a neighbourhood N of $1 \in G$ with the property $N \cdot V_1 \subset V$. This follows just by continuity of the action $\cdot : G \times M \to M$;
- Moreover, we can find a neighbourhood of x such that the points in its image cannot be further than by an element in N. More precisely, let

  $$M^* = \{(x, g \cdot x) \in M^2 : x \in M,\ g \in G\} \tag{I.1.13}$$

  and define

  $$\tau : M^2 \cap M^* \to G : (x, g \cdot x) \mapsto g \tag{I.1.14}$$

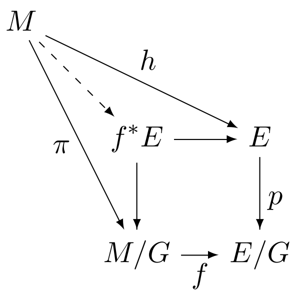
  this map is well defined by freeness of the action, and continuous: we can find a neighbourhood W_1 of x such that $\tau(W^2) \subset N$; if two elements in W_1 are in the same orbit of G, the $g \in G$ linking them is small;

- Consider the projection $\Pi(W_1 \times V_1) \subset (M \times E)/G$. This is an open set, so that its preimage $\mathcal{U}$ under the section $M/G \to (M \times E)/G$ is still open, and so is $\pi^{-1}\mathcal{U} \subset M$. Observe $x \in \pi^{-1}\mathcal{U}$, in fact

$$s(\pi(x)) = [x', h(x')], \;\; x' \in W_1, \;\; h(x') \in V_1 \tag{I.1.15}$$

  for x' such that $[x] = [x']$. This means $h(x) = h(g \cdot x') = g \cdot h(x')$: by construction, $g \in N$, so that $g \cdot h(x') \in V_1$. An analogous argument shows $h(\pi^{-1}\mathcal{U}) \subset V_1$, proving the claim.

Let's proceed. Observe $p \cdot h = f \cdot \pi$:



in fact

$$f \cdot \pi(x) = f([x]) = q \cdot s([x]) = q \cdot [(x, h(x))] = [h(x)] = p \cdot h(x) \tag{I.1.16}$$

we then obtain the dashed, continuous assignment $t = (\pi, h) : X \to f^*E$ by properties of the pullback. We claim that t is a homeomorphism, in fact:

- t is injective:

$$\begin{aligned} (\pi(x), h(x)) = (\pi(x'), h(x')) &\Rightarrow \begin{cases} \pi(x) = \pi(x') \\ h(x) = h(x') \end{cases} \Rightarrow \begin{cases} x = g \cdot x' \text{ for some } g \in G \\ h(x) = h(x') \end{cases} \\ &\Rightarrow h(x') = h(x) = h(g \cdot x') = g \cdot h(x') \\ &\Rightarrow g = \mathrm{id}, \;\; x = x' \end{aligned} \tag{I.1.17}$$

- t is surjective. Pick $([x], e) \in f^*E$, recall

$$f^*E = \{([x], e) \in X/G \times E : f([x]) = p(e)\} \tag{I.1.18}$$

  this means $p(h(x)) = p(e)$, i.e. $e = h(g \cdot x)$ for some $g \in G$ and $x \in [x]$. Then a preimage of $[(x, e)]$ is given by $g \cdot x$ for such g, in fact $\pi(g \cdot x) = [x]$, $h(g \cdot x) = e$ by construction.

- t^{-1} is continuous. To see this, pick any $x' \in f^*E'$ with its preimage $x \in M$, and consider an open neighbourhood V of x; we want to prove openness of t by constructing an open neighbourhood W of x' such that $t^{-1}(W)$ is contained in V. We do this in steps, similarly as we did for h:

- As before, there exists a neighbourhood V_1 of x inside V and a neighbourhood N of $1 \in G$ with the property $N \cdot V_1 \subset V$;
- Also analogously, we can find a neighbourhood W_1 of x' such that $\tau(W_1^2) \subset N$. Set $V_2 = V_1 \cap t^{-1}(W)$: this is an open neighbourhood of x satisfying $t(V_2) \subset W_1$;
- The next step is identifying an open neighbourhood W_2 of x' inside W_1 such that $f^*p(W) \subset \pi(V_2)$:

$$\begin{array}{ccc} V_2 \subset M & \xrightarrow{t} & f^*E \supset W_2 \\ & \searrow^{\pi} \quad \swarrow_{f^*p} & \\ & M/G & \end{array}$$

  just set $W_2 = W_1 \cap (f^*p)^{-1}\pi(V_2)$, and observe that we have $t(V_2) \subset W_2$, since by commutativity $t(V_2) \subset (f^*p)^{-1}\pi(V_2)$;
- Set $W = W_2$. It is an open neighbourhood of x', moreover for each $y' \in W$ we can find an $y \in V_2$ such that $f^*\pi(y) = \pi(y)$; since both y' and $t(y)$ are in W and in the same orbit of G, it holds $y' = g \cdot t(y)$ for $y \in S$.

  But $g \cdot t(y) = t(g \cdot y)$, then $t^{-1}(y') = g \cdot y \in N \times V_2 \subset V$. This shows $t^{-1}(W) \subset V$.

□

As a corollary, we have

Corollary I.1.3.5. *A principal $G-$bundle (E, G, p) is an universal $G-$bundle if and only if E is contractible.*

Proof. Suppose E is contractible, then the theorem above tells us how to express any principal $G-$bundle by pulling back E.

Conversely, we already saw that the total spaces of universal bundles necessarily share the same homotopy type. □

This statement reinforces the idea of the space E as a canvas, or, in other terms, of E/G as a footprint of G, as precise as we can get it. Since E is homotopically always the same, no matter what G we choose, we can think of all the information about principal $G-$bundles as being stored in the orbit space E/G. This leads us to the following definition:

Definition I.1.3.6. The space E/G, unique up to homotopy, is called *classifying space of G*.

I.1.4. Construction of an universal $G-$bundle

Recall that every compact Lie group has a faithful linear representation (see [12, p. 26]), i.e. it can be embedded as a subgroup of $U(n)$ for some $n \in \mathbb{N}$; it is then sufficient to find a contractible space E on which $U(n)$ acts freely, and then restrict its action to G.

Start by considering the Stiefel manifold V_n^k, $k > n$. This is the set of all orthonormal $n-$frames in $\mathbb{C}^k$; recall that V_n^k inherits a CW structure from $SO(k)$ via the projection π_k of the last n columns of the matrices of $SO(k)$ (see [16, p. 301]).

Observe

Lemma I.1.4.1. *There is a free action $U(n) \curvearrowright V_n^k \ \forall k > n$.*

Proof. Define for $A = (a_{ij})_{ij} \in U(n)$ and any $(v_1, \dots, v_n) \in V_n(\mathbb{C}^k)$

$$A \cdot (v_1, \dots, v_n) = (\sum_j a_{1j} v_j, \dots, \sum_j a_{nj} v_j) \tag{I.1.19}$$

this action is well defined, for

$$\langle (A \cdot v)_i, (A \cdot v)_j \rangle = \sum_{k,k'} a_i k \bar{a}_{jk'} \delta_{kk'} = (AA^*)_{ij} = \delta_{ij} \tag{I.1.20}$$

It is also free. Suppose $Av = v$, then

$$\delta_{ij} = \langle v_i, v_j \rangle = \langle (Av)_i, v_j \rangle = \sum_k a_{ik} \delta_{kj} = a_{ij} \tag{I.1.21}$$

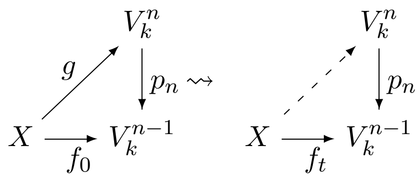
□

Now, if V_n^k were contractible, we would have found our universal bundle. Denote by $\pi_j(V_n^k) = 0$ the j^{th} homotopy group of V_n^k, and observe

Lemma I.1.4.2. $\pi_j(V_n^k) = 0$ *for* $2k \geqslant 2n + j$.

Proof. We prove the claim by induction on n. For $n = 1$ we get $V_1^k \simeq S^{2k-1}$, so that $\pi_j(V_1^k) = 0$ for $j < 2k - 1$, and the claim holds.

Let's show $n - 1 \Rightarrow n$. Observe that the map $p_n : V_k^n \to V_k^{n-1} : (v_1, \dots, v_n) \mapsto (v_1, \dots, v_{n-1})$ is a fibration: in fact, given a homotopy lifting problem



with $f_t(x) = (f_t^1(x), \dots, f_t^{n-1}(x))$, $g(x) = (g^1(x), \dots, g^n(x))$, we can lift f by $g_t(x) = GS(f_t^1(x), \dots, f_t^{n-1}(x), g_n(x))$, GS being the Gram-Schmidt algorithm. The fiber is a sphere:

$$p_n^{-1}(x) = \{v \in \mathbb{C}^k : \|v\| = 1, \ v \perp x_1, \dots, x_{n-1}\} \simeq S^{2(k-n)+1} \tag{I.1.22}$$

now choose $X = S^j$, with $j \leqslant 2(k - n + 1)$. Pick any $g_0 : S^j \to V_k^n$, induce an f by commutativity and solve the homotopy lifting problem with input the homotopy h_t from f to the trivial map. Then g_1 maps into the fiber, and is nullhomotopic for $j < 2k - 2n + 1$, i.e. $j \leqslant 2k - 2n$. □

The idea is then to construct for fixed n an infinite Stiefel complex V_n via the chain of inclusions

$$\dots \hookrightarrow V_n^k \underset{\iota_k}{\hookrightarrow} V_n^{k+1} \hookrightarrow \dots$$

sending $(v_1, \dots, v_n)$ to $((0, v_1), \dots, (0, v_n))$. This is formally accomplished by declaring V_n to be the *direct limit* of the V_n^k's, i.e. the space

$$V_n = \coprod_{k>n} V_n^k / \sim \tag{I.1.23}$$

where $\sim$ is constructed at follows:

- Denote ι_k^{i+1} the composition $\iota_{k+i} \cdot \dots \cdot \iota_k \ \forall i \geqslant 0$;
- Denote $\iota_k^0 = \mathrm{id}_{V_n^k}$;
- Identify two points $x \in V_n^k$, $y \in V_n^{k+i+1}$ if and only if $y = \iota_k^{i+1}$.

This equivalence relation is only well defined as long as the inclusions are compatible, i.e. $\iota_{k+i}^j \cdot \iota_k^i = \iota_k^{i+j}$, which is clearly the case: we're just adding a tuple of zeros before each component of $v \in V_n^k$, and it doesn't matter whether we do it in one or two steps. V_n also inherits a CW structure, topologizing it with the final topology with respect to the inclusions and taking as characteristic maps those of the V_n^k's.

All homotopy groups of V_n are then zero: the image of any map $f : S^i \to V_n$ is contained into some subcomplex V_n^k, whose homotopy group is trivial for k large enough. Then V_n is contractible, and $(V_n, V_n/G, \pi)$ a universal $G-$bundle.

Example I.1.4.3. (i) If we take $G = S^1 = U(1)$, we have $V_1^k = S^k$, and ι_k is the equatorial inclusion; then $V_1 = S^\infty$. The action of S^1 on each V_1^k yields the principal bundles $(S^k, \mathbb{C}P^k, \pi)$, and we can identify the $k-$skeleton of S^∞/G with $\mathbb{C}P^k$. Then $S^\infty/S^1 = \mathbb{C}P^\infty$, and $(S^\infty, \mathbb{C}P^\infty, \pi)$ is a universal S^1-bundle;

(ii) Analogously, $(S^\infty, \mathbb{R}P^\infty, \pi)$ is a universal $\mathbb{Z}_2-$bundle;

(iii) It is easy to see that $E(G \times H) \simeq EG \times EH$. A universal bundle for the $n-$torus is then just given by $\Pi_{i=1}^n (S_i^\infty, \mathbb{R}P_i^\infty, \pi_i)$.

I.1.5. The homotopy quotient

We are now ready to define the equivariant cohomology of a $G-$space M. The idea is to substitute

$$M/G \rightsquigarrow (M \times E)/G \tag{I.1.24}$$

and then study the cohomology of this last object. There are several advantages:

- E is contractible, so we're not changing the homotopy type of M;
- the action of G on $M \times E$ is free no matter the patologies of the original action, since $E \to E/G$ is a principal bundle;
- last but not least, E is unique up to homotopy: this definition doesn't have any inconsistencies.

Definition I.1.5.1.

For a $G-$space M, we call $M_G = (M \times E)/G$ the *homotopy quotient* of M, and

$$H_G^*(M) = H^*(M_G) \tag{I.1.25}$$

the *equivariant cohomology ring* of M.

Remark I.1.5.2.

- If M_G really is the appropriate substitute of M/G when the action is not free, we expect it to have the same homotopy type of M/G when the action is already free. This is true: we showed during the proof of Lemma I.1.3.1 that $\Pi : M_G \to M/G$ is a fibration with contractible fiber, in particular a homotopy equivalence;

- To get a handle on the theory, it is instructive to have a look at the cohomology of a single point. If we set $M = *$, we get $M_G \simeq E/G$. This object is already quite complicated for $G = S^0$: we get the cohomology of $\mathbb{R}P^\infty$, i.e. $\mathbb{Z}_2[x]$, with $|x| = 2$.

  This largeness can be somehow justified by observing that points of M/G are orbits of the action, and that the orbit containing some $x \in M$ corresponds to the quotient G/G_x, G_x being the stabilizer of x. Now, if x lies on a free orbit, we get $G_x = 0$, and computing its the equivariant cohomology yields

  $$H^*_G(G) = H^*(G_G) = H^*((G \times E)/G) \simeq H^*(E) = 0 \qquad \text{(I.1.26)}$$

  in general, it holds $H^*_G(G/G_x) = H^*(E/G_x) = H^*_{G_x}(*)$: the equivariant cohomology becomes more and more interesting as the action degenerates, and, in this sense, measures how pathological the action is.

  The equality $H^*_G(G/G_x) = H^*(E/G_x)$ follows from observing that two elements $(g \cdot G_x, e)$, $(g' \cdot G_x, e')$ of $G/G_x \times E$ are in the same orbit if and only if $g \cdot G_x = g' \cdot G_x$, equivalently $e = h \cdot e'$ for $h \in G_x$. Then $(G/G_x \times E)/G \simeq E/G_x$, which proves the claim;

- Equivariant cohomology is not a cohomology theory: the cohomology of a point is general too "large". Still, several well-know properties of e.g. singular cohomology, such as the Mayer-Vietoris sequence or the sequence of a pair, can be translated into an equivariant context.

  As an example, let's try to derive the exact sequence of a pair of spaces (M, Z). We have a diagram of the form

  $$\begin{array}{ccc} Z & \xrightarrow{\iota} & M \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ Z \times E & \xrightarrow{\iota \times \mathrm{id}} & M \times E \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi_M} \\ (Z \times E)/G & \dashrightarrow & (M \times E)/G \end{array}$$

  in order to induce an inclusion on the quotient, we need Z to be an *equivariant* subset of M: we can then use the sequence for a pair of spaces applied to Z_G, M_G to get the result. One needs to adapt similarly the Mayer-Vietoris argument, using equivariant open subsets;

- Consider the fibration $M \hookrightarrow (M \times E)/G \to E/G$:
  - Looking at the inclusion of the fiber, we can interpret equivariant cohomology as extending topological information from the manifold to its homotopy quotient;

- The projection map induces

$$\pi^* : H^*_G(*) \simeq H^*(E/G) \to H^*((M \times E)/G) \simeq H^*_G(M) \tag{I.1.27}$$

which we can use to regard $H^*_G(M)$ as a module over $H^*_G(*)$: for an $a \in H^*_G(*)$, $\omega \in H^*_G(M)$, one defines $a \cdot \omega := \pi^*(a) \cdot \omega$, using the ring structure of $H^*_G(M)$.

Let's see an example of computation. We consider equivariant cohomology with complex coefficients.

Example I.1.5.3. Consider the action of S^1 on S^2 given by rotation with respect to a fixed axis:

$$f_\theta : S^2 \to S^2 : \begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi) \\ \sin(\psi)\cos(\phi) \end{bmatrix} \mapsto \begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi+\theta) \\ \sin(\psi)\cos(\phi+\theta) \end{bmatrix} \tag{I.1.28}$$

this action is free on every point but the north and the south pole, which are fixed points. We can use the equivariant Mayer-Vietoris sequence to compute the equivariant cohomology of the sphere: choose as equivariant neighbourhood U^+, U^- two overlapping spherical caps, each containing one of the poles

$$U^+ = \left\{ \begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi) \\ \sin(\psi)\cos(\phi) \end{bmatrix} \in S^2 : \ \psi < \frac{3\pi}{4} \right\}, \quad U^- = \left\{ \begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi) \\ \sin(\psi)\cos(\phi) \end{bmatrix} \in S^2 : \ \psi > \frac{\pi}{4} \right\} \tag{I.1.29}$$

their intersection is then a belt V around the equator, which doesn't contain either of the poles. Observe that U^+ and U^- deformation retract to the fixed points, and V to the equator, so that

$$\begin{cases} H^*_{S^1}(U^+) = H^*_{S^1}(+) = \mathbb{C}[+], \ \deg[+] = 2 \\ H^*_{S^1}(U^-) = H^*_{S^1}(-) = \mathbb{C}[-], \ \deg[-] = 2 \\ H^*_{S^1}(V) = H^*(S^1/S^1) = H^*(*) \end{cases} \tag{I.1.30}$$

for $k > 1$, the Mayer-Vietoris sequence reads

$$0 \longrightarrow H^k_{S^1}(S^2) \longrightarrow (\mathbb{C}[+])^k \oplus (\mathbb{C}[-])^k \longrightarrow 0 \tag{I.1.31}$$

while for $k = 0, 1$ we have

$$0 \longrightarrow H^0_{S^1}(S^2) \longrightarrow \mathbb{C} \oplus \mathbb{C} \xrightarrow{j^*} \mathbb{C} \longrightarrow H^1_{S^1}(S^2) \longrightarrow 0 \tag{I.1.32}$$

where j^* is the restriction, acting as $j^*(z^1, z^2) = z_1 + z_2$. Thus $H^0_{S^1}(S^2) = \mathbb{C}$, $H^1_{S^1}(S^2) = 0$. We obtain:

$$H^k_{S^1}(S^2) = \begin{cases} \mathbb{C}, \ k = 0 \\ \mathbb{C}^2, \ k \text{ even} \\ 0, \ \text{otherwise} \end{cases} \tag{I.1.33}$$

Now consider the trivial action of S^1 on S^2. In this case

$$(S^2 \times ES^1)/S^1 \simeq S^2 \times BS^1 \tag{I.1.34}$$

so that $H^*_{S^1}(S^2) = H^*(S^2) \otimes H^*(\mathbb{C}P^\infty)$. Taken grade-by-grade, these two cohomology ring are the same: but these object have a more natural interpretation as $H^*_G(*)-$modules, as shown in the last remark, or $H^*_G(*)-$*superalgebras*, which will be introduced in the second section. We postpone the discussion on the isomorphism until the end of the chapter.

I.2. The Weil model

The purpose of this section is to show how to compute the equivariant cohomology of M by a pure algebraic construction. As remarked in the introduction, rather than defining a new space and computing the cohomology of the associated complex, we will just define a new complex from scratch, and define its cohomology as the equivariant cohomology of M. Some algebraic machinery is required, and provided in the next pages.

Remark I.2.0.1. In this section, M is always assumed to be a manifold. G is, as before, a compact and connected Lie group.

I.2.1. Some motivations

We motivate, and prepare the setting for, the algebraic definitions in the following subsections.

Consider a $G-$space M, and consider the map $\phi_g : M \to M : m \mapsto g \cdot m$ for fixed $g \in G$. Then the assignment

$$g \mapsto \left(\phi_g^{-1}\right)^* \tag{I.2.1}$$

defines a representation of G on $\Omega(M)$, the de Rham complex of M. We need to invert the pullback in order to satisfy the group homomorphism condition, which arises combining the two involutive operations:

$$gh \mapsto \left(\phi_{gh}^{-1}\right)^* = \left((\phi_g \phi_h)^{-1}\right)^* = \left(\phi_g^{-1}\right)^* \left(\phi_h^{-1}\right)^* \tag{I.2.2}$$

We can use the exponential mapping to induce a representation of $\mathfrak{g}$ on $\Omega(M)$. Explicitly,

$$\mathfrak{g} \ni \xi \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \left(\phi^{-1}_{\exp(t\xi)}\right)^* \tag{I.2.3}$$

regarding $t \mapsto \exp(t\xi)$ as a path having tangent vector ξ in $t = 0$. Now observe that this operation coincides with the Lie derivative with respect to the vector field having $\phi^{-1}_{\exp(t\xi)}$ as flux, namely

$$\hat{\xi}(\cdot) = -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \phi_{\exp t\xi}(\cdot) \tag{I.2.4}$$

Definition I.2.1.1. Let $\xi \in \mathfrak{g}$. We call the vector field

$$M \ni x \mapsto -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \phi_{\exp t\xi}(x) \in T_x M \tag{I.2.5}$$

the *fundamental vector field associated to* ξ. Also this vector field will be denoted by ξ.

Similarly, we define the contraction by ξ as the contraction with the associated fundamental vector field, and denote it by ι_ξ. Lie derivative, contraction and exterior differential satisfy the *Weil equations*:

$$\begin{cases} [L_\xi, d] = 0 \\ [L_\xi, \iota_\eta] = \iota_{[\xi,\eta]} \\ [L_\xi, L_\eta] = L_{[\xi,\eta]} \\ \{d, \iota_\xi\} = L_\xi \\ \{\iota_\xi, \iota_\eta\} = 0 \\ \{d, d\} = 0 \end{cases} \tag{I.2.6}$$

Where $[,]$ denotes the commutator and $\{,\}$ the anticommutator. Notice that whenever the operation of degree zero ($L_\xi : \Omega^k(M) \to W^k(M)$) is involved, we have the usual commutator; for the other two derivations, having degree ± 1, the anticommutator appears.

All in all, $\mathfrak{g}$ acts on $\Omega(M)$ via two derivations, related one another and with the derivation already present on $\Omega(M)$ either via commutators or anticommutators, according to their degree. We would like to turn the tables, and transform these equations into defining properties: a rich structure, called *Lie superalgebra*, which models the action of a Lie algebra on $\Omega(M)$. Untangling the relations I.2.6 will lead us to define a rather baroque construction, with two spaces isomorphic to $\mathfrak{g}$, but on different levels, and a third space which we'll equip with an algebra structure, only to ask it to be trivial: nonetheless, every single piece of information we define will turn out to be necessary; the multiple algebraic features of these Lie superalgebras just show how intrisically rich the $G-$space structure is.

I.2.2. Superalgebras

We need a preliminary notion:

Definition I.2.2.1. A *supervector space* is a vector space V with a $\mathbb{Z}_2-$gradation:

$$V = V_0 \oplus V_1 \tag{I.2.7}$$

Elements of V_0 are called *even*, elements of V_1 are called *odd*.

Remark I.2.2.2. (i) This definition work regardless of the field the vector space is defined on. But henceforth we will specialize the discussion to $\mathbb{R}$ and $\mathbb{C}$;

(ii) A vector space V can be made into a supervector space by setting $V_0 = V$, $V_1 = \emptyset$. In this sense, the former definition is just a generalization of the concept of vector space;

(iii) Given a $\mathbb{Z}-$graded vector space $V = \oplus_{i\in\mathbb{Z}} V_i$, we can induce a supervector space structure by setting

$$V_0 = \oplus_{i\in\mathbb{Z}} V_{2i},\ V_1 = \oplus_{j\in\mathbb{Z}} V_{2j+1} \tag{I.2.8}$$

We'll refer to such supervector spaces as $\mathbb{Z}-$*graded supervector spaces*, taking this identification for granted.

Now the plan is to let the $\mathbb{Z}_2-$grading propagate in usual algebraic definitions and identities, obtaining their *super* version.

Definition I.2.2.3. (i) A *superalgebra* A is a supervector space equipped with a bilinear product $\cdot$ satisfying

$$A_i \cdot A_j \subset A_{i+j},\ i = 0, 1 \tag{I.2.9}$$

with $i + j$ to be intended as a modulo 2 sum;

(ii) The *supercommutator* on A is defined as

$$[a, b] = a \cdot b - (-1)^{ij} b \cdot a,\ \forall a \in A_i, b \in A_j,\ \forall i, j \tag{I.2.10}$$

and A is called *super commutative* if $[a, b] = 0\ \forall a, b \in A$;

(iii) The *super anticommutator* on A is defined as

$$\{a,b\} = a \cdot b + (-1)^{ij} b \cdot a, \;\; \forall a \in A_i, b \in A_j, \;\; \forall i,j \tag{I.2.11}$$

and A is called *super anticommutative* if $\{a,b\} = 0 \; \forall a,b \in A$;

(iv) A (super) anticommutative (super) algebra $(A, [,])$ satisfies the *superJacobi identity* if and only if

$$(-1)^{ik}[a,[b,c]] + (-1)^{jk}[c,[a,b]] + (-1)^{ij}[b,[c,a]] = 0, \;\; \forall a \in A_i, b \in A_j, c \in A_k, \;\; \forall i,j,k \tag{I.2.12}$$

Such an A is called a *Lie superalgebra*;

(v) The endomorphisms of a superalgebra A satisfying the superLeibniz rule

$$D(a \cdot b) = (Da) \cdot b + (-1)^{ik} a \cdot (Db), \;\; \forall a \in A_i, b \in A_j, \forall i,j \tag{I.2.13}$$

for a $k \in \mathbb{Z}_2$ are called *even* or *odd derivations*, according to k being 0 or 1.

Remark I.2.2.4. (i) $\mathbb{Z}-$graded superalgebras are defined accordingly;

(ii) Derivations in $\mathbb{Z}-$graded superalgebras have degree in $\mathbb{Z}$, and are called even or odd according to their degree being 0 or 1 mod 2.

(iii) The mechanical procedure of substituting commutativity by graded commutativity goes under the name of *Quillen's law.*

Example I.2.2.5. $(\Omega(M), \wedge)$ is a commutative ($\mathbb{Z}-$graded) superalgebra:

$$[v,\omega] = v\wedge\omega - (-1)^{ij}\omega\wedge v = v\wedge\omega - (-1)^{2ij} v\wedge\omega = 0, \;\; \forall v \in \Omega^i(M), \omega \in \Omega^j(M), \;\; \forall i,j \tag{I.2.14}$$

Armed with such notions, we turn to the old problem of including the properties $I.2.6$, together with the overlying structure, in an unique algebraic object.

Definition I.2.2.6. Consider a Lie group G and its Lie algebra $\mathfrak{g}$. Then the *Lie superalgebra* $\tilde{\mathfrak{g}}$ of G is the Lie superalgebra

$$\tilde{\mathfrak{g}} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \tag{I.2.15}$$

with the grading $\mathfrak{g}_0 = \mathfrak{g}_0$, $\mathfrak{g}_1 = \mathfrak{g}_- \oplus \mathfrak{g}_+$, and:

(i) $\mathfrak{g}_-$ isomorphic to $\mathfrak{g}$ as a vector space via $a \mapsto i_a$;

(ii) $\mathfrak{g}_0$ isomorphic to $\mathfrak{g}$ as an algebra via $a \mapsto l_a$;

(iii) $\mathfrak{g}_+$ a $1-$dimensional vector space.

A multiplication is defined as follows:

$$\begin{cases} [,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0 : (l_a, l_b) \mapsto l_{[a,b]} \\ [,] : \mathfrak{g}_0 \times \mathfrak{g}_- \to \mathfrak{g}_- : (l_a, i_b) \mapsto l_{ad_a(b)} \\ [,] : \mathfrak{g}_0 \times \mathfrak{g}_+ \to \mathfrak{g}_+ \text{ trivial} \\ [,] : \mathfrak{g}_- \times \mathfrak{g}_- \to \mathfrak{g}_0 \text{ trivial} \\ [,] : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathfrak{g}_0 : (d, i_a) \mapsto l_a \\ [,] : \mathfrak{g}_+ \times \mathfrak{g}_+ \to \mathfrak{g}_0 \text{ trivial} \end{cases} \tag{I.2.16}$$

observe it respects the grading, it is anticommutative and satisfies the superJacobi identity.

Remark I.2.2.7. The set of rules I.2.16 mimics the Weil equations I.2.6. The Lie superalgebra of G expresses how its Lie algebra acts on the de Rham complex $\Omega(M)$ of some $G-$space M: we just need to connect the elements of $\mathfrak{g}$ to the corresponding fundamental vector fields of M.

The only new element in this system of equations is ad_a, the *adjoint representation* of $\mathfrak{g}$ on itself. This is given just by the commutator:

$$\mathfrak{g} \ni \xi \mapsto ad_\xi = [\xi, \cdot] \tag{I.2.17}$$

and can be shown to be the differential at 1 of the adjoint representation of G on $\mathfrak{g}$, i.e.

$$G \ni g \mapsto Ad_g = d_1(\psi_g) \tag{I.2.18}$$

with ψ_g the conjugation by g in G.

I.2.3. G^*-algebras

So far we managed to generalize the idea of action on the de Rham complex: as we saw, the Lie superalgebra of G expresses the action of $\mathfrak{g}$ on $\Omega(M)$ when the fundamental vector fields are given. The next step is to abstract from $\Omega(M)$: this will lead to the concept of G^*-*algebra*.

Definition I.2.3.1. (i) A G^*-module V is a supervector space together with a linear representation ρ of G on V and an action γ of the Lie superalgebra $\tilde{\mathfrak{g}}$ on V as endomorphisms.

We require the two operations to be compatible, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\rho(\exp{(t\xi)})(v) = \gamma(l_\xi)(v) \ \forall \xi \in \mathfrak{g}, \ \forall v \in V \tag{I.2.19}$$

(ii) A G^*-algebra A is a commutative superalgebra A together with a G^*-module structure, in which G acts by automorphisms and $\tilde{\mathfrak{g}}$ by derivations.

Remark I.2.3.2. (i) It is not a priori clear how the derivative in I.2.19 is defined. This can be done in two different ways: either if the G^*-module (or algebra) already has a topology in which the limit can be taken, or if each of the orbit spaces are finite dimensional vector spaces. We will always find ourselves in one of these two hypotheses;

(ii) When $\tilde{\mathfrak{g}}$ acts on A as derivations, we require the action it to mirror the grading of A: elements from $\mathfrak{g}_0$ should correspond to even derivation, elements from $\mathfrak{g}_\pm$ to odd derivations. But we also want the (possible) $\mathbb{Z}-$grading to be respected:

$$\begin{cases} \mathfrak{g}_- \ni i_\xi \mapsto \gamma(i_\xi) : A_i \to A_{i-1} \\ \mathfrak{g}_0 \ni l_\xi \mapsto \gamma(i_\xi) : A_i \to A_i \\ \mathfrak{g}_+ \ni i_\xi \mapsto \gamma(i_\xi) : A_i \to A_{i+1} \end{cases} \tag{I.2.20}$$

Morphisms of G^*-algebras and modules are maps which preserve the action:

Definition I.2.3.3. (i) Let A and B be G^*-modules. $f : A \to B$ is a *morphism of G^*-modules* if and only if $\forall x \in A, \xi \in \mathfrak{g}$

$$\begin{cases} [l_\xi, f] = 0 \\ [i_\xi, f] = 0 \\ [d, f] = 0 \end{cases} \tag{I.2.21}$$

(ii) A morphism of G^*-modules has *degree k* if and only if

$$f : A_i \to B_{i+k} \ \forall i \tag{I.2.22}$$

(iii) Let A and B be G^*-algebras. $f : A \to B$ is a *morphism of G^*-algebras* if and only if f is an algebra homomorphism and satisfies equations I.2.21. Degree of G^*-algebras morphisms is defined accordingly.

Remark I.2.3.4. (i) At first sight, it may seem that the morphism doesn't need to respect the ρ action; as a byproduct of I.2.19, however, it does. In fact, from I.2.19 and observing $[\mathrm{id} = \rho(1), f] = 0$ we obtain

$$[\rho(\exp(t\xi)), f] = 0 \tag{I.2.23}$$

and for our choice of G - compact and connected - the exponential mapping is a surjection. We obtain

$$[\rho(g), f] = 0 \ \forall g \in G \tag{I.2.24}$$

i.e. preservation of the G action.

(ii) As for derivations, morphisms of $\mathbb{Z}-$graded G^*-algebras and modules have degree in $\mathbb{Z}$, and are defined to be even or odd according to their degree mod 2 being 0 or 1;

With this definition, the set of G^*-algebras and G^*-modules is a category. We will find the algebraic equivalent of our universal bundle E in this category, giving an algebraic equivalent of the conditions of free action and contractibility.

I.2.4. Cohomology of G^*-algebras. Aciclicity.

Observe that $0 = \gamma([d, d]) = \gamma(d)^2$, and $\gamma(d)$ has degree 1 by hypothesis. Then a $\mathbb{Z}-$graded G^*-algebra together with the differential $\gamma(d)$ is a cochain complex, and we can consider its cohomology $H^*(A)$.

Remark I.2.4.1. (i) $H^*(A)$ with the cup product has a superalgebra structure, which is $\mathbb{Z}-$graded if A is;

(ii) We know that for a morphism of G^*-algebras $f : A \to B$ it holds $[d, f] = 0$. Then f descends to a morphism on cohomology: $f_* : H^*(A) \to H^*(B)$. Notice it is not contravariant;

(iii) One may wonder whether $H^*(A)$ inherits a G^* structure: since $[\iota_\xi, d] \neq 0$, it is in general not possible for the $\tilde{\mathfrak{g}}$ action on A to descend on cohomology; a $G-$action can, however, be defined.

To do this, observe

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(\exp(t\xi)) = \frac{\mathrm{d}}{\mathrm{d}\tau}|_{\tau=0}\rho(\exp((t+\tau)\xi)) = \rho(\exp(t\xi)) \cdot L_\xi \tag{I.2.25}$$

then

$$\begin{cases} 0 = \frac{\mathrm{d}}{\mathrm{d}t}(\rho(\exp(t\xi))\rho(\exp(-t\xi))) = \rho(\exp(t\xi)) \cdot L_\xi \cdot \rho(\exp(-t\xi)) - L_\xi \\ \frac{\mathrm{d}}{\mathrm{d}t}(\rho(\exp(t\xi)) \cdot d \cdot \rho(\exp(-t\xi))) = \rho(\exp(t\xi)) \cdot L_\xi \cdot d \cdot \rho(\exp(-t\xi)) \\ \qquad\qquad - \rho(\exp(t\xi)) \cdot d \cdot \rho(\exp(-t\xi)) \cdot L_\xi \end{cases} \tag{I.2.26}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho(\exp(t\xi)) \cdot d \cdot \rho(\exp(-t\xi))) = \rho(\exp(t\xi)) \cdot \underbrace{[L_\xi, d]}_{=0} \cdot \rho(\exp(-t\xi)) = 0 \tag{I.2.27}$$

Finally, surjectivity of exp for compact, connected Lie groups shows that $\rho(g)d = d\rho(g)\ \forall g \in G$, i.e. a $G-$action is well defined.
It is not an interesting action, however, since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho(\exp(t\xi)\omega) = d(\rho(\exp(t\xi)\iota_\xi\omega) + \rho(\exp(t\xi))\iota_\xi d\omega \tag{I.2.28}$$

meaning that on the level of cocycles ($d\omega = 0$) the derivative is 0. Since $\rho(\exp(0\xi))\omega = \omega$, we conclude that the action is trivial.

Contractibility - one of the key properties of the universal bundle - translates in the G^*-algebras language to acyclicity.

Definition I.2.4.2. A G^*-algebra A over a field $\mathbb{K}$ is *acyclic* if and only if

$$H^m(A) = \begin{cases} \mathbb{K}, & m = 0 \\ 0, & \text{otherwise} \end{cases} \tag{I.2.29}$$

Remark I.2.4.3. If M is a $G-$space, $\Omega(M)$ is a $\mathbb{Z}-$graded G^*-algebra. By construction, its cohomology $H^*(\Omega(M))$ coincides with $H^*(M)$, letting $\gamma(d)$ correspond to the usual differential, and we see that the aciclity condition mimics the contractibility condition.

The fact that the cohomology of $\Omega(M)$ as a G^*-algebra coincides with the usual de Rham cohomology of M also justifies, a posterior, the fact that there is no sensible action of either G or $\mathfrak{g}$ on the complex - this information just get lost when restricting to cochains.

I.2.5. Type (C) G^*−algebras

In this subsection we provide an algebraic version of the freeness of the action: we'll give the definition, and then try to show how it relates to the corresponding geometric notion.

Definition I.2.5.1. A G^*−algebra A is said to be *of type (C)* if there are elements $\theta^i \in A_1$, $i = 1, \dots, \dim G$, such that

$$\gamma(i_a)\theta^b = \delta_a^b \tag{I.2.30}$$

Such elements are called *connection elements* of A.

We expect this definition to generalize the concept of the de Rham complex of a G−space on which G acts freely; however, strictly speaking, this is not the case: when working with forms, it is easier to work on the infinitesimal level, passing from G to $\mathfrak{g}$, and reformulating the freeness condition in terms of $\mathfrak{g}$. This yields a weaker definition:

Definition I.2.5.2. Let M be a G−space. The action of G is *locally free* if and only $\forall \xi \neq 0 \in \mathfrak{g}$, the corresponding fundamental vector field ξ is nowhere vanishing.

Remark I.2.5.3. As promised, this is just an infinitesimal version of the freeness condition. Given a free action, we have

$$\exp(-t\xi) \cdot x = x \Rightarrow \xi = 0 \tag{I.2.31}$$

differentiating with respect to t, we get the condition

$$\xi(x) = 0 \Rightarrow \xi = 0 \tag{I.2.32}$$

Beware: on the left hand side we have the fundamental vector field of ξ, and not ξ itself. This is precisely our definition of locally free action.

Observe that we can associate to a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$ a set of 1−forms $\theta^1, \dots, \theta^n$ on M: we do so by requiring

$$\iota_{\xi_i}\theta^j = \delta_i^j \tag{I.2.33}$$

with ξ^i here denotes the fundamental vector field relative to the element of the basis. Observe that such an assignment is only made possible from the fundamental vector fields being non-vanishing everywhere: the local freeness of the action is equivalent to the existence of such θ's, which we call *connection forms* of M. The connection elements defined above generalize this notion, giving an algebraic generalization of the locally free action condition.

This generalization - surprisingly, perhaps - captures an important property of the connection elements, which only arises when we endow the manifold with a G−invariant metric.

We can do this without loss of generality: every manifold is metrizable (see [27, p. 166]), so that we can select some metric $\tilde{g}$ on M, and by compactness of G induce a G−invariant metric g via

$$g(v, w) = \int_G \tilde{g}(D_x L_h(v), D_x L_h(w))d\mu(h) \ \forall v, w \in T_x M, \forall x \in M \tag{I.2.34}$$

with μ a Haar measure for G (see e.g. [12]). Invariance follows from invariance of the measure:

$$\begin{aligned} g(D_x L_{h'} v, D_x L_{h'} w) &= \int_G \tilde{g}(D_x L_{h\cdot h'}(v), D_x L_{h\cdot h'}(w)) d\mu(h) = \\ &= \int_G \tilde{g}(D_x L_{h\cdot h'}(v), D_x L_{h\cdot h'}(w)) d\mu(h\cdot h') = g(v,w) \end{aligned} \tag{I.2.35}$$

The good thing about having a metric is that we receive a notion of orthogonality: the connection elements of M identify a subbundle $\mathcal{C}$ of T^*M, and by considering the orthogonal subspaces in each point we obtain a complementary subbundle, the *horizontal bundle* of M. Forms ω in the horizontal bundle are characterized by the requirements

$$\iota_{\xi_i}\omega = 0, \ i = 1, \dots, n \tag{I.2.36}$$

and called, accordingly, *horizontal.* Observe now

$$\begin{aligned} L_{\xi_i}\iota_{\xi_j}\theta^k &= ([L_{\xi_i}, \iota_{\xi_j}] + \iota_{\xi_j} L_{\xi_i})\theta^l \\ &= \iota_{[\xi_i,\xi_j]}\theta^l + \iota_{\xi_j} L_{\xi_i}\theta^l \\ &= c_{ij}^k \iota_{\xi_k}\theta^l + \iota_{\xi_j} L_{\xi_i}\theta^l \\ &= c_{ij}^l + \iota_{\xi_j} L_{\xi_i}\theta^l \end{aligned} \tag{I.2.37}$$

with $\{c_{ij}^k\}$ structure constants of $\mathfrak{g}$. On the other hand

$$L_{\xi_i}\iota_{\xi_j}\theta^k = L_{\xi_i}\delta_j^k = 0 \tag{I.2.38}$$

then we can write

$$L_{\xi_i}\theta^l = -\sum_j c_{ij}^l \theta^j + \omega \tag{I.2.39}$$

for a horizontal ω. Now, suppose that the horizontal bundle is $G-$invariant; since g is $G-$invariant as well, $\mathcal{C}$ is $G-$invariant, so that ω in I.2.39 must be zero: in fact, sub-bundle invariance reads

$$v \in \mathcal{C} \Rightarrow (\phi_{e^{t\xi}})^* v \in \mathcal{C} \tag{I.2.40}$$

then we can express $(\phi_{e^{t\xi}})^* v$ as a linear combination of the θ's:

$$(\phi_{e^{t\xi}})^* v = \sum_j f_j(e^{t\xi})\theta^j \tag{I.2.41}$$

deriving with respect to t yields the Lie derivative on one side, and on the right again an element of $\mathcal{C}$: comparing with I.2.39 yields then $\omega = 0$.

In conclusion, we obtain the equation

$$L_{\xi_i}\theta^l = -\sum_j c_{ij}^k \theta^j \tag{I.2.42}$$

and as a consequence, using Cartan's formula,

$$\iota_{\xi_i} d\theta^l = -\sum_j c_{ij}^k \theta^j \tag{I.2.43}$$

hence

$$d\theta^l = -\frac{1}{2}\sum_j c^k_{ij}\theta^i\theta^j + \mu^l \tag{I.2.44}$$

where μ^l is a *two*-form satisfying $\iota_{\xi_i}\mu^l = 0,\ i = 1, \dots, n$. The μ's are called *curvature forms* associated to the connection forms of M.

The next proposition shows that we can proceed analogously in the context of G^*-algebras.

Proposition I.2.5.4. *Let A be a type (C) G^*-algebra, and $(\theta^1, \dots, \theta^n)$ its connection elements. Then*

$$\gamma(l_{\xi_b})(\theta^a) = \sum_d c^a_{bd}\theta^d \tag{I.2.45}$$

Proof. We want to mirror the reasoning of the geometric case. Start by building an assignment from the basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$ to the connection elements $\theta^1, \dots, \theta^n$ of the G^*-algebra A: pick the basis $\xi^1, \dots, \xi^n$ of $\mathfrak{g}^*$ dual to the chosen basis of $\mathfrak{g}$, and then consider the homomorphism given by extension of the assignments

$$\xi^i \mapsto \theta^i \ \ i = 1, \dots, n \tag{I.2.46}$$

call this map h, and observe that we may rewrite I.2.30 as

$$\gamma(i_a)(h(\xi^b)) = \left\langle \xi_a, \xi^b \right\rangle \tag{I.2.47}$$

Now we want to reconstruct $I.2.42$. Observe that the homomorphism

$$h_g = \rho(g) \cdot h \cdot coAd_{g^{-1}} \tag{I.2.48}$$

also satisfies I.2.47. In fact:

$$\begin{aligned}\gamma(i_a)(h_g(\xi^b)) &= \rho(g) \cdot \iota_{Ad_{g^{-1}}\xi_a} h(coAd_{g^{-1}}\xi^b) \\ &= \rho(g) \cdot \left\langle Ad_{g^{-1}}\xi_a, coAd_{g^{-1}}\xi^b \right\rangle \\ &= \rho(g) \cdot \left\langle \xi_a, \xi^b \right\rangle = \delta^b_a\end{aligned} \tag{I.2.49}$$

Where we used:

- $\gamma(i_v)(h(w)) = \langle v, w\rangle \ \forall v \in \mathfrak{g},\ w \in \mathfrak{g}^*$. This follows from I.2.47;
- the compatibility relation $\iota_\xi \cdot \rho(g) = \rho(g) \cdot \iota_{Ad_{g^{-1}}\xi}$. This follows from I.2.19: in fact

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\rho(e^{t\xi})\iota_{Ad_{e^{-t\xi}}}\rho(e^{-t\xi}) = \\ &\quad = (\rho(e^{t\xi})L_\xi)\iota_{Ad_{e^{-t\xi}}}\rho(e^{-t\xi}) + \rho(e^{t\xi})\iota_{ad_\xi(Ad_{e^{-t\xi}}\eta)}\rho(e^{-t\xi}) + \rho(e^{t\xi})\iota_{Ad_{e^{-t\xi}}}(\rho(e^{-t\xi})L_{-\xi}) = \\ &\quad = \rho(e^{t\xi})([L_\xi, \iota_{Ad_{e^{-t\xi}}}] - \iota_{[\xi, Ad_{e^{-t\xi}}\eta]})\rho(e^{-t\xi}) = 0\end{aligned} \tag{I.2.50}$$

- The fact that the action $\rho(g)$ is trivial on A_0.

Consider now the average $\hat{h}$ of h over the group with respect to the Haar measure:

$$\hat{h}(w) = \int_G h_g(w) d\mu_g \tag{I.2.51}$$

the resulting $\hat{h}$ is such that $\hat{h}_g = \hat{h}$, by its very construction. This means $\rho(e^{t\xi_b}) \cdot \hat{h} = \hat{h} \cdot coAd_{e^{t\xi_b}}$: deriving on both sides in $t = 0$ we conclude

$$\gamma(l_{\xi_b})(h(\xi^a)) = \hat{h} \cdot coad_{\xi_b}(\xi^a) = -\hat{h}(\xi^a(ad_{\xi_b}(\cdot))) = -\hat{h}(\sum_i c^a_{bd}\xi^d) \tag{I.2.52}$$

that is

$$\gamma(l_{\xi_b})(\theta^a) = \sum_d c^a_{bd}\theta^d \tag{I.2.53}$$

□

Remark I.2.5.5. This is exactly I.2.42! From here, we can proceed analogously and define *curvature elements* for the G^*−algebra A. Notice that in this derivation there isn't an equivalent of the assumption that the horizontal bundle be G−invariant: this is not needed either when we move from the space of all horizontal differential forms to the horizontal *closed* differential forms - which is exactly what A should be generalizing.

I.2.6. Equivariant cohomology of G^*−algebras

We are now equipped with the algebraic equivalent A of a contractible space E on which G acts freely. Remember the idea of the geometric construction:

$$M \rightsquigarrow M \times E \rightsquigarrow (M \times E)/G \tag{I.2.54}$$

here we are working with $\Omega(M)$, rather than M, and we need to give an algebraic version of the procedure above. This will be

$$\Omega(M) \rightsquigarrow \Omega(M) \otimes A \rightsquigarrow (\Omega(M) \otimes A)_{bas} \tag{I.2.55}$$

it is not very surprising that the tensor product is the right substitute for the cartesian product of spaces: one already appreciates such an interplay between the two concepts when studying the singular homology of a product of spaces. The new ingredient are the *basic* elements of the tensor product of G^*−algebras, which should convey the concept of equivalence classes of forms with respect to the G−action.
As before, we give the algebraic definition and then show its relation to the geometric notion.

Definition I.2.6.1. Let A be a G^*−algebra. An element $\omega \in A$ is called *basic* if and only if it satisfies the equations

$$\begin{cases} \gamma(l_{\xi_i})\omega = 0, \ i = 1, \dots, n \\ \gamma(i_{\xi_i})\omega = 0, \ i = 1, \dots, n \end{cases} \tag{I.2.56}$$

We denote by A_{bas} the set of basic elements of A.

Remark I.2.6.2. Basic elements form a subcomplex of A. In fact:

$$\begin{cases}\gamma(l_{\xi_i})(\gamma(d)\omega) = \gamma(d)(\gamma(l_{\xi_i})\omega) = 0,\ i = 1, \dots, n \\ \gamma(i_{\xi_i})(\gamma(d)\omega) = \gamma(l_{\xi_i})\omega - \gamma(d)(\gamma(i_{\xi_i})\omega) = 0,\ i = 1, \dots, n\end{cases} \tag{I.2.57}$$

the cohomology of this subcomplex is the *basic cohomology* of A:

$$H^*_{bas}(A) = H^*(A_{bas}, \gamma(d)) \tag{I.2.58}$$

similarly, one shows that a morphism of G_*-algebras induces a morphism on the level of basic cohomology.

The next proposition gives I.2.56 a geometric interpretation:

Proposition I.2.6.3. *Suppose G acts freely on M, let $\pi : M \to M/G$ be the orbit map. Then*

$$\pi^*\Omega(M/G) = \{\omega \in \Omega(M) : \iota_{\xi_i}\omega = L_{\xi_i}\omega = 0,\ i = 1, \dots, n\} \tag{I.2.59}$$

Proof. Suppose $\omega \in \pi^*\Omega(M/G)$. Then $\omega = \pi^*\eta$ for some $\eta \in \Omega(M/G)$, and

$$\iota_{\xi_i}\omega = \iota_{\xi_i}\pi^*(\eta) = \pi^*(\iota_{\pi_*\xi_i}\eta) = 0 \tag{I.2.60}$$

since

$$\pi_*(\xi_i)_x = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\pi(\phi_{\exp(t\xi_i)}(x)) = 0\ \forall x \in M \tag{I.2.61}$$

by Cartan's formula, we also get $L_{\xi_i}\omega = 0$.

Suppose now that ω satisfies $\iota_{\xi_i}\omega = L_{\xi_i}\omega = 0,\ i = 1, \dots, n$. By connectedness of G, the condition $L_{\xi_i}\omega = 0$ for each i is equivalent to the $G-$invariance $(\phi_g)^*\omega = \omega\ \forall g \in G$.
By the slice theorem (see [6, p. 17]), $\pi : M \to M/G$ is a fibration with fiber G, so that it suffices to show $\omega \in \ker(j^* : \Omega(M) \to \Omega(G))$. To see this, denote multiplication in G by ψ

$$j \cdot (\psi_{h^{-1}}) = \phi_{h^{-1}} \cdot j\ \forall h \in G \tag{I.2.62}$$

where we defined j to take $g \in G$, seen as the fiber over $x \in M$, to $\phi_{g^{-1}}(x)$. Now, use $G-$invariance of ω:

$$j^*(\omega) = j^*(\phi^*_{h^{-1}}(\omega)) = (j \cdot \psi_{h^{-1}})^*\omega\ \forall h \in G \tag{I.2.63}$$

in particular, this implies $j^*(\omega)_h = j^*(\omega)_e\ \forall h \in G$. We know that a basis of T_eG is given by $\xi_1, \dots, \xi_n$; if we can show that $j_*\xi_i = \xi_i$ - on the right we have the fundamental vector field associated to ξ_i -, we are done:

$$j^*(\omega)_e(\xi_{i_1}, \dots, \xi_{i_k}) = \omega_x(\xi_{i_1}, \dots, \xi_{i_k}) = 0 \tag{I.2.64}$$

since $\iota_{\xi_i}\omega = 0$. To see why $j_*\xi_i = \xi_i$, observe that

$$j_*\xi_i = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}j(\exp(t\xi_i)) = -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\phi_{\exp t\xi}(x) \tag{I.2.65}$$

exactly the definition of the fundamental vector field related to ξ_i. □

In conclusion, in the case where the action is already free basic elements really coincide with equivalence classes of forms, in the sense specified by the proposition. For $A = \Omega(M)$, basic elements are usually called *basic forms*.

The last step of the construction is - finally! - the definition of this algebraic equivariant cohomology of M.

Definition I.2.6.4. Let A be a type (C), acyclic G^*-algebra. We call $(\Omega(M) \otimes A)_{bas}$ the *homotopy basic subcomplex* of M, and

$$H^*_G(M) = H^*_{bas}(\Omega(M) \otimes A) \tag{I.2.66}$$

the *equivariant cohomology ring* of M.

Some questions naturally arise:

- Is the definition well posed? Namely, is it true that there isn't any dependence on the specific A we choose?
- The name *homotopy basic subcomplex* suggests that, in the case where the action is already free, $(\Omega(M) \otimes A)_{bas}$ coincides with $(\Omega(M))_{bas}$. Is this the case?
- Last but not least: does this algebraic equivariant cohomology coincides with the geometric equivariant cohomology defined in the first section?

All these questions have positive answer, but will not be discussed in these pages. The interested reader may consult [14]: the first two points are discussed at page 48, while the last point, the Equivariant de Rham Theorem, is shown to hold at page 28.

Our aim is now to give a concrete example of type (C), acyclic algebra, which will result in the construction of the Weil model. Together with the Cartan model, this version has the advantage of taking into play $S(\mathfrak{g}^*)$, the algebra of polynomials over $\mathfrak{g}$: this allows, in turn, a more concrete visualization of equivalence classes of cochains.

I.2.7. The Weil model

The idea here is to define an acyclic complex, and later on add the $G-$structure. The type (C) property will follow as a byproduct.

Definition I.2.7.1. Let V be an $n-$dimensional vector space, consider its exterior algebra $\Lambda(V)$ and its symmetric algebra $S(V)$.

(i) The *Koszul algebra* $\mathcal{K}$ of V is the tensor product $\Lambda(V) \otimes S(V)$, graded assigning degree k to elements of $\Lambda^k(V) \otimes S^0$, and $2k$ to elements of $\Lambda^0(V) \otimes S^k(V)$;

(ii) The *Koszul operator* d_K is the derivation extending the assignments

$$\begin{cases} \Lambda^1(V) \otimes S^0(V) \ni x \otimes 1 \mapsto 1 \otimes x \in \Lambda^0(V) \otimes S^1(V) \\ \Lambda^0(V) \otimes S^1(V) \ni 1 \otimes x \mapsto 0 \end{cases} \tag{I.2.67}$$

this extension is unique, since such elements generate $\mathcal{K}$;

(iii) The *Koszul complex* of V is the couple $(\mathcal{K}, d_K)$.

Lemma I.2.7.2. *The Koszul complex of V is an acyclic chain complex.*

Proof. To see $d_K^2 = 0$, observe that d_K^2 is itself a derivation:

$$\begin{aligned} d_K^2(uv) &= d_K(d_K(u)v + (-1)^{|u|}u(d_K(v))) = \\ &= d_K^2(u)v + (-1)^{(|u|+1)}d_K(u)d_K(v) + (-1)^{|u|}d_K(u)d_K(v) + u(d_K^2(v)) = \\ &= d_K^2(u)v + u(d_K^2(v)) \end{aligned} \tag{I.2.68}$$

as such, it suffices to check whether it vanishes on generators: but it is clear from the definition of d_K that $d_K^2(1 \otimes x) = d_K^2(x \otimes 1) = 0$ for x either in $\Lambda^1(V)$ or $S^1(V)$.

Now let's show that this complex is acyclic. To compute the cohomology in degree greater than zero, consider the derivation H extending the assignments

$$\left\{\Lambda^1(V) \otimes S^0(V) \ni x \otimes 1 \mapsto 0 \ \Lambda^0(V) \otimes S^1(V) \ni 1 \otimes x \mapsto x \otimes 1\right. \tag{I.2.69}$$

one directly checks that $Q = H \cdot d_K + d_K \cdot H$ is an even derivation, satisfying $Q(x \otimes 1) = x \otimes 1$, $Q(1 \otimes x) = 1 \otimes x$. But then, in general $Q(u) = |u| \cdot u$, which in turn implies

$$u = \frac{1}{|u|}Q(u) = \frac{d_K(H(u))}{|u|} + \frac{H(d_K(u))}{|u|} \tag{I.2.70}$$

when working with cycles we have $u \in \ker d_K$, and the equation above provides a preimage of u: then cohomology in degree greater than 0 must be zero.

Cohomology in degree zero equals the field: by properties of derivations $d_K(1) = 0$. □

Now we need to make G act on $\mathcal{K}$, to get the G^*-algebra structure. We still have a somewhat large degree of freedom, the choice of V: we set it to be equal to $\mathfrak{g}^*$, just as a vector space, and then try to define an action.

Definition I.2.7.3. The *Weil algebra* $\mathcal{W}$ of G is the Koszul algebra of $\mathfrak{g}^*$:

$$\mathcal{W} = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \tag{I.2.71}$$

Now we need to put a G^*-algebra structure on $\mathcal{W}$, that is, an action of G on $\mathcal{W}$ via automorphisms, and of the super Lie algebra $\tilde{\mathfrak{g}}$ via derivations (cf. I.2.3). There is a natural choice for the action of G on $\mathcal{W}$, the extension of the coadjoint representation on $\mathfrak{g}^*$; we still denote by Ad^*. Since the action γ of $\tilde{\mathfrak{g}}$ should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \mathrm{Ad}^*_{e^{t\xi}} = \gamma(l_\xi) \tag{I.2.72}$$

we can set $\gamma(l_\xi)$ as the derivation extending the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$: then right hand side and left hand side are derivations agreeing on generators, hence identical everywhere.

We already have a differential, d_K, and we need to check

$$\gamma(l_\xi) \cdot d_K = d_K \cdot \gamma(l_\xi) \tag{I.2.73}$$

Fix a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$, and denote respectively by $\theta^1, \dots, \theta^n$ and $z^1, \dots, z^n$ the induced generators $\xi^1 \otimes 1, \dots, \xi^n \otimes 1$ and $1 \otimes \xi^1, \dots, 1 \otimes \xi^n$; to check I.2.73, write the action of $\gamma(l_\xi)$ on generators in terms of the structure constants, i.e.

$$\begin{cases} \gamma(l_{\xi_i})\theta^j = -c^j_{ik}\theta^k \\ \gamma(l_{\xi_i})z^j = -c^j_{ik}z^k \end{cases} \tag{I.2.74}$$

then I.2.73 is seen to be satisfied on generators, and hence everywhere. Also the relation $[\gamma(l_{\xi_i}), \gamma(l_{\xi_j})] = \gamma(l_{[\xi_i,\xi_j]})$ follows by construction, making use of the Jacobi identity.
It remains to define the action $\gamma(i_\xi)$, in such a way that it is coherent with all the rest. Since in the end we will want a type (C) structure, we require from the very beginning that some elements of $\mathcal{W}$ play the role of the connection elements: we already prepared the notation in this sense, and now set, by definition,

$$\gamma(i_{\xi_a})\theta^b = \delta^b_a \tag{I.2.75}$$

the value of $\gamma(i_{\xi_a})$ on the z's is forced by compatibility, in particular by Cartan's formula:

$$\gamma(i_{\xi_a})z^b = \gamma(i_{\xi_a})d\theta^b = (d\gamma(i_{\xi_a}) + \gamma(i_{\xi_a})d)\theta^b = \gamma(l_{\xi_a})\theta^b = -c^b_{ak}\theta^k \tag{I.2.76}$$

and now we can extend $\gamma(i_{\xi_a})$ as a derivation over the whole $\mathcal{W}$. There are still a pair of equations to be checked, namely $\{\gamma(\iota_{\xi_i}), \gamma(\iota_{\xi_j})\} = 0$ and $[\gamma(l_{\xi_i}), \gamma(\iota_{\xi_j})] = \gamma(\iota_{[\xi_i,\xi_j]})$: these identities can be seen to hold by using the structure constants and the Jacobi identity.

The discussion above shows:

Proposition I.2.7.4. *The Weil algebra $\mathcal{W}$ is an acyclic, type (C) G^*-algebra.*

So far, we obtained at least that there exist type (C), acyclic G^*-algebras; taking exactly the Weil algebra as such a G^*-algebra has other advantages, though: for example, we get some insight on the cohomology of a single point, which we already realized to be quite complex.

We don't need much more machinery: let's make explicit a concept we already touched, that of horizontal elements:

Definition I.2.7.5. Let A be a G^*-algebra. An element $\omega \in A$ satisfying

$$\gamma(i_{\xi_i})\omega = 0, \ i = 1, \dots, n \tag{I.2.77}$$

is called *horizontal*. We denote the set of horizontal elements by A_{hor}.

After a "change of variables", $\mathcal{W}$ is seen to have a nice characterization in terms of horizontal elements. We assume the Einstein convention until the end of the section: whenever a latin index is repeated, a sum is implied.

Theorem I.2.7.6. *There is an isomorphism*

$$\mathcal{W} = \Lambda(\mathfrak{g}^*) \otimes \mathcal{W}_{hor} \tag{I.2.78}$$

Proof. As remarked above, this follows essentially from a change of variables. Define

$$\mu^i = z^i + \frac{1}{2}c^i_{jk}\theta^j\theta^k \tag{I.2.79}$$

we can express the z's in terms of the μ's: then we can pick $\theta^1, \dots, \theta^n, \mu^1, \dots, \mu^n$ as a set of generators of $\mathcal{W}$. The μ's have the advantage of being horizontal:

$$\gamma(\iota_{\xi_l})\mu^i = \gamma(\iota_{\xi_l})\theta^i + \frac{1}{2}c^i_{jk}\gamma(\iota_{\xi_l})(\theta^j\theta^k) = -c^i_{lj}\theta^j + \frac{1}{2}c^i_{lj}\theta^j - \frac{1}{2}c^i_{jl}\theta^j = 0 \tag{I.2.80}$$

moreover, the horizontal elements of $\mathcal{W}$ are exactly those generated by the μ's: in fact

$$\iota_{\xi_a}(\sum_{i,j,k,k'} c_{ij}\theta^{i_1}\dots\theta^{i_k}\mu^{j_1}\dots\mu^{j_{k'}}) = 0 \Rightarrow \iota_{\xi_a}(\theta^{i_1}\dots\theta^{i_k}) = 0 \tag{I.2.81}$$

and the latter can hold for each a if and only if $\theta^{i_1}\dots\theta^{i_k} = 0$. This proves the claim. □

This results produces a neat visualization of the equivariant cohomology of a point:

Corollary I.2.7.7. *$H^*_G(*) = (\mathbb{C}[\mu^1, \dots, \mu^n])^G$, the $G-$invariant polynomials in $\mu^1, \dots, \mu^n$.*

Proof. We know from the theorem above $W_{hor} = S(\mathfrak{g}^*) = \mathbb{C}[\mu^1, \dots, \mu^n]$; then $\mathcal{W}_{bas} = (\mathbb{C}[\mu^1, \dots, \mu^n])^G$, the elements in W which are horizontal and $G-$invariant. Furthermore

$$H^*_G(*) = H^*(\mathcal{W}_{bas}) \tag{I.2.82}$$

so that we only need to show $d_K\omega = 0\ \forall\omega \in \mathcal{W}_{bas}$. Let's start by computing $d_K(\mu^k)$: from the very definition, we have

$$\begin{aligned} d\mu^k &= dz^k + \frac{1}{2}c^k_{ij}d(\theta^i\theta^j) = \frac{1}{2}c^k_{ij}(d\theta^i)\theta^j - \frac{1}{2}c^k_{ij}\theta^i(d\theta^j) = \\ &= \frac{1}{2}c^k_{ij}(\mu^i - \frac{1}{2}c^i_{pq}\theta^p\theta^q)\theta^j - \frac{1}{2}c^k_{ij}\theta^i(\mu^j - \frac{1}{2}c^j_{pq}\theta^p\theta^q) = \\ &= -c^k_{ij}\theta^i\mu^j + \frac{1}{2}(c^i_{jp}c^k_{iq} + c^i_{qj}c^l_{ip})\theta^p\theta^q\theta^j + \frac{1}{2}c^i_{pq}c^k_{ij}\theta^p\theta^q\theta^j = \\ &= -c^k_{ij}\theta^i\mu^j \end{aligned} \tag{I.2.83}$$

using the Jacobi identity twice. One similarly computes

$$\gamma(l_{\xi_i})\mu^k = -c^k_{ij}\mu^j \tag{I.2.84}$$

Then $d = \theta^i\gamma(l_{\xi_i})$ when acting on some μ^j; but every basic element η is generated by the μ's, and satisfies $\gamma(l_{\xi_i})\eta = 0$, so that

$$d\eta = \theta^i\gamma(l_{\xi_i})\eta = 0 \tag{I.2.85}$$

This concludes the proof. □

Remark I.2.7.8. We may think of $(\mathbb{C}[\mu^1, \ldots, \mu^n]) = S(\mathfrak{g}^*)$ as the tensor product $S(\mathfrak{g}^*) \otimes \mathbb{C}$ or equivalently $S(\mathfrak{g}^*) \otimes \Omega(*)$, using $\mathbb{C}$ as ground field: then $H^*_G(*) = H^*((S(\mathfrak{g}^*) \otimes \Omega(*))^G)$.

This is a particular instance of a theorem of Cartan (see [14, p. 45]):

$$H^*_G(M) = H^*(C_G(M), d_G) \tag{I.2.86}$$

on the left we have our usual equivariant cohomology, and on the right another algebraic formulation, the *Cartan model*: in this model we pick as chain complex

$$C_G(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G \tag{I.2.87}$$

the $G-$invariant elements of $S(\mathfrak{g}^*) \otimes \Omega(M))$, which we may regard as $G-$invariant polynomials from $\mathfrak{g}$ to $\Omega(M)$; as before, elements of $S(\mathfrak{g}^*)$ have their grading doubled, while $\Omega(M)$ keeps the usual grading. The $G-$invariance for an element $\omega \in S(\mathfrak{g}^*) \otimes \Omega(M)$ reads

$$\omega(Ad_{g^{-1}}(\xi)) = g^*\omega(\xi) \tag{I.2.88}$$

and the differential d_G of ω is the element

$$\eta : \xi \mapsto d(\omega(\xi)) - \iota_\xi(\omega(\xi)) \tag{I.2.89}$$

one computes $d^2_G(\omega)(\xi) = -L_\xi(\omega(\xi))$, and this is seen to be zero by deriving I.2.88 in $t = 0$.

We conclude the analysis of $H^*_{S^1}(S^2)$ for two different S^1 actions (see I.1.5.3).

Example I.2.7.9. The first action of S^1 on S^2 we defined is a rotation around the equator:

$$f_\theta : S^2 \to S^2 : \begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi) \\ \sin(\psi)\cos(\phi) \end{bmatrix} \mapsto \begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi+\theta) \\ \sin(\psi)\cos(\phi+\theta) \end{bmatrix} \tag{I.2.90}$$

this action has the northern and southern poles as fixed point, and with the help of the Mayer-Vietoris sequence we computed

$$H^k_{S^1}(S^2) = \begin{cases} \mathbb{C}, \ k = 0 \\ \mathbb{C}^2, \ k \text{ even} \\ 0, \ \text{otherwise} \end{cases} \tag{I.2.91}$$

The plan now is to use the Cartan model to interpret elements of the ring as polynomials over the Lie algebra of S^1, $\mathbb{R}$. We want to enforce condition I.2.88; observe that $Ad_g(\xi) = \xi, \ \forall g \in S^1, \forall \xi \in \mathbb{R}$, since S^1 is abelian, so that we require our polynomials $\omega(\xi)$ to be constant on each orbit of the action.

We can then use the isomorphisms in the Mayer-Vietoris sequence to determine what the generators looks like. The map

$$(\iota_+)^*_{S^1} - (\iota_-)^*_{S^1} : H^k_{S^1}(S^2) \to H^k_{S^1}(N) \oplus H^k_{S^1}(S) \tag{I.2.92}$$

is injective for each k, and isomorphism for $k > 0$. Here $+, -$ are the northern and southern pole and $\hat{\iota} = \iota_+, \ \iota_-$ the inclusions.
Both $(\iota_+)^*_{S^1}$ and $(\iota_-)^*_{S^1}$ are morphisms of $\mathbb{C}[\xi]-$algebra, and so must be $(\iota_+)^*_{S^1} - (\iota_-)^*_{S^1}$.

We can then describe $H^*_{S^1}(S^2)$ as a subalgebra of $\mathbb{C}[N] \oplus \mathbb{C}[S]$ by analyzing the image $\hat{\iota}$: in grade 0 we have an exact sequence

$$0 \to H^0_{S^1}(S^2) \to^{\hat{\iota}} H^0_{S^1}(N) \oplus H^0_{S^1}(S) \to^{j^*} H^0(*) \to 0 \qquad \text{(I.2.93)}$$

with $j^*(z_1, z_2) = z_1 - z_2$. By exactness, the image of $\hat{\iota}$ in degree zero is then given by those polynomials in $C[N] \oplus C[S]$ whose constant components coincide; $\hat{\iota}$ is an isomorphism on all the other levels, so that this is the only restriction we need to impose. We can obtain the value of the constant component of the polynomials by evaluating them at 0, so that

$$H^*_{S^1}(S^2) \simeq \{(f, g) \in \mathbb{C}[N] \oplus \mathbb{C}[S] : f(0) = g(0)\} \qquad \text{(I.2.94)}$$

Now analyze the case of the trivial action, for which we know $H^*_{S^1}(S^2) = \mathbb{C}[\xi] \otimes H^*(S^2)$. Pick a nonzero element $\nu \in H^2(S^2)$, then a pair of generators of the $C[\xi]-$algebra $H^k_{S^1}$ is given by

$$\begin{cases} \phi = 1 \otimes 1 \in \mathbb{C}[\xi]^0 \otimes H^0(S^2) \\ \gamma = 1 \otimes \nu \in \mathbb{C}[\xi]^0 \otimes H^2(S^2) \end{cases} \qquad \text{(I.2.95)}$$

We conclude that the cohomology rings are isomorphic when considered as modules over $\mathbb{C}[\xi]$, just via the assignment

$$(f(N), g(S)) \mapsto \frac{(f(\xi) - g(\xi))}{2\xi} \otimes 1 + \frac{(f(\xi) + g(\xi))}{2} \otimes \nu \qquad \text{(I.2.96)}$$

having as inverse

$$a(\xi)\phi + b(\xi)\nu = (a(N) - b(N) \cdot N, a(S) + b(S) \cdot S) \qquad \text{(I.2.97)}$$

It is, anyway, impossible to construct an isomorphism respecting the $\mathbb{C}[\xi]-$algebra structure. In fact we have

$$\begin{cases} \gamma^2 = 0 \\ (f, g)^2 = (f^2, g^2), \ (f, g) \in \mathbb{C}[N] \oplus \mathbb{C}[S] \end{cases} \qquad \text{(I.2.98)}$$

so that an isomorphism ψ should satisfy $0 = \psi(\nu^2) = (f^2, g^2)$ for $f, g \neq 0$, hence a contradiction.

II. The localization theorem and its consequences

Now that all the basic definitions are in place, we can move forward and prove the localization theorem: in its algebraic formulation, the theorem states that the equivariant cohomology of the space is the same as that of its fixed points, as long as we forget about torsion components. When we add geometric information on the manifold, the theorem yields an explicit integration formula.

The first part of the chapter is dedicated to the proof of the theorem and of this formula. In the second part, we examine the consequences of the localization theorem in the setting of symplectic geometry.

II.1. The localization theorem

For the moment, we restrict our attention to torus actions, setting $G = T$. Moreover, the manifold M on which we work is always assumed to be compact and orientable, unless otherwise stated.

When considering a torus, $H_T^*(M)$ can be regarded as a $\mathbb{C}[u_1, \ldots, u_l]-$module, l being the dimension of T: we start by introducing the concepts of support and localization for such objects.

II.1.1. Support and localization of a $\mathbb{C}[u_1, \ldots, u_l]-$module

The notion of support of a $\mathbb{C}[u_1, \ldots, u_l]-$module is introduced in order to handle torsion information on such modules.

Definition II.1.1.1. Let H be a $\mathbb{C}[u_1, \ldots, u_l]-$module. The *support of* H is the set

$$\operatorname{supp} H = \bigcap_{f : fH = 0} V_f \subset \mathbb{C}^l \tag{II.1.1}$$

with $V_f = \{z \in C^l : f(z) = 0\}$.

Remark II.1.1.2. The kind of information this notion conveys is similar to that of support of a function: it provides the points where, in a suitable sense, the object we're considering is not zero.

For H, this can be clearly seen when we have something of the form

$$H = \mathbb{C}[u_1, \ldots, u_l]/p^1 \otimes \cdots \otimes \mathbb{C}[u_1, \ldots, u_l]/p^k \tag{II.1.2}$$

with $p^1, \ldots, p^k \in \mathbb{C}[u_1, \ldots, u_l]$. Notice this is not the most general case, for H has here a natural algebra structure.

For an element $[a_1] \otimes \dots \otimes [a_k] \in H$, the evaluation map

$$[a_1] \otimes \dots \otimes [a_k] \mapsto a_1(x) \dots a_k(x) \tag{II.1.3}$$

is well defined for all $x \in \operatorname{supp} H$, and it yields zero if and only if $[a_1] \otimes \dots \otimes [a_k] = 0$. In this case, the support of H is the locus of points where no nontrivial element of H vanishes, expressing the usual concept of "points where the object is not zero".
Finally, observe that the definition of support for functions consists in the closure of the set where the function is not zero; a closure is here not necessary, since all of the V_f's are already closed sets.

We list some key features and properties of the support:

Proposition II.1.1.3. *Let H be a $\mathbb{C}[u_1, \dots, u_l]-$module. Then:*

(i) $\operatorname{supp} H = \mathbb{C}^l \iff H$ *is free;*

(ii) $\operatorname{supp} H = \emptyset \iff H = 0$;

(iii) if H is graded, $\operatorname{supp} H$ *is $\mathbb{C}-$invariant;*

(iv) Let F, G be $\mathbb{C}[u_1, \dots, u_l]-$modules such that

$$F \to^{\iota} G \to^{\pi} H \tag{II.1.4}$$

is exact. Then

$$\operatorname{supp} G \subset (\operatorname{supp} F \cup \operatorname{supp} H) \tag{II.1.5}$$

Proof. Ad (i). $\operatorname{supp} H = \mathbb{C}^l \iff (fH = 0 \Rightarrow f = 0)$, which is equivalent to freeness.
Ad (ii). $\Rightarrow$: there exist $z_1, z_2 \in \mathbb{C}^l$ such that the polynomials $f(z) = (z - z_1)$, $g(z) = (z - z_2)$ satisfy $fH = 0 = gH$. Let then $\omega = z_1 - z_2$ and observe

$$H \ni h = \frac{\omega}{\omega} h = \frac{\omega}{\omega} h + \frac{x - z_1}{\omega} h = \frac{x - z_2}{\omega} h = 0 \; \forall h \in H \tag{II.1.6}$$

so that $H = 0$. Conversely, $H = 0$ is annihilated by all the polynomials, so that its support is empty.
Ad (iii). If $H = \oplus_{i \in \mathbb{N}} H$ is a graded module, any $f \in \mathbb{C}[u_1, \dots, u_l]$ such that $fH = 0$ must be homogeneous in grade. In fact, the module structure must be compatible with the grading, so that

$$u_j^k h_i \in H_{i+2k} \; \forall h_i \in H_i, \forall i \tag{II.1.7}$$

consider for simplicity $f = u_1^{j_1} \dots u_l^{j_l} + u_1^{k_1} \dots u_l^{k_l}$ such that $fH = 0$, let $j = \sum j_i$ and $k = \sum k_i$. Then $\forall h_i \in H_i$, $\forall i$ it holds

$$H_{i+2j} \ni (u_1^{j_1} \dots u_l^{j_l}) h_i = (u_1^{k_1} \dots u_l^{k_l}) h_i \in H_{i+2k} \tag{II.1.8}$$

so that either the two elements are separately zero or $j = k$. In any case, the torsion polynomial must be homogeneous in grade, and the presence of more terms or coefficients in f does not invalidate the argument. But the zeroes of such polynomials are linear subspaces

of $\mathbb{C}^l$, hence so is their intersection.
Ad (iv). We show, equivalently,

$$z \notin (\operatorname{supp} F \cap \operatorname{supp} H) \Rightarrow z \notin \operatorname{supp} G \ \forall z \in \mathbb{C}^l \tag{II.1.9}$$

to see this, pick $f, h \in \mathbb{C}[u_1, \ldots, u_l]$ such that $fF = 0 = hH$, $f(z) \neq 0 \neq h(z)$. Then for each $\omega \in G$, $\pi(h\omega) = 0$, and, by exactness, there exists $\eta \in F$ such that $\iota(\eta) = h\omega$. But $f\eta = 0$, so that $0 = \iota(f\eta) = f\iota(\eta) = fh\omega$. Since ω was arbitrary we conclude $(fh)G = 0$, and by construction $fh(z) \neq 0$. Thus, $z \notin \operatorname{supp} G$. □

As a corollary we get

Corollary II.1.1.4. *Let $H \neq 0$ be a torsion $C[u]$−module. Then* $\operatorname{supp} H = 0$.

Proof. The only $\mathbb{C}$−invariant subspaces of $\mathbb{C}$ are 0 and $\mathbb{C}$ itself. □

The next concept we introduce is that of localization: it is the algebraic equivalent of restricting to an open set, and the tool we use to selectively ignore some of the torsion sub-modules of H.

Definition II.1.1.5. Let H be a $\mathbb{C}[u_1, \ldots, u_l]$−module, $f \in \mathbb{C}[u_1, \ldots, u_l]$. The *localization of H at f* is the $\mathbb{C}[u_1, \ldots, u_l]$−module given by

$$H_f = H \otimes \mathbb{C}[u_1, \ldots, u_l]_f \tag{II.1.10}$$

with $\mathbb{C}[u_1, \ldots, u_l]_f$ the ring of rational functions having a power of f as denominators.

Remark II.1.1.6. (i) "Localization of H at f" is short for "Localization of H at $\mathbb{C}^l - V_f$": we take away from the domain the points on which the rational functions wouldn't be defined. An analogous procedure generates $\mathbb{Q}$ from $\mathbb{Z}$, by localizing outside zero;

(ii) Suppose $fH = 0$ for some polynomial f. Then

$$H_f \ni a \otimes b = a \otimes f\frac{b}{f} = fa \otimes \frac{b}{f} = 0 \tag{II.1.11}$$

so that $H_f = 0$. Localizing outside of the support kills the module, and we can use this fact to ignore specific torsion component, as promised before.

II.1.2. The algebraic localization theorem

The plan is now to take $H^*_T(M)$ as our $\mathbb{C}[u_1, \ldots, u_l]$−module, and relate its support to the Lie algebra of T. As a first step in this direction, recall that the $u_1, \ldots, u_n$ may be interpreted as coordinate on the Lie algebra of T, as stated in I.2.7.7; observe then that the complexification of the Lie algebra of T, $\mathfrak{t} \simeq \mathbb{R}^l$, is $\mathfrak{t}^c = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^l$, so that we may think of $\operatorname{supp} H^*_T(M)$ as contained in $\mathfrak{t}^c$.
Let's see a consequence of this new perspective:

Lemma II.1.2.1. *If there is a T−map $M \to T/K$, where K is a closed subgroup of T, then*

$$\operatorname{supp} H^*_T(X) \subset \mathfrak{k}^c \tag{II.1.12}$$

Proof. Consider the following diagram:

$$\begin{array}{ccc} X & \longrightarrow & T/K \\ & \searrow & \downarrow \\ & & \{*\} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} H_T^*(X) & \longleftarrow & H_K^*(*) \\ & \nwarrow & \uparrow \\ & & H_T^*(*) \end{array} \tag{II.1.13}$$

The inclusion of the base ring $H_T^*(*) \to H_T^*(X)$ factors through $H_K^*(*)$. As a closed subgroup of T, K is the product of a torus with finite groups; the finite groups only generate torsion components and their contribution is lost when considering complex coefficients, and we can regard $H_T^*(X)$ as a module over $C[u_1, \dots, u_k]$, where k is the dimension of the torus component of K.

By commutativity of the diagram, the H_T^*-module and H_K^*-module structures are compatible; the support of $H_T^*(X)$ as a H_T^*-module, obtained by intersections, must therefore be included in the support of $H_T^*(X)$ as a H_K^*-module, which is in turn included in $\mathfrak{k}^c$. □

Remark II.1.2.2. Topologically, such $T-$maps arise when X is a orbit of T, having K as stabilizer. For fixed points, the lemma does not provide any useful information; on the other hand, when the stabilizer is not the whole group we gain some insight about where the torsion takes place.

Now we have all we need to prove the algebraic localization theorem:

Theorem II.1.2.3. *Let M be a compact manifold on which T acts smoothly, F the set of fixed points of the action. Then the kernel and cokernel of the pullback*

$$\iota^* : H_T^*(M) \to H_T^*(F) \tag{II.1.14}$$

have support in $\cup \mathfrak{k}^c$, where the union runs over all the proper isotropy subgroups of the action.

Proof. As a first step, we claim

$$\operatorname{supp} H_T^*(M - F) \subset \cup \mathfrak{k}^c \tag{II.1.15}$$

We can use the slice theorem to remove an equivariant tubular neighbourhood of F and obtain a compact $T-$space $M - U$ homotopy equivalent to $M - F$. Cover $M - U$ with equivariant neighbourhoods of its orbits, and extract a finite subcover $V_1, \dots, V_m$.

Consider the equivariant Mayer-Vietoris sequence of the pair (V_1, V_2):

$$\dots \to H_T^{k-1}(V_1 \cap V_2) \to H_T^k(V_1 \cup V_2) \to H_T^k(V_1) \oplus H_T^k(V_2) \to \dots \tag{II.1.16}$$

We know from II.1.2.1 that both $\operatorname{supp} H_T^{k-1}(V_1 \cap V_2)$ and $\operatorname{supp} H_T^k(V_1) \oplus H_T^k(V_2)$ are contained in $\cup \mathfrak{k}^c$, and applying II.1.1.3(iv) we obtain

$$\operatorname{supp} H_T^k(V_1 \cup V_2) \subset \operatorname{supp} H_T^{k-1}(V_1 \cap V_2) \cup \operatorname{supp} H_T^k(V_1) \oplus H_T^k(V_2) \subset \cup \mathfrak{k}^c \tag{II.1.17}$$

the claim follows now by induction. As a direct consequence, and again using II.1.1.3(iv), we obtain that the same result holds for pairs of spaces contained in $M - F$, and by excision it holds

$$H_T^*(M, F) = H_T^*(M - U, \partial U) \tag{II.1.18}$$

with U an equivariant tubular neighbourhood of F. Then $\operatorname{supp} H^*_T(M,F) \subset \cup \mathfrak{k}^c$. On the other hand, we have the exact sequence of the pair (M,F), namely:

$$\dots \quad \to H^k_T(M,F) \xrightarrow{\eta} H^k_T(M) \xrightarrow{\iota^*} H^k_T(F) \xrightarrow{\mu} H^{k-1}_T(M,F) \longrightarrow \dots \tag{II.1.19}$$

We conclude observing that by exactness

- $fH^*_T(M,F) = 0 \Rightarrow f \ker \iota^* = 0$, since $\forall a \in \ker \iota^* \ \exists b \in H^*_T(M,F)$ such that $\eta(b) = a$, so that $fa = f\eta(a) = \eta(fa) = 0$;

- $fH^*_T(M,F) = 0 \Rightarrow f \operatorname{coker} \iota^* = 0$, since $\forall [\alpha] \in \operatorname{coker} \iota^* \ \tilde{\mu}([\alpha]) = \mu(\alpha) = 0 \Rightarrow [\alpha] = 0$, so that $\tilde{\mu}(f[\alpha]) = f\tilde{\mu}([\alpha]) = 0 \Rightarrow f[\alpha] = 0$;

so that $\operatorname{supp} \ker \iota^* \subset \cup \mathfrak{k}^c$, $\operatorname{supp} \operatorname{coker} \iota^* \subset \cup \mathfrak{k}^c$ □

Remark II.1.2.4. (i) This first form of the localization theorem tells us that all the information regarding the free part of $H^*_T(M)$ is already contained in $H^*_T(F)$. We can go further: observe that

$$H^*_T(F) = H^*_T(*) \otimes H^*(F) \tag{II.1.20}$$

since $H^*_T(F) = H^*_{bas}(\mathcal{W} \otimes \Omega(F))$, and, as we're working on the fixed point set, $\Omega(F)_{bas} = \Omega(F)$. Then the rank of $H^*_T(M)$, i.e., the number of free generators it has, can be identified with the dimension of $H^*(F)$, the ordinary cohomology ring of F, usually way easier to compute;

(ii) We also know where the torsion takes place, and with respect to what kind of polynomials we need to localize to delete it: it suffices to pick any $f \in \mathbb{C}[u_1, \dots, u_l]$ vanishing on $\cup \mathfrak{k}^c$ to obtain a free localized module $(H^*_T(M))_f$.

Example II.1.2.5. We computed in the last chapter the cohomology ring of $H^*_{S^1}(S^2)$ with two different actions. Using the localization theorem, we could have immediately seen that both of them have two generators: when the action is a rotation, the two generators arise from the geometric fact that the fixed point set has two connected components; for a trivial action, the fixed set has only one component - but this is more complex than a point, and this fact is expressed by that fact that it provides alone both the generators.

II.1.3. The equivariant Thom class

In the following subsections we prepare the ground for the topological localization theorem: we need to connect the algebraic result we just proved with the topology of the space. One of the objects realizing this connection are the characteristic classes, which try to measure the nontriviality of a vector bundle; another is the Thom class, a sort of bump function around the submanifold with some pleasant integration properties.

Both Thom class and characteristic classes arise first in a non-equivariant context; in both cases, we'll describe the classical definition and then discuss its equivariant generalization.

We start with the Thom class. The classical recipe reads as follows:

1. Given an oriented, $n-$dimensional manifold M, we have a powerful tool available, Poincaré duality. In its De Rham formulation, it takes the form of a non denegerate pairing between the cohomology of the space and its compactly supported cohomology:

$$H^k(M) \times H_0^{n-k}(M) \to \mathbb{C}: \ ([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta \tag{II.1.21}$$

2. Now pick a compact, oriented submanifold N with codimension k. Integration of $k-$forms ω of M over N is defined by restriction:

$$\int_N \alpha := \int_N \iota^* \alpha \in \mathbb{C} \tag{II.1.22}$$

   and we obtain a linear map over $H^d(M)$:

$$[\omega] \mapsto \int_N \omega \tag{II.1.23}$$

3. Now we put the two things together: we use Poincaré Duality to find the unique cohomology class $[\tau(N)]$ in $H_0^{n-k}(M)$ such that

$$\int_M \omega \wedge \tau(N) = \int_N \omega, \ \forall [\omega] \in H^d(M) \tag{II.1.24}$$

   $[\tau(N)]$ is then called the *Thom class* of N, and any form η such that $[\eta] = [\tau(N)]$ is called a *Thom form* of N.

As it turns out, Thom forms have a natural interpretation in terms of integration along the fiber: consider a tubular neighbourhood U of the submanifold N, and regard it as a subset of the the normal bundle $\pi : \mathcal{N}N \to U$, when we think of $\mathcal{N}N$ itself as a subset of M. The integration along the fiber is a map $\pi_* : \Omega^l(U)_0 \to \Omega^{l-k}(N)_0$ characterized by the equality

$$\int_U \pi^* \beta \wedge \mu = \int_N \beta \wedge \pi_* \mu, \ \forall \beta \in \Omega_0^{n-l}(U) \tag{II.1.25}$$

π_* descends to a map $\pi_* : H^l(U)_0 \to H^{l-k}(N)_0$:

$$\begin{aligned} \int_N \beta \wedge \pi_* d\mu = \int_U \pi^* \beta \wedge d\mu = (-1)^{n-l-1} \int_U \pi^* d\beta \wedge \mu \\ = (-1)^{n-l-1} \int_N d\beta \wedge \pi_* \mu = \int_N \beta \wedge d\pi_* \mu, \ \forall \beta \in \Omega_0^{n-l}(U) \end{aligned} \tag{II.1.26}$$

If we pick a $\hat{\tau} \in \Omega^k(U)_0$, $\pi_* \hat{\tau}$ is a function over the submanifold; moreover, U deformation retracts into N, so that the inclusion ι of N as the zero section of the normal bundle has the property $\pi^* \iota^* \simeq \text{id}$. Then

$$\int_U \alpha \wedge \hat{\tau} = \int_U \pi^* \iota^* \alpha \wedge \hat{\tau} = (\pi_* \hat{\tau}) \int_N \alpha, \ \forall \alpha \in \Omega^{n-k}(U) \tag{II.1.27}$$

The form $\hat{\tau} \in \Omega^k(U)_0$ may be extended to all of M by just setting it to be zero outside of U, thus defining $\tau \in \Omega^k(M)$. If $\pi_* \hat{\tau} = 1$ we obtain from II.1.27

$$\int_M \alpha \wedge \tau = \int_N \alpha, \ \forall \alpha \in \Omega^{n-k}(N) \tag{II.1.28}$$

i.e., if τ is closed, it is a Thom form of N. We may then visualize the Thom form of a submanifold as a closed form with support in a neighbourhood of the submanifold itself, whose integrals over the fibers are normalized to 1.

Abstracting from the construction of the integration over the fiber, we can define the push-forward of a class:

Definition II.1.3.1. Let M, N be two compact, orientable manifolds, respectively of dimension m and n, $f : N \to M$ a map. The *push-forward* of f is the unique homomorphism $f_* : H^*(N) \to H^{*+m-n}(M)$ satisfying

$$\int_N f^*\alpha \wedge \beta = \int_M \alpha \wedge f_*\beta, \ \forall \alpha \in H^k(M), \beta \in H^{n-k}(N) \tag{II.1.29}$$

Remark II.1.3.2. (i) From II.1.29 we immediately obtain functoriality: $(f \cdot g)_* = f_* \cdot g_*$;

(ii) Notice that integration over the fiber was defined over forms, while the pushforward is in general defined only on cohomology, that is, on closed forms. The reason for this is that while uniqueness can be enforced in the same way - namely, by equation II.1.29 -, and existence on the level of closed forms is automatically provided by Poincaré Duality, existence of morphisms over general forms must be proven directly.
For the integration over the fiber, this is done by explicitly integrating out the "degrees of freedom" of the fiber in local coordinates, and then patching the maps given in such a way via a partition of unity (see [10, p. 61] for details). Such a concrete construction is clearly not always possible;

(iii) It is instructive to consider the case $M = U$, a tubular neighbourhood of N. If $\iota : N \to U$, is the inclusion and $\pi : U \to N$ the fibration map, observe $\pi^*\iota^* \simeq \mathrm{id}$, so that fora Thom form τ of N

$$\int_U \beta \wedge \pi^*\alpha \wedge \tau = \int_U \pi^*(\iota^*\beta \wedge \alpha) \wedge \tau = \int_N \iota^*\beta \wedge \alpha, \ \forall \beta \in \Omega^l_0(U), \alpha \in \Omega^{n-(l+k)}(N) \tag{II.1.30}$$

so that $\iota_*(\alpha) = \pi^*(\alpha) \wedge \tau$. As a byproduct, we obtain the identity $\iota_*(1) = \tau$, and by functoriality $\pi_* \cdot \iota_* = \mathrm{id}$: $\iota_* : H^*(N) \to H^{*+k}_0(U)$ is known as the *Thom isomorphism*;

(iv) Denote by $e(N)$ the Euler class of the normal bundle induced by a tubular neighbourhood around N. $e(N)$ is related to the Thom class of N via the identity $\iota^*\tau(N) = e(N)$ (see e.g. [10, p. 132]), and putting this together with the last remark we obtain the identity

$$\iota^*\iota_*(1) = e(N) \tag{II.1.31}$$

Now move to the equivariant setting. Here we cannot produce the Thom class with the same construction, because we miss the main tool, the isomorphism given by Poincaré Duality; however, fiber integration is still well-defined, and descends to a map on the equivariant cohomology of the spaces:

Lemma II.1.3.3. *Consider a $G-$manifold M and a $G-$invariant submanifold N of M. Integration on the fibers of the normal bundle of N, regarded as an equivariant tubular neighbourhood U of N, descends to a morphism*

$$\pi_* : H^l_{G,0}(U) \to H^{l-k}_{G,0}(N) \tag{II.1.32}$$

Proof. Making use of the Cartan model, we regard elements of $H^l_{G,0}(U)$ as equivalence classes of invariant polynomials from $\mathfrak{g}$ to $\Omega(U)_0$. Given an $[\omega(\xi)] \in H^l_{G,0}(U)$, there are two things we need to check:

(i) $\pi_*(\omega(\xi))$ is still invariant;

(ii) $d_G\pi_*(\omega(\xi)) = \pi_* d_G(\omega(\xi))$.

Ad (i). Observe that $g \cdot \pi = \pi \cdot g \ \forall g \in G$. The polynomial obtained by letting π_* act on $\omega(\xi)$ is

$$(\pi_*\omega)(\xi) = \pi_*(\omega(\xi)) \ \forall \xi \in \mathfrak{g} \tag{II.1.33}$$

so that the invariance condition I.2.88 reads

$$\pi_*(g^*\omega(\xi)) = g^*(\pi_*\omega(\xi)) \tag{II.1.34}$$

which follows in turn from equivariance of U: $g \cdot \pi = \pi \cdot g$.

Ad (ii). The differential d_G acts on $\omega(\xi)$ as follows:

$$(d_G\omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi)) \tag{II.1.35}$$

we know that $d\pi_* = \pi_* d$, so it remains to check $\pi_*(\iota_\xi(\omega(\xi))) = \iota_\xi(\pi_*(\omega(\xi)))$. Write just ω for $\omega(\xi)$ - the ξ does not play any role in the following - and observe that for $\beta \in \Omega^{n_l}(N)_0$, $\omega \in \Omega^{l+1}(U)_0$

$$\int_N \beta \wedge \pi_*\iota_\xi\omega = \int_U \pi^*\beta \wedge \iota_\xi\omega = (-1)^{n-l}\int_U (\iota_\xi\pi^*\beta) \wedge \omega \tag{II.1.36}$$

where we used that fact that ι_ξ acts as a derivation, and $\deg(\pi^*\beta \wedge \omega) = n+1$. Moreover, π^* commutes with ι_ξ, and

$$(-1)^{n-l}\int_U \pi^*(\iota_\xi\beta) \wedge \omega = (-1)^{n-l}\int_N (\iota_\xi\beta) \wedge \pi_*\omega = \int_N \beta \wedge \iota_\xi\pi_*\omega \tag{II.1.37}$$

and (ii) follows. □

Fiber integration is essentially the only tool that we can still employ in equivariant cohomology, and we use it to define both Thom forms and pushforwards:

- The equality $\pi_*\tau = 1$ is taken as a characterization of a Thom form. The authors of [14] construct it explicitly in Chapter 10;
- Every map $f : N \to M$ can be thought as a composition of a inclusion and a projection:

$$\begin{aligned} f : N &\hookrightarrow N \times M \ \to M \\ n &\mapsto (n, f(n)) \mapsto f(n) \end{aligned} \tag{II.1.38}$$

  so that $f_* = \pi_* \cdot \iota_*$ and it suffices to define ι_*. We pick one of the previous remarks as a definition, and set $\iota_*(\cdot) = \pi^*(\cdot) \wedge \tau$, with τ a Thom form. There is an intermediate passage that should be emphasized: here we denote by π^* the composition

$$H^*_T(N) \to H^{*+k}_{T,0}(U) \to H^{*+k}_T(M) \tag{II.1.39}$$

where the first step is the usual fiber integration, and in the second step we extend the forms to the whole manifold by zero.

As before, for $M = U$, an equivariant tubular neighbourhood of M, we obtain by functoriality an isomorphism $\iota_* : H^*_T(N) \to H^{*+k}_{T,0}(U)$, the *equivariant Thom isomorphism.*

Putting the last two definitions together we obtain

$$\int_N \beta = \int_N \iota^*\beta \wedge 1 = \int_M \beta \wedge \iota_*(1) = \int_M \beta \wedge \tau \tag{II.1.40}$$

thus recovering property II.1.24.

II.1.4. Equivariant characteristic classes

Let's start with a definition.

Definition II.1.4.1. Consider a $G-$manifold M and the map $\pi : M \to *$. The induced map

$$\pi^* : H^*_G(*) \simeq S(\mathfrak{g}^*)^G \to H^*_G(M) \tag{II.1.41}$$

is called *Chern-Weil map* and denoted by $\mathfrak{k}$.

Now we consider a particular $\mathfrak{k}$: pick a complex vector bundle $E \to X$ over a manifold X. Call M its bundle of unitary frames: the elements of M are pairs $(x, (e_1, \ldots, e_n))$, with $(e_1, \ldots, e_n)$ an orthonormal basis of E_x, and $U(n)$ acts on M on the right, via $A \cdot (x, (e_1, \ldots, e_n)) = (x, (e_1, \ldots, e_n) \cdot A)$.

Observe that this action is free. The Chern-Weil map reads

$$\mathfrak{k} : S(\mathfrak{u}(n)^*)^{U(n)} \to H^*_{U(n)}(M) \simeq H^*(X) \tag{II.1.42}$$

and the element of its image are called *characteristic classes of E*. The same reasoning applies to real vector bundles, just by considering orthogonal frames and the action of the orthogonal group $O(n)$.

Remark II.1.4.2. Characteristic classes are natural, in the following sense: pick two complex vector bundles $p_1 : E_1 \to X_1$ and $p_2 : E_2 \to X_2$ such that there exists a vector bundle morphism (ϕ, f):

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow[f]{} & X_2 \end{array} \tag{II.1.43}$$

This induces an equivariant map $\varphi : M_1 \to M_2$, so that we get a commuting diagram

$$\begin{array}{ccc} H^*_{U(n)}(M_1) & \xleftarrow{\varphi^*} & H^*_{U(n)}(M_2) \\ {\scriptstyle \mathfrak{k}_1} \nwarrow & & \nearrow {\scriptstyle \mathfrak{k}_2} \\ & S(\mathfrak{u}(n)^*)^{U(n)} & \end{array} \tag{II.1.44}$$

The characteristic classes of the two bundles are then related by φ^*, i.e. $\mathfrak{k}_1(a) = \varphi^*\mathfrak{k}_2(a)\ \forall a \in S(\mathfrak{u}(n)^*)^{U(n)}$. The remark holds unchanged for a pair of real vector bundles.

Let's stick to the complex case and consider a first example. Elements of $\mathfrak{u}(n)$ are matrices satisfying the condition $C = -C^*$, which we may write as $C = iA$, with A self-adjiont. The adjoint representation is just given by conjiugation, and if we let elements of $\mathfrak{u}(n)^*$ act on $\mathfrak{u}(n)$ via $A(B) = \mathrm{tr}(AB)$, we see that the same holds for the coadjoint representation: an element $p \in S(\mathfrak{u}(n)^*)^{U(n)}$ should satisfy

$$p(UAU^{-1}) = p(A) \tag{II.1.45}$$

this equation already hints at a choice of p, the determinant. We move in this direction, and consider the characteristic polynomial:

$$p(A) = \det(\lambda - A) = \lambda^n - c_1(A)\lambda^{n-1} + \cdots + (-1)^n c_n(A) \tag{II.1.46}$$

$p(A)$ is clearly an invariant polynomial, and so must be the coefficients $c_i(A)$. They are called *Chern classes* of the vector bundle E.

One of the main ingredients of the geometric localization theorem is the equivariant Euler class of the bundle. Before introducing it, let's have a look at the usual Euler class.
Consider then a real, orientable vector bundle of even dimension $2l$: we have a notion of orientation preserving action, expressed by an action of $SO(2l)$; recall that elements of $\mathfrak{so}(2l)$ are $2l \times 2l$ matrices characterized by the equality $A^t = -A$. The invariant we look for may be constructed as follows:

- Associate to every matrix $A \in \mathfrak{so}(2l)$ the linear map

  $$\omega_A : \mathbb{R}^{2l} \times \mathbb{R}^{2l} \to \mathbb{R} : \ (v, w) \mapsto \langle v, Aw \rangle \tag{II.1.47}$$

  it is an antisymmetric map: $\omega_A \in \Lambda^2(\mathbb{R}^{2l})$.

- Then consider the map $\omega_A^n \in \Lambda^{2n}(\mathbb{R}^{2n})$. It must be proportional to the volume element $e_1^* \wedge \cdots \wedge e_n^*$, which is a $SO(2l)$ invariant: we define the *Pfaffian* of A via the equation

  $$\frac{1}{n!\sqrt{(2\pi)^n}}\omega_A^n = \mathrm{Pfaff}(A) e_1^* \wedge \cdots \wedge e_n^* \tag{II.1.48}$$

  Notice that to every A corresponds an unique ω_A, so that the definition is well-posed;

- Finally, define the Euler class of the vector bundle:

  Definition II.1.4.3. Let $\pi : E \to X$ be a real, orientable vector bundle of even dimension $2l$. The *Euler class* of E is the characteristic class

  $$\not{k}(\mathrm{Pfaff}) \in H^{2l}(E) \tag{II.1.49}$$

  and it is denoted by $e(E)$.

The quantity Pfaff(A) is in principle quite a mysterious object. We can make it more familiar by finding a suitable coordinate change for A, so that

$$A = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & 0 & \dots & 0 \\ 0 & \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \begin{pmatrix} 0 & \lambda_l \\ -\lambda_l & 0 \end{pmatrix} \end{pmatrix} \tag{II.1.50}$$

Then ω_A takes the simple form $\lambda_1 e_1^* \wedge e_2^* + \dots + \lambda_l e_{2l-1}^* \wedge e_{2l}^*$, and we get Pfaff$(A) = \frac{\lambda_1 \cdot \dots \cdot \lambda_l}{\sqrt{(2\pi)^n}}$. It is easy to check that $\det A = \lambda_1^2 \cdot \dots \cdot \lambda_n^2$, hence the relation $\sqrt{(2\pi)^n}\,\mathrm{Pfaff}^2 = \det$.

Now let's consider the equivariant setting. Pick a $G-$manifold X, and consider a complex vector bundle $\pi : E \to X$ such that G acts as a vector bundle automorphism, i.e. making the following diagram commute $\forall g \in G$:

$$\begin{array}{ccc} E & \xrightarrow{g_*} & E \\ \pi\downarrow & & \downarrow\pi \\ X & \xrightarrow[g]{} & X \end{array} \tag{II.1.51}$$

this will always be the case when working with the normal bundle induced by an equivariant tubular neighbourhood. Now pass to the bundle M of unitary frames: we get an induced action of G and the $U(n)$ action we had before, and they commute.

We would like to obtain equivariant characteristic classes, elements of $H_G^*(X)$: we consider then the action of $U(n)$ on the bundle $(E \times EG)/G \to (X \times EG)/G$, which is still free, and consider the Chern-Weil map of this bundle:

$$k : S(\mathfrak{u}(n)^*)^{U(n)} \to H^*((X \times EG)/G) = H_G^*(X) \tag{II.1.52}$$

this is the *equivariant Chern-Weil map* of the bundle $E \to X$.

Remark II.1.4.4. In the same spirit of equivariant cohomology, for which already points can give large contributions, equivariant characteristic classes of bundles of the form $E \to *$ are not necessarily trivial, and generated by the map

$$k : S(\mathfrak{u}(n)^*)^{U(n)} \to S(\mathfrak{g}^*)^G \tag{II.1.53}$$

in this specific case, the Chern-Weil homomorphism can be obtained easily by considering the action of G on E given by representing G in $U(n)$. We then have a homomorphism $G \to U(n)$, which induces the Chern-Weil map by naturality of the Chern classes:

$$\begin{array}{ccc} (E \times EG)/G & \longrightarrow & (E \times EU(n))/U(n) \\ \downarrow & & \downarrow \\ (* \times EG)/G \simeq BG & \longrightarrow & BU(n) \simeq (* \times EU(n))/U(n) \end{array} \tag{II.1.54}$$

II.1.5. The geometric localization theorem

In this last section we apply the notions we have introduced to derive the geometric localization theorem. With the new machinery, we can prove the following corollary of II.1.2.3:

Corollary II.1.5.1. *Let M be a compact manifold on which T acts smoothly, F the set of fixed points of the action. Then the kernel and cokernel of*

$$\iota_* : H^*_T(F) \to H^{*+k}_T(M) \tag{II.1.55}$$

have support in $\cup \mathfrak{k}^c$, where the union runs over all the proper isotropy subgroups of the action.

Proof. Consider the long exact sequence of the pair $(M, M - F)$:

$$\cdots \to H^{k-1}_T(M - F) \to H^k_T(M, M - F) \to^{e^*} H^k_T(M) \to H^k_T(M - F) \to \dots \tag{II.1.56}$$

during the proof of II.1.2.3 we saw

$$\operatorname{supp} H^*_T(M - F) \subset \cup \mathfrak{k}^c \tag{II.1.57}$$

and following the same argumentation we obtain that kernel and cokernel of e^* have support in $\cup \mathfrak{k}^c$.

Moreover, by excision

$$H^k_T(M, M - F) \simeq H^k_T(U, U - F) \tag{II.1.58}$$

where U is an equivariant tubular neighbourhood of F. Now we can construct, following the idea of [16, p. 244], an increasing sequence of compact, equivariant tubular neighbourhoods F_k which deformation retract to F and such that every compact set is eventually contained in one of these sets. We obtain then an isomorphism

$$\varinjlim H^*_T(U, U - F_k) \simeq H^*_{T,0}(U) \tag{II.1.59}$$

with $\varinjlim$ denoting the direct limit. On the other hand, the maps induced by the inclusions $(U, U - F_k) \to (U, U - F_{k-1})$ are all isomorphisms, so that

$$H^k_T(U, U - F) \simeq H^*_{T,0}(U) \tag{II.1.60}$$

and by the equivariant Thom isomorphism $H^*_T(N) \simeq H^{*+k}_{T,0}(U)$. Then kernel and cokernel of the map

$$\phi : H^*_T(N) \to^{\simeq} H^*_{T,0}(U) \to^{\simeq} H^{*+k}(U, U - F) \to^{\simeq} H^{*+k}(M, M - F) \to^{e^*} H^{*+k}_T(M) \tag{II.1.61}$$

have support in $\cup \mathfrak{k}^c$. We claim that $\phi = \iota_*$, and to verify it we need to check that the last two compositions amount to extending by the zero form a class in $H^*_{T,0}(U)$ to a class in $H^{*+k}_T(M)$. But this just follows from the definition of cohomology of a pair in the De Rham setting: it is the homology of the complex with elements

$$C^k_T(M, M - F) = (C^k_T(M) \oplus C^{k-1}_T(M - F), d_{(M,F)}) \tag{II.1.62}$$

with $d_{(M,F)}(\omega, \theta) = (d\omega, \iota^*\omega - d\theta)$, and $C_T^k(M)$ the Cartan complex of M. The map e^* is induced by the projection on the first factor,

$$\beta : C_T^k(M, M - F) \to C_T^k(M) : (\omega, \theta) \to \omega \tag{II.1.63}$$

and the extension by zero of the form takes place when we work with the excision, in the isomorphism $H^{*+k}(U, U - F) \to^{\simeq} H^{*+k}(M, M - F)$. Then the claim holds, and the thesis follows. □

Now, recall that $\iota^*\iota_*(1) = e(F)$, the equivariant Euler class of the normal bundle at F: the corollary, together with the algebraic localization theorem, tell us that $e(F)$ is invertible up to torsion - that is, after localizing with respect to a suitable polynomial.
The geometric localization theorem goes further, giving us a precise formula for the polynomial we need to pick: to prove it, we need some concepts from the representation theory of a Lie group. We recall the basic concepts:

Definition II.1.5.2. Let G be a Lie group, V a vector space. A *representation* of G on V is a group homomorphism

$$\psi : G \to \mathrm{Aut}(V) \tag{II.1.64}$$

A representation is called *irreducible* if it doesn't have proper invariant subsets, i.e. proper linear subspaces W of V such that $\psi(g)W \subseteq W \ \forall g \in G$.

We are interested in the case $G = T$, an $n-$torus. Consider generators $t_1, \dots, t_n$ of T: if ψ is a representation of T on an $m-$dimensional $\mathbb{C}-$vector space V, abelianity of T implies

$$\psi(t_i) \cdot \psi(t_j) = \psi(t_j) \cdot \psi(t_i), \ \forall i, j \tag{II.1.65}$$

Regarding $\mathrm{Aut}(V)$ as $Gl(m, \mathbb{C})$, we have that the matrices $\{\psi(t_i)\}_i$ all commute, and can be simultaneously diagonalized: we get a splitting of ψ into m irreducible representations of T on 1 dimensional vector spaces.

How do these irreducible representations look like? Start with the case $n = 1$, i.e. $T = S^1$. A $1-$dimensional representation of S^1 is a homomorphism $S^1 \to GL(1, \mathbb{C}) \simeq \mathbb{C} - \{0\}$; such representations can always be made unitary by renormalization, so that we obtain an homomorphism $\theta : S^1 \to U(1)$. Considering the differential in 0 we induce a homomorphism between the Lie algebras:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\theta_*} & \mathbb{R} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ S^1 & \xrightarrow[\theta]{} & U(1) \end{array} \tag{II.1.66}$$

the exp maps on the sides send $t \in \mathbb{R}$ to e^{it}, either seen as an element of $S^1 \subset \mathbb{C}$ or of $U(1)$. The map θ_* is a linear map from $\mathbb{R}$ to $\mathbb{R}$, i.e. it must be $\theta_*(u) = au, \alpha \in \mathbb{R} \ \forall u \in \mathbb{R}$. By imposing commutativity we obtain

$$1 = \theta(1) = \theta(\exp(2\pi)) = \exp(2a\pi) = e^{2\pi i a} \tag{II.1.67}$$

so that $a \in \mathbb{Z}$, a fact that we can use to characterize all the θ's: they have the form

$$\theta_n : S^1 \to U(1) : e^{iu} \mapsto e^{inu}, \ l \in \mathbb{Z} \tag{II.1.68}$$

1−dimensional representations τ of an $n-$torus are realized by fixing a 1−dimensional representation for each of the S^1-components:

$$\tau : T \to U(1) : (e^{iu_1}, \dots, e^{iu_n}) \mapsto (e^{il_1 u_1}, \dots, e^{il_n u_n}) \tag{II.1.69}$$

and are thus uniquely identified by a tuple $(l_1, \dots, l_n) \in \mathbb{Z}^n$. Splitting the $m-$dimensional representation ψ into m 1−representations amounts then to assigning to ψ m $n-$tuples of integers:

$$\psi \iff \begin{cases} \tau_1 \sim (l_1^1, \dots, l_n^1) \\ \vdots \\ \tau_m \sim (l_1^m, \dots, l_n^m) \end{cases} \tag{II.1.70}$$

Denote $\ell_i = \sum_j l_i^j$. Then ψ can be explicitly described with the following assignment:

$$\psi(e^{iu_1}, \dots, e^{iu_n}) = \begin{pmatrix} e^{i\ell_1 u_1} & 0 & \dots & 0 \\ 0 & e^{i\ell_2 u_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\ell_n u_n} \end{pmatrix} \tag{II.1.71}$$

the elements $\ell_1, \dots, \ell_n$ are called *weights* of the representation ψ. The corresponding map ψ_* on the Lie algebra is

$$\psi_* : \mathbb{R}^n \to \mathfrak{u}(n) : (u_1, \dots, u_n) \mapsto \mathrm{diag}\,(\ell_1 u_1, \dots, \ell_n u_n) \tag{II.1.72}$$

Now we have everything we need to prove the geometric localization theorem. Let us fix some notation:

- The spaces $(F_1, \dots, F_k)$ are the connected components of F;
- The maps $\iota_i : F_i \to M$ are the inclusions of such components in M;
- The maps $\pi^i : F_i \to *$ and $\pi^M : M \to *$ are the the maps collapsing their domain to a point;
- e_i is the equivariant Euler class corresponding to the normal bundle of F_i in M, and its inverse is formally denoted by $\frac{1}{e_i}$.

Then the theorem can be stated as follows:

Theorem II.1.5.3. *Let M be a compact $n-$dimensional manifold, T an $l-$torus acting on M. Let F be the set of fixed points of the action.*

Then there exists a nontrivial polinomial $f \in \mathbb{C}[u_1, \dots, u_l]$ such that $\forall \omega \in H_T^*(M)_f$

$$\pi_*^M \omega = \sum_i \pi_*^i \left(\frac{\iota_i^* \omega}{e_i} \right) \tag{II.1.73}$$

Proof. We start by showing that all the Euler classes e_i can be simultaneously inverted in a nontrivial localization. Consider then the equivariant normal bundle of F_i in M, let f_i be the codimension of F_i in M: the action of T on each fiber doesn't fix any vector, so that the representation of T splits, as discussed before, into nontrivial 1−dimensional complex representations. Notice that this forces f_i to be even. Set $f_i = 2l_i$.

The equivariant Euler class is an element of $H_T^{f_i}(F_i)$, and since the action of T on F_i is trivial we have an isomorphism $H_T^*(F_i) \simeq H_T^*(*) \otimes H^*(F_i)$. Let f_{l_i} be the pure polynomial component of e_i: then localizing with respect to $f = f_{l_i}$ we obtain a well defined inverse

$$\frac{1}{e_i} = \frac{1}{f_{l_i}}\left(1 + \frac{\alpha}{f_{l_i}} + \frac{\alpha^2}{f_{l_i}^2} + \cdots + \frac{\alpha^q}{f_{l_i}^q}\right) \tag{II.1.74}$$

where $\alpha = f_{l_i} - e_i$ and q is the largest integer smaller than or equal to $\dim(F_i)/2$, and we use the fact that $H^k(F_i) = 0$ for $k > \dim(F_i)$, since the fixed point set is a manifold (see I.1.1.2).

Now show that f_{l_i} is not the trivial polynomial: we can obtain the component $H_T^{2l_i}(*) \otimes H^0(F_i)$ of e_i by considering the map induced by the inclusion $j : \{x_0\} \to F_i$, for some $x_0 \in F_i$, so that $f_{l_i} = j^*(e_i)$. On the other hand

$$j^*(e_i) = j^*(e(F_i)) = e(\{x_0\}) \tag{II.1.75}$$

by naturality of characteristic classes, and we can shift to problem to studying the equivariant Euler class of a normal bundle over a point: as discussed in II.1.4.4 it is a polynomial defined by the composition of the Chern-Weil map k with the Pfaffian. But the Chern-Weil map is induced by II.1.71: it picks a polynomial over $U(f_i)$,

$$A \mapsto p(A) \tag{II.1.76}$$

and it precomposes it with ψ to obtain a polynomial over $\mathbb{R}^l$:

$$(u_1, \dots, u_l) \mapsto \operatorname{diag}(\ell_1 u_1, \dots, \ell_n u_n) \mapsto p(\operatorname{diag}(\ell_1 u_1, \dots, \ell_n u_n)) \tag{II.1.77}$$

As for the Pfaffian, we derived in the section on characteristic classes the equation $(2\pi)^n \,\mathrm{Pfaff}^2 =$ det, where the determinant was considered for matrices over $\mathbb{R}$.
Now, when considering matrices acting between even dimensional real spaces, we may regard them as acting over $\mathbb{C}$ vector spaces by halving the dimensions; the determinants over the two different fields are related by the equation

$$\det_{\mathbb{R}}(\cdot) = |\det_{\mathbb{C}}(\cdot)|^2 \tag{II.1.78}$$

so that, up to sign, the Pfaffian corresponds to the norm of the determinant over $\mathbb{C}$. We can dispose of the sign by coherently choosing an orientation of the normal bundle over the point, so that the equivariant Euler class is given by the polynomial

$$(u_1, \dots, u_l) \mapsto \det(\operatorname{diag}(\ell_1 u_1, \dots, \ell_n u_n)) = \Pi_{i=1}^l \ell_i u_i \tag{II.1.79}$$

i.e. the product of the weights of the representation of T over the normal bundle of x_0. Nontriviality of the representation guarantees that none of the ℓ_i is zero, so that $f_{l_i} = e(\{x_0\}) \neq 0$.

Then the right hand side of II.1.73 is well defined: we just need to localize with respect to

$$f = \Pi_{i=1}^k f_{l_i} \tag{II.1.80}$$

Linking it to the the left hand side is considerably easier: the maps ι_*, ι^* induced by the inclusion $\iota : F \to M$ may be written as

$$\begin{cases} \iota_* = \sum_i \iota_{i*} \\ \iota^* = \sum_i \iota_i^* \end{cases} \tag{II.1.81}$$

and $\iota^*\iota_* = \sum_i \iota_i^*\iota_{i*}$. Moreover

$$\iota_i^*\iota_{i*}(\omega) = \iota_i^*(\pi^*(\omega) \wedge \tau_i) = (\pi \cdot \iota)^*(\omega) \wedge \iota^*\tau_i = \omega \wedge e_i \tag{II.1.82}$$

τ_i being a Thom class of F_i. We obtain

$$\sum_i \iota_{i*}\frac{\iota_i^*}{e_i} = \mathrm{id} \tag{II.1.83}$$

apply π_*^M to both sides, and observe $\pi^M \cdot \iota_i = \pi^i$ to obtain the claim:

$$\pi_*^M\omega = \sum_i \pi_*^i\left(\frac{\iota_i^*\omega}{e_i}\right) \quad \forall \omega \in H_T^*(M)_f \tag{II.1.84}$$

□

We turn back to our beloved example of the S^1 action on S^2 to show an easy application of the theorem. (see I.1.5.3, I.2.7.9)

Example II.1.5.4. We want to use formula II.1.73 to compute the surface area of the sphere. Recall that the volume form of the sphere may be obtained by contracting the volume form $dx_1 \wedge dx_2 \wedge dx_3$ in $\mathbb{R}^3$ with a radial vector field. We obtain a form

$$\nu = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2 \tag{II.1.85}$$

This form is invariant under the action of S^1, so that $\nu(\xi) = 1 \otimes \nu \in \mathbb{C}[u] \otimes (\Omega^*(S^2))^{S^1}$. If it were closed, we would get

$$(\pi_*^M(\nu))(\xi) = \pi_*^M(\nu(\xi)) = \int_{S^2} \nu(\xi) = \mathrm{vol}(S^2) \tag{II.1.86}$$

But $1 \otimes \nu$ is not closed. In fact

$$(d_{S^1}\nu)(\xi) = d(\nu(\xi)) - \iota_\xi\nu(\xi) = -\iota_\xi\nu(\xi) \tag{II.1.87}$$

and the fundamental vector field at a point is given by the assignment

$$\begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi) \\ \sin(\psi)\cos(\phi) \end{bmatrix} \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_0 \begin{bmatrix} \cos(\psi) \\ \sin(\psi)\sin(\phi + t) \\ \sin(\psi)\cos(\phi + t) \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(\psi)\cos(\phi) \\ -\sin(\psi)\sin(\phi) \end{bmatrix} \tag{II.1.88}$$

or, in cartesian coordinates, $\xi_{(x_1,x_2,x_3)} = (0, x_3, -x_2)$. Then one computes

$$\iota_\xi\nu(\xi) = x_1(x_1dx_1 + x_2dx_2 + x_3dx_3) - dx_1 \neq 0 \tag{II.1.89}$$

To solve this problem, we want to add a term $\xi \otimes \omega$, with $\omega \in \Omega^0(S^2)$, such that $d(\omega(\xi)) = \iota_\xi \nu(\xi)$:

$$d_{S^1}(\nu + \omega)(\xi) = d(\nu(\xi)) - \iota_\xi \nu(\xi) + d(\omega(\xi)) - \iota_\xi \omega(\xi) = -\iota_\xi \nu(\xi) + d\omega(\xi) \tag{II.1.90}$$

What kind of function could ω be? We know that $\xi = 0$ at the north and south pole, which should then be critical points; moreover, contracting with ξ takes care of vectors tangent to the orbits: the function ν should then grow only orthogonally to them. A reasonable ansatz could be

$$\omega : S^2 \to \mathbb{R} : (x_1, x_2, x_3) \to ax_1 + b, \ \ a, b \in \mathbb{R} \tag{II.1.91}$$

if we consider ω as the restriction of a function $\hat{\omega} : \mathbb{R}^3 \to \mathbb{R}$, we can obtain $d\omega$ by considering the pointwise projection of $d\hat{\omega} = adx_1$ to the tangent space of S^2. For every point, the projection of $d\hat{\omega}$ in the radial direction is just given by $\langle (a, 0, 0), (x_1, x_2, x_3) \rangle = ax_1$; we obtain the projection to the tangent space of S^2 by subtracting this component to $d\hat{\omega}$:

$$d\omega = dx_1 - ax_1(x_1 dx_1 + x_2 dx_2 + x_3 dx_3) \tag{II.1.92}$$

this is exactly $\iota_\xi \nu(\xi)$ if we set $a = 1$! Notice that the component $\xi \otimes \omega$ is in the kernel of π_*^M, which sends $\Omega^k(S^2)$ into $\Omega^{k-2}(*)$. Then for $\nu_{eq}(\xi) = 1 \otimes \nu + \xi \otimes \omega$ we still have the equality we want, $(\pi_*^M(\nu_{eq}))(\xi) = \mathrm{vol}(S^2)$.

Now look at the left hand side of II.1.73: the action of S^1 on the normal bundles of the poles is just given by considering the differential of the action at the poles: for the northern one, the action of an $e^{i\tau} \in S^1$ has differential

$$\frac{\mathrm{d}}{\mathrm{d}t}|_0 e^{i\tau} \cdot \begin{bmatrix} \cos(vt) \\ \sin(vt)\sin(\phi) \\ \sin(vt)\cos(\phi) \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}|_0 \begin{bmatrix} \cos(vt) \\ \sin(vt)\sin(\phi+\tau) \\ \sin(vt)\cos(\phi+\tau) \end{bmatrix} = v \begin{bmatrix} 0 \\ \sin(\phi+\tau) \\ \cos(\phi+\tau) \end{bmatrix} = e^{i\tau} \begin{bmatrix} 0 \\ v\sin(\phi) \\ v\cos(\phi) \end{bmatrix} \tag{II.1.93}$$

if we regard the tangent space at the northern pole as $\mathbb{C}$, we see that the action of S^1 is just multiplication by a phase: $e^{i\tau} \cdot z = ze^{i\tau}$ $\forall z \in \mathbb{C}$. Since $\det(e^{i\tau}) = e^{i\tau}$, we conclude that the equivariant Euler class e_N at the northern pole is $\frac{1}{2\pi}$.

The only difference when considering e_S is in the computation of the differential:

$$\frac{\mathrm{d}}{\mathrm{d}t}|_0 e^{i\tau} \cdot \begin{bmatrix} \cos(\pi - vt) \\ \sin(\pi - vt)\sin(\phi) \\ \sin(\pi - vt)\cos(\phi) \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}|_0 \begin{bmatrix} \cos(\pi - vt) \\ \sin(\pi - vt)\sin(\phi+\tau) \\ \sin(\pi - vt)\cos(\phi+\tau) \end{bmatrix} = -v \begin{bmatrix} 0 \\ \sin(\phi+\tau) \\ \cos(\phi+\tau) \end{bmatrix} = -e^{i\tau} \begin{bmatrix} 0 \\ v\sin(\phi) \\ v\cos(\phi) \end{bmatrix} \tag{II.1.94}$$

that is, an element $e^{i\tau}$ acts as multiplication by the complex phase $-e^{i\tau}$, so that $e_S = -\frac{1}{2\pi}$. Observe that we need to localize with respect to the polynomial $\xi \cdot (-\xi) = -\xi^2$, but this does not affect the computation, since ν_{eq} is of degree 1.

Lastly, $\iota_N^*(\nu_{eq})$ and $\iota_S^*(\nu_{eq})$ are restrictions to the point. The process kills the ν component, and yields respectively

$$\begin{cases} \omega(1, 0, 0) = 1 + b \\ \omega(-1, 0, 0) = -1 + b \end{cases} \tag{II.1.95}$$

and π_*^i, induced by the trivial fibration, is the identity. Then the left hand side reads

$$\frac{1 + b}{1/2\pi} + \frac{-1 + b}{-1/2\pi} = 4\pi \tag{II.1.96}$$

We also see that the choice of b is not relevant.

The map ω is called *moment map* of the action: we will study these objects in greater detail in the next section.

II.1.6. The case $G \neq T$

So far we only talked about abelian, compact and connected Lie groups; in this section we let the abelianity hypothesis fall, and discuss generalizations of the previous results. We need some preliminary definitions:

Definition II.1.6.1. Let G be a compact, connected Lie group.

(i) A *maximal torus* of G is a maximal connected Abelian subgroup of G;

(ii) The *Weyl group* of G is the quotient $W(G) = N_G(T)/T$, where $N_G(T)$ is the normalizer of a maximal torus T in G.

Remark II.1.6.2. It is not a priori clear that the Weyl group $W(G)$ does not depend on the chosen maximal torus. This is a consequence of a theorem from Cartan (for the proof, see e.g. [12, p. 119]):

Theorem II.1.6.3. *Let G be a compact, connected Lie group, and let T be a maximal torus. Then every maximal torus is conjugate to T, and every element of G is contained in a conjugate of T.*

As a consequence, if we pick another maximal torus T' we get $T' = gTg^{-1}$ for $g \in G$, hence $N(T') = gN(T)g^{-1}$, and we can identify the quotients $N(T')/T'$ and $N(T)/T$.

We sketch the proof of an important property of the Weyl group: finiteness.

Proposition II.1.6.4. *$W(G)$ is finite.*

Sketch of the proof. Let T be a maximal torus, and consider the action of its normalizer $N(T) \to \mathrm{Aut}(T)$ given by conjiugation. Every element ϕ of $\mathrm{Aut}(T)$ induces a map on the level of Lie algebras, and we have a commutative diagram ($n = \dim T$)

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{d\phi} & \mathbb{R}^n \\ \exp \downarrow & & \downarrow \exp \\ T & \xrightarrow[\phi]{} & T \end{array} \tag{II.1.97}$$

Regard T as $\mathbb{R}^n/\mathbb{Z}^n$. By commutativity, $d\phi$ sends $\mathbb{Z}^n \simeq \ker\exp$ into $\mathbb{Z}^n$, so that we can identify it with a map in $GL(n,\mathbb{Z})$, and this characterizes ϕ as well: thus $\mathrm{Aut}(T) \simeq GL(n,\mathbb{Z})$. Since $GL(n,\mathbb{Z})$ is discrete, the identity component $N(T)^0$ must be sent into the trivial map:

$$\forall g \in N(T)^0, t \in T \; gtg^{-1} = t \tag{II.1.98}$$

so that elements of $N(T)^0$ commute with those of T. Now, if there existed $g \in N(T)^0 - T$, the exponential mapping from the Lie algebra of $N(T)^0$ would give us a $1-$parameter subgroup

$g(t)$: the closure of the group generated by $g(t)$ and T is still Abelian and connected. One directly shows it is a submanifold of G, hence a torus of dimension bigger than the maximal torus: a contradiction.

Thus $N(T)^0/T \simeq *$; by compactess $N(T)$ has only finitely many connected components, and the thesis follows. □

There is some further input we need, regarding the cohomology of G/T:

Theorem II.1.6.5. *Let G be a compact, connected Lie group, and T a maximal torus of G. Then the odd Betti numbers of G/T are all zero, and its Euler characteristic equals $|W|$, the order of $W(G)$.*

Proof. See [9, p 66] □

We can use this theorem to relate the equivariant cohomology of G with that of one of its maximal tori:

Theorem II.1.6.6. *Let G be a compact, connected Lie group, T a maximal torus of G and W its Weyl group. Let M be a simply connected, orientable $G-$manifold. Then*

$$H^*_G(M) \simeq (H^*_T(M))^W \tag{II.1.99}$$

the subring of $W-$invariant elements of $H^*_T(M)$.

Proof. We make use of the fibrations (recall $ET = EN(T) = EG$, since $T \subseteq N(T) \subseteq G$)

$$\begin{cases} W \to G/T \to G/N(T) \\ W \to (M \times ET)/T \to (M \times EN(T))/N(T) \\ G/N(T) \to (M \times EN(T))/N(T) \to (M \times EG)/G \end{cases} \tag{II.1.100}$$

The first one is actually a covering, since W is finite: the induced map $\pi^* : H^*_{dR}(G/N(T)) \to H^*_{dR}(G/T)$ is injective, since by the Thom isomorphism

$$\pi^*(\omega) = d\eta \Rightarrow \omega = \pi_*(\pi^*(\omega) \wedge \tau) = \pi_*(d\eta \wedge \tau) = d\pi_*(\eta \wedge \tau) \tag{II.1.101}$$

its image is given by the $W-$invariant elements of $H^*_{dR}(G/T)$, so that we get

$$H^*_{dR}(G/N(T)) \simeq (H^*_{dR}(G/T))^W \tag{II.1.102}$$

moreover (see again [9]) W acts on $H^*_{dR}(G/T)$ as the regular representation, so that we can relate the Betti numbers of the two spaces:

$$b^i(G/T) = |W| b^i(G/N(T)) \tag{II.1.103}$$

we can use this to compute the Euler characteristic of $G/N(T)$:

$$\chi(G/N(T)) = \frac{1}{|W|}\chi(G/T) = 1 \tag{II.1.104}$$

The odd Betti numbers of $G/N(T)$ vanish, and we obtain that the cohomology of $G/N(T)$ with coefficients in a field is acyclic.

We can use the same argument in the second fibration to obtain the isomorphism

$$H^*_{N(T)}(M) \simeq (H^*_T(M))^W \tag{II.1.105}$$

Now pass to the last fibration: since M is simply connected, so is $(M \times EG)/G$, and the associated spectral sequence computes the equivariant cohomology of M with respect to $N(T)$. We showed that $H^*(G/N(T);\mathbb{R})$ is acyclic, thus $H^*_{N(T)}(M) \simeq H^*_G(M)$. This concludes the proof. □

Equipped with the isomorphism II.1.99, we can defined the support of the H^*_G-module $H^*_G(M)$ as the support of a module over $(H^*_T)^W = (\mathbb{C}[u_1, \ldots, u_l])^W$, and run through the same proofs. There is, however, one point where having $G = T$ was quite important, namely in discussing the representation theory of T: if we want to compute equivariant characteristic classes, we should consider in the general case the characteristic map

$$\not{k} : S(\mathfrak{u}(n)^*)^{U(n)} \to S(\mathfrak{g}^*)^G \simeq S(\mathfrak{t}^*)^W \tag{II.1.106}$$

The computation of the equivariant Euler characteristic class doesn't change much: we pick a polynomial over $\mathfrak{u}(n)$ and precompose it with the map

$$T \hookrightarrow G \to U(n) \tag{II.1.107}$$

the crucial point comes when considering non-triviality of the equivariant Euler class. We argued by saying that points on the fiber of the normal bundle of the fixed point set could not be, by definition, fixed points for the action of T: but if we consider the fixed point space of the $G-$action, it is a priori possible to find points in fiber which are not fixed points not for G, but for T. Then the theorem holds provided that the fixed point of the $G-$action and of the restricted $T-$action coincide.

Example II.1.6.7. Again to rotations of S^2. This time, instead of considering just the action of S^1, we consider rotations around any possible axis: this defines an action of $SO(3)$ on S^2.

A maximal torus of $SO(3)$ is given (see [25] for details) by rotations around a fixed axis, for example

$$T = \left\{ A \in SO(3) \ : \ A = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right), \ R \in SO(2) \right\} \tag{II.1.108}$$

The normalizer of this set is given by matrices satisfying

$$A^t \cdot \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right) \cdot A = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R' \end{array}\right), \ R, R' \in SO(2) \tag{II.1.109}$$

one obtains

$$N(T) = \left\{ A \in SO(3) \ : \ A = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array}\right), \ T \in O(2) \right\} \tag{II.1.110}$$

and finally

$$W = \frac{N(T)}{T} = \left\{ w_0 = \left[\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\right], w_1 = \left[\left(\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)\right] \right\} \simeq \mathbb{Z}_2 \tag{II.1.111}$$

observe that there is a well defined action of W on invariant polynomials $f \in H^*_T(S^2)$:

$$f(\mathrm{Ad}_{(nt)^{-1}}\,\xi) = (nt)^* f(\xi) \iff f(\mathrm{Ad}_{t^{-1}}(\mathrm{Ad}_{n^{-1}}\,\xi)) = t^*(n^* f(\xi)) \iff f(\mathrm{Ad}_{n^{-1}}\,\xi) = n^* f(\xi) \tag{II.1.112}$$

where $n \in N(T)$, $t \in T$. We already computed in examples I.1.5.3, I.2.7.9 $H^*_T(S^2) \simeq H^*_T(S^2)$, and we know

$$H^*_{SO(3)}(S^2) \simeq (H^*_{S^1}(S^2))^W \tag{II.1.113}$$

now, how do the $W-$invariant polynomials look like? The adjoint action of w_1 can be computed observing

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(vt) & \sin(vt) \\ -\sin(vt) & \cos(vt) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(vt) & -\sin(vt) \\ \sin(vt) & \cos(vt) \end{pmatrix} \tag{II.1.114}$$

so that $\mathrm{Ad}_{w_1^{-1}}\,\xi = -\xi$. The action on elements of the sphere is a reflection:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi)\sin(\varphi) \\ \sin(\psi)\cos(\varphi) \end{pmatrix} = \begin{pmatrix} -\cos(\psi) \\ \sin(\psi)\cos(\varphi) \\ \sin(\psi)\sin(\varphi) \end{pmatrix} \tag{II.1.115}$$

The $W-$invariance condition links the value of orbits of the upper and lower hemisphere which have the same distance from the equator; combining this restriciton with S^1-invariance, we obtain polynomials which are either symmetric or antisymmetric with respect to the equator. In other words, condition

$$f(-\xi) \begin{pmatrix} \cos(\psi) \\ \frac{1}{\sqrt{2}}\sin\psi \\ \frac{1}{\sqrt{2}}\sin\psi \end{pmatrix} = f(\xi) \begin{pmatrix} -\cos(\psi) \\ \frac{1}{\sqrt{2}}\sin\psi \\ \frac{1}{\sqrt{2}}\sin\psi \end{pmatrix}, \ \forall\psi \tag{II.1.116}$$

implies

$$f(-\xi) \begin{pmatrix} \cos(\psi) \\ \sin(\psi)\sin(\varphi) \\ \sin(\psi)\cos(\varphi) \end{pmatrix} = f(\xi) \begin{pmatrix} -\cos(\psi) \\ \sin(\psi)\sin(\varphi) \\ \sin(\psi)\cos(\varphi) \end{pmatrix}, \ \forall\psi,\varphi \tag{II.1.117}$$

Now, such polynomials define a cohomology class in $H^*_{S^1}(S^2)$ only if they are closed under the Cartan differential C_{S^1}. The typical element of $C^k_{S^1}$, $k > 0$, has the form

$$\xi^k \otimes f + \xi^{k-1} \otimes \nu, \ f \in C^\infty(M), \ \nu \in \Omega^2(M) \tag{II.1.118}$$

and closure reads

$$df = \iota_\xi \nu \tag{II.1.119}$$

Now, passing from f to df flips the symmetry of the polynomial:

$$\frac{\mathrm{d}}{\mathrm{d}t}|_0 f_{-\xi}(\gamma(t)) = \pm\frac{\mathrm{d}}{\mathrm{d}t}|_0 f_\xi(-\gamma(t)) = \mp\frac{\mathrm{d}}{\mathrm{d}t}|_0 f_{-\xi}(\gamma(t)) \tag{II.1.120}$$

and the same happens when passing from ν to $\iota_\xi\nu$:

$$\iota_{-\xi}\nu_{-\xi}(x) = \pm\iota_{-\xi}\nu_\xi(-x) = \mp\iota_{-xi}\nu_\xi(x) \tag{II.1.121}$$

but f and ν have opposite symmetries: if k is even, $k-1$ is odd, and vice versa. Thus II.1.119 equates a symmetric and an antisymmetric polynomial, which should then separately be constant. This implies $k = 0$, i.e., there aren't closed polynomials of degree greater than or equal to 1. In degree zero, however, everything still works. We get

$$H^k_{SO(3)}(S^2) = \begin{cases} \mathbb{C}, & k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{II.1.122}$$

The result shouldn't be surprising: $SO(3)$ acts transitively on S^2, and without fixed points. Thus

$$H^*_{SO(3)}(S^2) \simeq H^*(S^2/SO(3)) \simeq H^*(*) \tag{II.1.123}$$

II.2. Applications

Now we turn to some applications. We'll restrict to symplectic manifolds, and later on to Kähler manifolds: the richer structure we impose on the manifold will successfully lead to an easier employment of the localization formula.

II.2.1. The symplectic setting

In the next two sections we study an application of the localization theorem to symplectic manifolds, and related symplectic actions. For starters, recall the definitions:

Definition II.2.1.1. (i) A *symplectic manifold* is a pair (M, ω), where M is an even-dimensional manifold and ω a closed, non-degenerate $2-$form;

(ii) We call an action $\phi : G \to \mathrm{Diff}(M) : g \mapsto \phi_g$ of a Lie group G on the symplectic manifold (M, ω) *symplectic* if it preserves the symplectic form ω, i.e.

$$\phi_g^*(\omega) = \omega, \ \forall g \in G \tag{II.2.1}$$

For a symplectic manifold (M, ω), non-degeneracy of ω induces a pairing between T_*M and T^*M:

$$T_*M \ni \chi \longleftrightarrow \iota_\chi \omega \in T^*M \tag{II.2.2}$$

We can use the pairing to analyze forms in terms of the associated vector fields, and vice versa. An example of the first instance is given by the symplectic gradient:given a smooth function f on M, we can consider the vector field χ_f associated to its gradient, i.e. satisfying

$$\iota_{\chi_f}\omega = df \tag{II.2.3}$$

This vector field χ_f is called *symplectic gradient* of f, or *Hamiltonian vector field* associated to f. A bilinear operation, the Poisson bracket, is then defined as follows:

Definition II.2.1.2. Let (M, ω) be a symplectic manifold. The *Poisson bracket* of two functions f, g on M is the function $\{f, g\}$ defined by

$$\{f, g\} = \omega(\chi_f, \chi_g) \tag{II.2.4}$$

By directly computing the Jacobi identity one shows

Lemma II.2.1.3. *$(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra.*

Manifolds M such that $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra are called *Poisson manifolds.*

The characterization of vector fields in terms of the associated forms leads to the definition of (locally) Hamiltonian vector fields:

Definition II.2.1.4. Let (M, ω) be a symplectic manifold, $\chi \in T_*M$.

(i) We say that χ is *Hamiltonian*, and write $\chi \in \mathcal{H}(M)$, if $\iota_\chi\omega$ is exact;

(ii) We say that χ is *locally Hamiltonian*, and write $\chi \in \mathcal{H}_{loc}(M)$, if $\iota_\chi\omega$ is closed.

The spaces $\mathcal{H}(W)$ and $\mathcal{H}_{loc}(M)$ sit in a short exact sequence with the first de Rham cohomology group of M.

Lemma II.2.1.5. *Let (M, ω) be a symplectic manifold. Then there is a short exact sequence of vector spaces*

$$0 \to \mathcal{H}(M) \hookrightarrow \mathcal{H}_{loc}(M) \to H^1_{dR}(M) \mapsto 0 \tag{II.2.5}$$

Proof. Consider the map $\phi : H_{loc}(M) \to H^1_{dR}(M) : \chi \to \iota_\chi \omega$. It is well defined, since $\iota_\chi \omega$ is closed by definition of $H_{loc}(M)$, and its kernel is exactly $H(M)$, the vector fields such that $\iota_\chi \omega$ is exact. By non-degeneracy of the pairing, the map is also surjective. ☐

Moreover, symplectic group actions from compact Lie groups act via locally Hamiltonian vector fields:

Proposition II.2.1.6. *Let G be a compact Lie group acting on the symplectic manifold (M, ω). Then the action is symplectic if and only if the fundamental vector fields of the actions are locally hamiltonian.*

Proof. Suppose that the action is symplectic, let $\xi \in \mathfrak{g}$. We have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}|_0 (e^{t\xi})^* \omega = \mathcal{L}_\xi \omega \tag{II.2.6}$$

Cartan's formula implies then $d\iota_\xi \omega = 0$.

Conversely, $d\iota_\xi \omega = 0$ implies $\mathcal{L}_\xi \omega = 0$, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}|_\tau (e^{\tau\xi})^* \omega = (e^{\tau\xi})^* \mathcal{L}_\xi \omega = 0 \tag{II.2.7}$$

but $(e^{0\cdot\xi})^* \omega = \omega$, and for compact G the exponential map is surjective. The thesis follows. ☐

We saw that a symplectic manifold M automatically carries a Poisson manifold structure. The notion of Hamiltonian action describes a symplectic action that respects the Poisson brackets - this does not follow directly from the definition, though: we prove it directly afterwards.

Definition II.2.1.7. Consider a symplectic group action ϕ of the Lie group G on the manifold M, and the maps

$$\begin{cases} \mathfrak{g} \to \mathcal{H}_{loc}(M) : \xi \mapsto \underline{\xi} \\ C^\infty(M) \to \mathcal{H}(M) : f \mapsto \chi_f \end{cases} \tag{II.2.8}$$

where $\underline{\xi}$ denotes the fundamental vector field associated to ξ. Suppose there exists a Lie algebra morphism $\tilde{\mu}$ such that the diagram commutes:

$$\begin{array}{ccccccccc} & & C^\infty(M) & \overset{\tilde{\mu}}{\dashleftarrow} & \mathfrak{g} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{H}(M) & \hookrightarrow & \mathcal{H}_{loc}(M) & \longrightarrow & H^1_{dR}(M) & \longrightarrow & 0 \end{array} \tag{II.2.9}$$

then ϕ is called a *Hamiltonian action.*

Remark II.2.1.8. Observe that whenever an action is Hamiltonian, commutativity of II.2.9 implies that so are its fundamental vector fields; suppose, conversely, that a symplectic action has corresponding Hamiltonian vector fields: then the problem reduces to finding a lift of the map $\hbar : \xi \mapsto \underline{\xi}$:

$$\begin{array}{ccccccccc} & & & & \mathfrak{g} & & & & \\ & & & & \tilde{\mu} \downarrow & \searrow^{\hbar} & & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M) & \longrightarrow & \mathcal{H}(M) & \longrightarrow & 0 \end{array} \tag{II.2.10}$$

the obstruction being given by the $\mathbb{R}$ component on the left: exactness of the sequence can be proven by identifying constant functions with $\mathbb{R}$ and examining the assignment $f \to \chi_f$.

The relation between a Hamiltonian action and the Poisson manifold structure of M is clarified by the next proposition.

Proposition II.2.1.9. *Consider a Lie group G acting on a symplectic manifold M. Then*

(i) $\mathfrak{g}^*$ is a Poisson manifold;

(ii) If the action is Hamiltonian, there exists a Poisson manifolds morphism

$$\mu : M \to \mathfrak{g}^* \tag{II.2.11}$$

Proof. Ad (i). We can use the Lie bracket in $\mathfrak{g}$ to define a Lie algebra structure on $C^\infty(\mathfrak{g}^*)$ by setting

$$\{f, g\}(\eta) = \eta([df_\eta, dg_\eta]) \tag{II.2.12}$$

where we identified df_η with an element of $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$. Then $(C^\infty(\mathfrak{g}^*), \{\cdot, \cdot\})$ is a Lie algebra, and $\mathfrak{g}^*$ a Poisson manifold.

Ad (ii). If the action is Hamiltonian, we can find a Lie algebra morphism $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$. Define

$$\mu : M \to \mathfrak{g}^* : x \mapsto (\xi \mapsto \tilde{\mu}(g)(x)) \tag{II.2.13}$$

We need to show

$$\{f, g\} \cdot \mu = \{f \cdot \mu, g \cdot \mu\}, \ \forall f, g \in C^\infty(\mathfrak{g}^*) \tag{II.2.14}$$

we can compute this directly. Fix $x \in M$, then

$$\begin{aligned} \{f, g\}\mu(x) &= \mu(x)\left([df_{\mu(x)}, dg_{\mu(x)}]\right) = \tilde{\mu}([df_{\mu(x)}, dg_{\mu(x)}])(x) \\ &= \{\tilde{\mu}(df_{\mu(x)}), \tilde{\mu}(dg_{\mu(x)})\} = \omega(\chi_{\tilde{\mu}(df_{\mu(x)})}, \chi_{\tilde{\mu}(dg_{\mu(x)})})(x) \\ &= \omega(\chi_{f\cdot\mu}, \chi_{g\cdot\mu})(x) = \{f \cdot \mu, g \cdot \mu\}(x) \end{aligned} \tag{II.2.15}$$

where we used commutativity of diagram II.2.9 to obtain

$$\chi_{\tilde{\mu}(df_{\mu(x)})}(x) = \iota_{df_{\mu(x)}}\omega(x) = \chi_{f\cdot\mu}(x) \tag{II.2.16}$$

□

The maps $\tilde{\mu}$ and μ are called moment and comoment maps; it is clear from the definition that each moment map induces a comoment map, and viceversa.

Definition II.2.1.10. Consider a hamiltonian group action ϕ of the Lie group G on the manifold M. The derived Poisson manifold morphism

$$\mu : M \to \mathfrak{g}^* \tag{II.2.17}$$

is called *moment map* of the action ϕ.
The Lie algebra morphism $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ is called *comoment map* of ϕ.

II.2.2. The Duistermaat-Heckman formula

The main purpose of this subsection is proving the correspondence between comoment maps and equivariant extensions of a symplectic form.

Theorem II.2.2.1. *Let (M, ω) be a symplectic manifold, G a Lie group, ϕ a hamiltonian action of G on M. Then there is a bijection between equivariant closed extensions of ω and comoment maps of ϕ.*

Proof. We make use of the Weil model. Recall how it was defined: we use the chain complex

$$(\Omega(M) \otimes \mathcal{W})_{bas} \subset \Omega(M) \otimes \mathcal{W} \tag{II.2.18}$$

where $\mathcal{W} = \lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g})^*$. Given a basis $\xi_1, \ldots, \xi_n$ of $\mathfrak{g}$, we denoted by $\theta^1, \ldots, \theta^n$ and $z^1, \ldots, z^n$ the corresponding elements of $\lambda(\mathfrak{g}^*)$ and $S(\mathfrak{g})^*$, and the differential d_K on $\mathcal{W}$ was defined for all i by

$$\begin{cases} d_K \theta^i = z^i \\ d_K z^i = 0 \end{cases} \tag{II.2.19}$$

now denote by D the differential on $\Omega(M) \otimes \mathcal{W}$, and suppose we are given a comoment map $\tilde{\mu}$. Let $f_i = \tilde{\mu}(\xi_i)$. We claim that (summing over repeated indices)

$$\omega_{eq} = \omega - D(f_i \otimes \theta^i) \tag{II.2.20}$$

is a basic, closed extension of ω. Closedness is already apparent:

$$D\omega_{eq} = d\omega - D^2 \sum_i f_i \otimes \theta^i = 0 \tag{II.2.21}$$

To check that ω_{eq} is basic, observe

$$\omega_{eq} = \omega - \sum_i (df_i \otimes \theta^i + f_i \otimes z^i) \tag{II.2.22}$$

so that

$$\iota_{\xi_j} \omega_{eq} = \iota_{\xi_j} \omega - (\iota_{\xi_j} df_i) \otimes \theta^i - df_i \otimes \delta^i_j + c^i_{jk} f_i \otimes \theta^k \tag{II.2.23}$$

but by definition of comoment map we have $\iota_{\xi_j} \omega = df_j$ and

$$\iota_{\xi_j} df_i = df_i(\xi_j) = (\iota_{\xi_i} \omega)(\xi_j) = \omega(\xi_i, \xi_j) = f_{[i,j]} = c^k_{ij} f_k \tag{II.2.24}$$

hence $\iota_{\xi_j} \omega_{eq} = 0$, and by Cartan's formula $\mathcal{L}_{\xi_j} \omega_{eq} = 0$. Then ω_{eq} is closed and basic, and it corresponds in Cartan's model to an equivariant extension of ω.

Conversely, any closed, basic extension has the form

$$\omega_{eq} = \omega + g_{ij} \otimes (\theta^i \wedge \theta^j) + h_i \otimes z^i + \eta_i \otimes \theta^i \tag{II.2.25}$$

and respects

$$\begin{cases} 0 = \iota_{\xi_k}\omega_{eq} = \iota_{\xi_k}\omega + 2g_{ki} \otimes \theta^i + c^i_{kl} h_i \otimes \theta^l + \eta_k \\ 0 = D\omega_{eq} = (dg_{ij}) \otimes \theta^i \wedge \theta^j + (dh_i) \otimes z^i + (d\eta_i) \otimes \theta^i - \eta_i \otimes z^i \end{cases} \tag{II.2.26}$$

hence

$$\begin{cases} \iota_{\xi_k}\omega + \eta_k = 0 = 2g_{ij} + c^i_{kj} h_i \\ dg_{ij} = 0 = dh_i - \eta_i = d\eta_i \end{cases} \tag{II.2.27}$$

and we get a comoment map by setting $f_i = -h_i$. The other components reflect the Lie algebra homomorphism property and commutativity of II.2.9, and may be identified as

$$\begin{cases} \eta_i = -df_i \\ g_{ij} = -\frac{1}{2} f_{[i,j]} \end{cases} \tag{II.2.28}$$

□

Remark II.2.2.2. (i) The theorem gives an explicit formulation of the equivariant extension in the Weil model. It's not hard to find a formulation for the Cartan model as well, provided $G = T$, an $l-$torus.

Let then $u_1, \ldots, u_l$ be coordinates on the Lie algebra $\mathfrak{t}$ of T and $u^1, \ldots, u^n$ the dual coordinates; the equivariant extension of the symplectic form ω is given by

$$\omega_{eq} = 1 \otimes \omega - u^i \otimes f_i \tag{II.2.29}$$

where $f_i = \tilde{\mu}(u_i)$, the image of the basis vectors under the comoment map.

Notice that we can think of each $u^i \otimes f_i$ as a map from M to $\mathfrak{t}^*$:

$$f_i : x \mapsto \{a = a_i u^i \mapsto f_i(x) a_i\} \tag{II.2.30}$$

and this is nothing but the moment map of $\tilde{\mu}_i : \mathfrak{t} \to C^\infty(M) : \sum_j a^j u_j \mapsto \tilde{\mu}(a^i u_i)$, where summation is *not* implied in the second expression. We obtain the decomposition

$$\omega_{eq} = \omega - f \tag{II.2.31}$$

with f the moment map of the action.

(ii) We can also employ some reverse engineering: in II.1.5.4 we found a map completing the volume form to an equivariant closed form. S^2 has dimension 2, so that we can take the volume form to interpret it as a symplectic manifold: we then have that the action is Hamiltonian, with moment map given by the height map on the sphere. We'll discuss non-uniqueness later.

The Duistermaat-Heckman formula follows now as a corollary.

Corollary II.2.2.3. *Let ϕ be a Hamiltonian action of the $l-$torus T on the symplectic, compact $2m-$manifold (M,ω), and suppose that the fixed point set $\mathcal{P}$ of the action is discrete. Then*

$$\int_M \frac{\omega^m}{m!} e^{-f} = \sum_{p\in\mathcal{P}} \frac{e^{-f(p)}}{e_p} \tag{II.2.32}$$

where the e_p terms correspond to the equivariant Euler classes of the normal bundles around the points, and f is a moment map for ϕ.

Proof. We have from the previous remark that $\omega_{eq} = \omega - f$ is an equivariantly closed form. Consider the terms ω_{eq}

$$(e^{\omega_{eq}})_j = \left(\sum_{i=0}^{m} \frac{\omega^i}{i!}\right) \frac{(-f)^j}{j!} \tag{II.2.33}$$

and apply the integration formula II.1.73. π_*^M annihilates all the forms with degree different from the dimension of M, so that we get

$$\pi_*^M((e^{\omega_{eq}})_j) = \int_M \frac{\omega^m}{m!} \frac{(-f)^j}{j!} \tag{II.2.34}$$

by a similar reasoning, on the right hand side there isn't any ω component, leaving us with the evaluation of f at the fixed points. We obtain

$$\int_M \frac{\omega^m}{m!} \frac{(-f)^j}{j!} = \sum_{p\in\mathcal{P}} \frac{(-f)^j/j!}{e_p} \tag{II.2.35}$$

the thesis follows by summing over the j's. □

Remark II.2.2.4. In its original formulation, the Duistermaat-Heckman formula relates the Lebesgue measure on $\mathfrak{g}^*$, seen as some $\mathbb{R}^n$, to the volume form on a symplectic manifold.

To simplify the discussion, consider the case $T = S^1$. We pick a Hamiltonian action of S^1 on the symplectic manifold (M,ω), and define a measure m_{DH} on $\mathfrak{g}^* \simeq \mathbb{R}$ by

$$m_{DH}(U) = \int_{\mu^{-1}(U)} \frac{\omega^m}{m!} \tag{II.2.36}$$

where $\mu : M \to \mathfrak{g}^*$ is the moment map associated to the action and $2m$ the dimension of M. This is called the *Duistermaat-Heckman* measure on $\mathfrak{g}^*$, and the Duistermaat-Heckman theorem states that its Radon-Nykodim derivative with respect to the Lebesgue measure on $\mathfrak{g}^*$ is a polynomial when restricted to the regular values of μ.
For our case $\mathfrak{g}^* \simeq \mathbb{R}$, the differential of μ at a point $p \in M$ is the map

$$\frac{\mathrm{d}}{\mathrm{d}t}|_0 \gamma(t) \mapsto \{\ \xi \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_0 \tilde{\mu}(\xi)(\gamma(t))\ \} \tag{II.2.37}$$

where $\gamma(t)$ is a path in M with $\gamma(0) = p$, and $\tilde{\mu}$ is the comoment map of the action. Critical points are then identified with the points around which $\tilde{\mu}(\xi)$ is a constant for all $g \in \mathfrak{g}$; by definition of the comoment map, this means that the associated fundamental vector fields vanish in the critical points of μ, i.e., they are the critical points of the action.
In conclusion, the Radon-Nykodim derivative is a polynomial when restricted to non-fixed

points of the action. If the fixed points make up a discrete set $\{p_1, \ldots, p_n\}$ we get a piecewise polynomial: we can use the fundamental theorem of calculus to integrate each of the pieces and obtain

$$m_{DH}(U) = \int_{\mu^{-1}(U)} \frac{\mathrm{d}}{\mathrm{d}x} m_{DH}(U)dx = f(p_1, \ldots, p_n) \tag{II.2.38}$$

that is, the measure of the set only depends on the fixed points of the action.

II.2.3. Relation with the stationary phase approximation

In this section we explain the relation between formula II.2.32 and the stationary phase approximation. The formula then shows that the approximation is in fact exact in the case prescribed by the localization theorem, and we obtain in return a concrete interpretation of the kind of information it conveys.

Consider the case of an S^1 action, and look at II.2.32. If u is a generator of the Lie algebra of S^1, we have $f = fu$, with $f \in C^\infty(M)$. When read in chart domains, and with the help of a partition of the unity, the integral on the left hand side takes the form

$$\int_{\mathbb{R}^n} a(y)e^{f(y)u}dy \tag{II.2.39}$$

where $a(y) \in C^\infty(\mathbb{R}^n)$ is a smooth function with compact support, and we still denoted by f the coordinate version of the function on M. If we write $u = ik$, with $k \in \mathbb{R}$, we obtain a more familiar object - at least for physicists: it's a superposition of waves, whose amplitude and frequency is weighted by $a(y)$ and $f(y)$, and sharing a common parameter k.

Now, what's the approximation? Fix a $y_1 \in \mathbb{R}^n$, and suppose the parameter k is very large when compared with the growth of the f and a: if we regard the integral à la Riemann - a sum over volumes of $n-$rectangles - it follows that a little tilt in the value of y_0 will be enough to change the sign of the exponential, since the change in f is amplified by k, without changing significantly the value of a, which changes more slowly. In other words, the contributions given by the two $n-$rectangles will compensate one another.

This reasoning relies on the fact that a little tilt in y_1 yields a non-zero change in f, which is then amplified: when this does not happen, there is no balancing of the contribution in y_0, and we expect the value of the integrand in this point to give a leading contribution to the evaluation of the integral. All in all, we expect something like

$$\int_{\mathbb{R}^n} a(y)e^{if(y)k}dy \approx g(y_0, y_1, \ldots, y_n) \tag{II.2.40}$$

where $(y_1, \ldots, y_n)$ are the critical values of f, which we assume to be finite, and g is some smooth function.

This already gives an idea of where we're going: but we can do better, and formalize the process to obtain more details on g. We need a classical result in Morse theory (for the proofs see e.g. [23]): recall that a *Morse function* is a function whose critical points are not degenerate - the Hessian matrix H_f in those points should be invertible. We have the following:

Lemma II.2.3.1. *Let $f \in C^\infty(\mathbb{R}^n)$, fix $y_0 \in \mathbb{R}^n$. Then:*

(i) There exist functions $f^i, \dots, f^n$ such that

$$\begin{cases} f^i(y_0) = \frac{\partial f}{\partial x_i}|_{y_0} \ i = 1, \dots, n \\ f(y) = f(y_0) + y_i f^i(y), \ \text{where } y = (y_1, \dots, y_n) \end{cases} \tag{II.2.41}$$

(ii) (Morse's Lemma) If f is a Morse function and y_0 is a critical point of f there is a local change of coordinates $z = z(y)$ such that

$$\begin{cases} z(y_0) = 0 \\ f = f(y_0) - z_1^2 - \dots - z_l^2 + z_{l+1}^2 + \dots + z_n^2 \end{cases} \tag{II.2.42}$$

where l is the index of H_f at y_0, the maximal dimension of a linear subspace on which $H_f(y_0)$ is negative definite.

Now consider open local chart domains as in the lemma for the critical points of f, call them $U_1, \dots, U_n$. We can find an open set $V \supset \mathbb{R}^n - \cup_i U_i$ which does not contain the critical points of f, and consider a partition of unity $\{\lambda_i\}_{i=0,\dots,n}$ to express the integral as

$$\int_{\mathbb{R}^n} a(y)e^{ikf(y)}dy = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \lambda_i(y)a(y)e^{ikf(y)}dy + \int_{\mathbb{R}^n} \lambda_0(y)a(y)e^{ikf(y)}dy \tag{II.2.43}$$

We start by evaluating the first term. Referring to the second point of the lemma, we'll just write $f = f(y_0) + \frac{Q(z)}{2}$: we can change coordinates to obtain

$$e^{ikf(y_0)} \int_{\mathbb{R}^n} b(z)e^{i\frac{kQ(z)}{2}}dz \tag{II.2.44}$$

where $b(z) = \lambda_i(z)a(z)\left|\det\left[\frac{\partial y}{\partial z}\right]\right|$. If we use the first part of the lemma on b, we get two terms

$$b(y_i)e^{ikf(y_i)} \int_{\mathbb{R}^n} e^{i\frac{kQ(z)}{2}}dz + e^{ikf(y_i)} \int_{\mathbb{R}^n} b^i(z)z_i e^{i\frac{kQ(z)}{2}}dz \tag{II.2.45}$$

anyway, notice

$$z_i e^{i\frac{kQ(z)}{2}} = \pm\frac{1}{ik}\frac{\partial}{\partial z_i}e^{i\frac{kQ(z)}{2}} \tag{II.2.46}$$

so that we can integrate by parts, and the second summands yields a term of order $\frac{1}{k}$; applying again the lemma, we can iterate the procedure to obtain a term of order k^{-N}, for N arbitrarily large. The first summand is a product of elements

$$\int_{\mathbb{R}} \exp\left(\pm ik\frac{x^2}{2}\right)dx = \sqrt{\frac{2\pi}{\mp ik}} = \sqrt{\frac{2\pi}{k}}e^{\pm i\frac{\pi}{4}} \tag{II.2.47}$$

moreover, we can choose the partition of unity so that $\lambda_i(y_0) = 1$, and lastly

$$1 = |\det H_Q(y_i)| = \left|\det\left(\left[\frac{\partial y}{\partial z}\right]^t H_f(y_i)\left[\frac{\partial y}{\partial z}\right]\right)\right| \tag{II.2.48}$$

Then we have

$$\int_{\mathbb{R}^n} \lambda_i(y)a(y)e^{ikf(y)}dy = \frac{a(y_i)e^{ikf(y_i)}}{\sqrt{|\det\left(H_f(y_i)\right)|}}\left(\sqrt{\frac{2\pi}{k}}\right)^n e^{i\varepsilon_i\frac{\pi}{4}} + O(k^{-N}) \tag{II.2.49}$$

with ε_i equal to $n - 2l_i$, and l_i the index of $H_f(y_i)$. The remaining term

$$\int_{\mathbb{R}^n} \lambda_0(y)a(y)e^{ikf(y)}dy \tag{II.2.50}$$

can be estimated as follows: consider the vector field

$$X = \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} \tag{II.2.51}$$

and observe

$$X(e^{ikf(y)}) = ik\sum_i \left(\frac{\partial f}{\partial x^i}\right)^2 e^{ikf(y)} \tag{II.2.52}$$

then we can define a vector field η on the support of λ_0 by

$$\eta = \frac{1}{ik\sum_i \frac{\partial f}{\partial x^i}^2}\frac{\partial}{\partial x^i} \tag{II.2.53}$$

to obtain $\eta(e^{ikf(y)}) = e^{ikf(y)}$. Everything is well defined because there are no critical points of f in the support of λ_0 by construction; extend η to zero outside of the support, observe that $\lambda_0\eta$ is smooth. We have

$$\int_{\mathbb{R}^n} \lambda_0(y)a(y)e^{ikf(y)}dy = \int_{\mathbb{R}^n} \lambda_0(y)a(y)\eta(e^{ikf(y)})dy \tag{II.2.54}$$

integrating by parts yields again a dependence of order k^{-N}, and we obtain the global formula

$$\int_{\mathbb{R}^n} a(y)e^{ikf(y)}dy = \sum_i \frac{a(y_i)e^{ikf(y_i)}}{\sqrt{|\det(H_f(y_i))|}}\left(\sqrt{\frac{2\pi}{k}}\right)^n e^{i\varepsilon_i\frac{\pi}{4}} + O(k^{-N}) \tag{II.2.55}$$

we can patch all the contributions and obtain the approximation

$$\int_M \frac{\omega^m}{m!}e^{ikf} = \sum_i \frac{\omega^m(y_i)e^{ikf(y_i)}}{m!\sqrt{|\det(H_f(y_i))|}}\left(\sqrt{\frac{2\pi}{k}}\right)^n e^{i\varepsilon_i\frac{\pi}{4}} + O(k^{-N}) \tag{II.2.56}$$

Now compare this with II.2.32. Observe that the equivariant Euler class e_i is a polynomial of degree $m = \frac{n}{2}$ in $\mathbb{C}[u]$, so that it has the form $\epsilon_i u^m$, with $\epsilon_i \in \mathbb{C}$. Writing $u = ik$, we have

$$\int_M \frac{\omega^m}{m!}e^{ikf} = \sum_i \frac{e^{ikf(y_i)}}{\epsilon_i}\left(\sqrt{\frac{1}{k}}\right)^n \tag{II.2.57}$$

that is, the approximation is exact. It's also apparent from the formula the remarkable amount of information encoded from the equivariant Euler classes, taking contributions from the values of the volume form, the determinant of the Hessian of the moment map and its signature.

II.2.4. Existence and uniqueness of moment maps

We conclude the discussion on the Duistermaat-Heckman formula proving some results on moment maps: in fact, we conveniently shifted the problem of finding an equivariant extension for the symplectic form to that of finding a moment map for a symplectic action. It is then natural to ask when does such a map exist. We'll produce a criterion, and obtain some results about uniqueness on the way.

The main technical tool for this kind of investigation is the Lie algebra cohomology, defined as follows:

Definition II.2.4.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The *Lie algebra cohomology* of $\mathfrak{g}$ is defined as

$$H^*(\mathfrak{g};\mathbb{R}) = (\mathrm{Hom}(\Lambda\mathfrak{g}), \delta) \tag{II.2.58}$$

with $\delta_k : \mathrm{Hom}_k(\Lambda\mathfrak{g}) \to \mathrm{Hom}_{k+1}(\Lambda\mathfrak{g})$ given by

$$\delta_k(c)(\chi_1, \dots, \chi_k) = \sum_{i<j} (-1)^{i+j} c([\chi_i, \chi_j], \chi_1, \dots, \hat{\chi}_i, \dots, \hat{\chi}_j, \dots, \chi_k) \tag{II.2.59}$$

for $k > 0$, and 0 for $k = 0$.
$(\mathrm{Hom}_*(\Lambda\mathfrak{g}), \delta)$ is called the *Chevalley-Eilenberg complex.*

We can get sufficient conditions by studying the first Lie algebra cohomology groups of $\mathfrak{g}$. Let's start with uniqueness:

Proposition II.2.4.2. *Let G be a Lie group, and suppose $H^1(\mathfrak{g};\mathbb{R}) = 0$. Then moment maps for Hamiltonian actions of G are unique.*

Proof. Let G act on the manifold (M, ω). We have two Lie algebra homomorphisms $\tilde{\mu}_1$, $\tilde{\mu}_2$ making the diagram commute:

$$\begin{array}{ccccccc} & & & & \mathfrak{g} & & \\ & & & \tilde{\mu}_i \swarrow & & \searrow \hbar & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M) & \longrightarrow & \mathcal{H}(M) \end{array} \tag{II.2.60}$$

by exactness, their difference $\tilde{\nu} = \tilde{\mu}_1 - \tilde{\mu}_2$ may be regarded as an element of $\mathfrak{g}^* = \mathrm{Hom}_1(\Lambda g)$. It is closed, in fact

$$\delta\nu(X, Y) = -\nu([X, Y]) = \{\tilde{\mu}_1(X), \tilde{\mu}_1(Y)\} - \{\tilde{\mu}_2(X), \tilde{\mu}_2(Y)\} \tag{II.2.61}$$

which vanishes by definition of the Poisson bracket on $C^\infty(M)$ and commutativity of the diagram. Then $\nu \in \ker(\delta_1) = H^1(\mathfrak{g};\mathbb{R}) = 0$. □

Remark II.2.4.3. As we saw in the proof of the proposition, $H^1(\mathfrak{g};\mathbb{R})$ has a natural algebraic interpretation as $[\mathfrak{g}, \mathfrak{g}]^0 \subseteq \mathfrak{g}^*$, the *annihilator* of $[\mathfrak{g}, \mathfrak{g}]$. Thus, a nontrivial Abelian group will always have $H^1(\mathfrak{g};\mathbb{R}) \neq 0$.

To make sure a moment map exists, we require also the second cohomology group to be trivial.

Proposition II.2.4.4. *Let G be a Lie group, and suppose $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$. Then any symplectic action of G is Hamiltonian.*

Proof. As a first step, observe that the commutator of two fundamental vector fields $[\underline{\xi},\underline{\eta}]$ is a Hamiltonian vector field. In fact

$$\iota_{[\underline{\xi},\underline{\eta}]}\omega = [\mathcal{L}_{\underline{\xi}},\iota_{\underline{\eta}}]\omega = \mathcal{L}_{\underline{\xi}}\iota_{\underline{\eta}}\omega \tag{II.2.62}$$

since by symplecticity $\mathcal{L}_{\underline{\eta}}\omega = 0$. Applying Cartan's formula, we get

$$\mathcal{L}_{\underline{\xi}}\iota_{\underline{\eta}}\omega = d\iota_{\underline{\xi}}\iota_{\underline{\eta}}\omega + \iota_{\underline{\xi}}d\iota_{\underline{\eta}}\omega = d\iota_{\underline{\xi}}\iota_{\underline{\eta}}\omega - \iota_{\underline{\xi}}\iota_{\underline{\eta}}d\omega = d\iota_{\underline{\xi}}\iota_{\underline{\eta}}\omega \tag{II.2.63}$$

and since $H^1(\mathfrak{g};\mathbb{R}) = 0$ we have $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$. We can use this isomorphism to read the map $[\mathfrak{g},\mathfrak{g}] \to \mathcal{H}(M) : [\xi,\eta] \to [\underline{\xi},\underline{\eta}]$ as a map $\hbar$ from $\mathfrak{g}$ to $\mathcal{H}(M)$: we get again the diagram

$$\begin{array}{ccccc} & & \mathfrak{g} & & \\ & \tilde{\mu}' \swarrow & & \searrow \hbar & \\ C^\infty(M) & \longrightarrow & \mathcal{H}(M) & \longrightarrow & 0 \end{array} \tag{II.2.64}$$

by exactness we can find for each basis vector ξ_i of $\mathfrak{g}$ an element $f_i \in C^\infty(M)$ such that $\chi_{f_i} = \hbar(\xi_i)$: we define $\tilde{\mu}'(\xi_i) = f_i$ and extend the map linearly.
Now we need to check that this map is a Lie algebra homomorphism, that is, $\tilde{\mu}'([\xi,\eta]) = \{\tilde{\mu}'(\xi),\tilde{\mu}'(\eta)\}$. But we know

$$d\tilde{\mu}'([\xi,\eta]) = \iota_{\chi_{\tilde{\mu}'([\xi,\eta])}} = \iota_{[\underline{\xi},\underline{\eta}]}\omega = d\iota_{\underline{\xi}}\iota_{\underline{\eta}}\omega = -d\{\tilde{\mu}'(\xi),\tilde{\mu}'(\eta)\} \tag{II.2.65}$$

We define

$$c(\xi,\eta) = \tilde{\mu}'([\xi,\eta]) + \{\tilde{\mu}'(\xi),\tilde{\mu}'(\eta)\} \in \mathbb{R},\ \forall \xi,\eta \in \mathfrak{g} \tag{II.2.66}$$

which we can regard as an element of $\mathrm{Hom}_2(\Lambda\mathfrak{g},\mathbb{R})$. It is closed by the Jacobi identity:

$$\delta c(\xi,\eta,\nu) = -c([[\xi,\eta],\nu]) + c([[\xi,\nu],\eta]) - c([[\eta,\nu],\xi]) = 0 \tag{II.2.67}$$

and since $H^2(\mathfrak{g};\mathbb{R}) = 0$ we can express it as some in term of some $b \in \mathfrak{g}^*$: $c(\xi,\eta) = b([\xi,\eta])$. Now let $\tilde{\mu} = b - \tilde{\mu}'$, regarding $b : \mathfrak{g} \to \mathbb{R} \to C^\infty(M)$: then II.2.64 still commutes, and we have

$$(b-\tilde{\mu}'-b)([\xi,\eta]) = b([\xi,\eta]) - \tilde{\mu}([\xi,\eta]) = \{\tilde{\mu}'(\xi),\tilde{\mu}'(\eta)\} = \{(b-\tilde{\mu}')(\xi),(b-\tilde{\mu}')(\eta)\} \tag{II.2.68}$$

so that $\tilde{\mu}$ is a comoment map for the action. □

These criteria shift the problem of existence and uniqueness to the computation of the Lie algebra cohomology groups: when working with compact Lie groups, this amounts to a great simplification.

Lemma II.2.4.5. *Let G be a compact lie group. Then $H^*(\mathfrak{g};\mathbb{R}) \simeq H^*_{dR}(G)$.*

Proof. We divide the proof in two steps:

$$\Omega^*(G) \to \Omega^*_{eq}(G) \to \mathrm{Hom}_*(\Lambda\mathfrak{g}) \tag{II.2.69}$$

that is, first we relate forms to equivariant forms, and then we pass to the Chevalley-Eilenberg complex.

For the first step, observe that we can use the Haar measure on G to average forms: we obtain a map

$$\Omega^k(G) \ni \omega \mapsto \frac{1}{\mathrm{vol}(G)} \int_G (g^*\omega) d\mu(g) \in \Omega^k_{eq}(G) \tag{II.2.70}$$

and since $[d, g^*] = 0$, the assignment descends to a map in cohomology. Surjectivity is clear, to show injectivity observe

$$\int_G (g^*\omega) d\mu(g) = 0 \Rightarrow 0 = \int_G \left(\int_G (g^*\omega) d\mu(g) \right) dx = \int_G \left(\int_G (g^*\omega) dx \right) d\mu(g) \tag{II.2.71}$$

and the quantity $\int_G (g^*\omega) dx$ does not depend on $g \in G$, so that

$$\int_G (g^*\omega) d\mu(g) = 0 \Rightarrow \int_G \omega dx = 0 \tag{II.2.72}$$

which yields injectivity on cohomology.

For the second step, observe that any $\omega \in \Omega^k_{eq}(G)$ is completely determined by its value in e:

$$\omega(e) = \sum_i a_i dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \mathrm{Hom}_k(\Lambda\mathfrak{g}) \tag{II.2.73}$$

in coordinates, the differential of a generic form $\omega \in \Omega(M)$ is given by

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = &\sum_{i=0}^k (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \end{aligned} \tag{II.2.74}$$

for equivariant forms the first term vanishes, so that the assignment $\omega \to \omega(e)$ is a chain map and a bijection. The thesis follows. □

Remark II.2.4.6. (i) We can now address the problem of (non) uniqueness of the height map on the sphere (see II.1.5.4): we found many possible moment maps, which differed by an additive constant. Going through the proof of II.2.4.2, we can stop one step before the conclusion, and obtain that the difference between any two moment maps is an element of $H^1(\mathfrak{g}; \mathbb{R})$: in the case of S^1, this is exactly $\mathbb{R}$;

(ii) An example of Lie groups having $H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}; \mathbb{R}) = 0$ is given by special orthogonal groups, provided their dimension is big enough. Indeed, $SO(3) \simeq \mathbb{R}P^3$ satisfies the requirements, for $H^1(\mathbb{R}P^3; \mathbb{Z}) = \mathbb{Z}_2$, which disappears when taking coefficients in $\mathbb{R}$, and $H^2(\mathbb{R}P^3; \mathbb{Z}) = 0$ already with integral coefficients: to see that the same holds for all higher dimensional special orthogonal groups, just consider the fibration

$$SO(n) \to SO(n+1) \to S^n \tag{II.2.75}$$

and the related spectral sequence. For $n \geqslant 3$ the relevant elements are already stable on the second page, and the claim follows.

Example II.2.4.7. We studied the case $G = T$: from the previous lemma, the first and second Lie algebra cohomology groups of the torus are not trivial, so that equivariant extensions are not always granted to exist.

As an example, consider the S^1−action on a 2−torus given by

$$e^{i\theta} \cdot (e^{i\phi_1}, e^{i\phi_2}) = (e^{i(\phi_1+\theta)}, e^{i\phi_2}) \tag{II.2.76}$$

on the one hand, the action preserves the symplectic form $\omega = d\phi_1 \wedge d\phi_2$; on the other hand, the action is free, so that $H^2_{S^1}(T) = H^2(T/S^1) = H^2(S^1) = 0$. We conclude that there aren't any equivariant extensions of the symplectic form, so that the action is not hamiltonian.

II.2.5. Ehresmann connections

We present a different construction of the Chern-Weil map. This definition is more geometric, and gives some insight on the explicit expression of the characteristic classes of a bundle - and this is certainly a great advantage when looking for equivariant extensions.

The construction requires some background on connections. We recall the main definitions and introduce the notion of *Ehresmann connection*. A more detailed exposition can be found e.g. in [26].

Definition II.2.5.1. Let $\pi : E \to M$ be a vector bundle. A *connection* ∇ on a fiber bundle is a map

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M) \tag{II.2.77}$$

satisfying $\nabla(fs) = s \otimes df + f\nabla(s)\ \forall f \in C^\infty(M)$, where $\Gamma(\cdot)$ denotes the space of sections on the related vector bundle.

If we work locally, we can define an orthonormal frame of sections $\{s_\alpha\}_\alpha$ and look at the behaviour of ∇ on a generic section s in term of a decomposition $s = f^\alpha s_\alpha$:

$$\nabla(f^\alpha s_\alpha) = f^\alpha \nabla(s_\alpha) + s_\alpha \otimes df^\alpha = f^\alpha s_\beta \otimes \omega^\beta_\alpha + s_\alpha \otimes df^\alpha \tag{II.2.78}$$

The matrix of 1−forms $\omega = [\omega^\beta_\alpha]$ is called the *connection form* of ∇. A related concept is that of *curvature form*, a 2−forms matrix defined by

$$\Omega^\beta_\alpha = (d\omega)^\beta_\alpha + \omega^\beta_\gamma \wedge \omega^\gamma_\alpha \tag{II.2.79}$$

for simplicity, such expressions are formally written as

$$\Omega = d\omega + \omega \wedge \omega \tag{II.2.80}$$

where meaning II.2.79 is implied.

In our previous construction of the Chern-Weil map - consider the complex case -, we passed from the bundle $E \to M$ to the bundle of unitary frames $F(E) \to M$, and then worked with the action of $U(n)$ on such a bundle; we do something similar here, by considering local unitary frames of sections. Any two such frames $\{s_\alpha\}_\alpha$, $\{s'_\beta\}_\beta$ - when the intersection U of their domains of defintion is not empty - are related by a change of coordinates

$$s^\alpha(x) = s'_\beta(x) \cdot \mathcal{A}^\alpha_\beta(x) \tag{II.2.81}$$

with $\mathcal{A}(x) \in U(n)\ \forall x \in U$. Explicitly, we have a function

$$\mathcal{A} : U \to U(n) : x \mapsto \mathcal{A}(x) \tag{II.2.82}$$

The connection is called *compatible* with respect to the $U(n)-$bundle structure provided that the associated parallel transport takes unitary frames into unitary frames, and this is in turn seen to be equivalent to

$$\omega(\xi) \in \mathfrak{u}(n)\ \forall \xi \in T_*M \tag{II.2.83}$$

Remark II.2.5.2. (i) Since the connection form depends on the local frame we decide to work with, the definition only makes sense if compatibility in one frame implies compatibility in every other frame: this is indeed the case. To prove it, consider two local unitary frames $\{s_\alpha\}_\alpha$, $\{s'_\beta\}_\beta$ with a common domain of definition, and call ω and ω' the related connection forms. As before, $s(x) = s'(x) \cdot \mathcal{A}(x)$ with $\mathcal{A}(x) \in U(n)$; fix an $x \in M$ and let $A = \mathcal{A}(x)$, we have the system

$$\begin{cases} \nabla s_\alpha = s_\beta \otimes \omega^\beta_\alpha = s_\beta A^\gamma_\alpha \otimes \omega^\beta_\gamma \\ \nabla s_\alpha = \nabla(s'_\beta A^\beta_\alpha) = s'_\beta \otimes dA^\beta_\alpha + s'_\beta A^\beta_\gamma \otimes (\omega')^\gamma_\alpha \end{cases} \tag{II.2.84}$$

putting the two together, we obtain the matrix relation

$$\omega = dA \cdot A^{-1} + A\omega' A^{-1} = dA \cdot A^{-1} + \mathrm{Ad}_A\,\omega' = (\mathcal{A})^*(A^{-1}) + \mathrm{Ad}_A\,\omega' \tag{II.2.85}$$

so that $\omega'(\xi) \in \mathfrak{u}(n) \Rightarrow \omega(\xi) \in \mathfrak{u}(n)$, and similarly the converse.
Notice that the A^{-1} is actually the map induced from the multiplication $U(n) \to U(n)$ on the level of Lie algebras, and, since $\mathcal{A}(x) = A$, it lands exactly in $T_e(U(n)) \simeq \mathfrak{u}(n)$: we can then define the assignment

$$T_A U(n) \ni \xi \mapsto (A^{-1})_*(\xi) \tag{II.2.86}$$

which globally defines a $\mathfrak{u}(n)-$valued $1-$form on $U(n)$, called the *Maurer-Cartan form* of $U(n)$;

(ii) The same strategy applies to find the coordinate change law for the curvature form:

$$\Omega = \mathrm{Ad}_A\,\Omega' \tag{II.2.87}$$

It is also clear that the curvature matrix of a compatible connection is $\mathfrak{u}(n)-$valued;

(iii) Denote by ω_A the assignment obtained by restricting the Maurer-Cartan form of $U(n)$ to $T_A U(n)$. Since M is compact, we can cover it by domains of local unitary frames; for any such local frame s_U and domain U, let

$$\begin{cases} \pi_{U(n)} : U \times U(n) \to U(n) : (x, A) \mapsto A \\ \pi_U : U \times U(n) \to U : (x, A) \mapsto x \end{cases} \tag{II.2.88}$$

the canonical projections, and define

$$\tilde{\omega}_{(x,A)} = \mathrm{Ad}_{A^{-1}}\,\pi_U^*(\omega(s_U)) + \pi_{U(n)}^*\omega_A \tag{II.2.89}$$

where $\omega(s_U)$ is the connection form of ∇ expressed in the reference frame U. We can obtain the unitary frame bundle by patching together the neighbourhoods $U \times U(n)$ via the equivalence relation

$$U \times U(n) \ni (x, A) \sim (y, B) \in V \times U(n) \iff s_U = s_V \cdot \mathcal{A}_{UV}, \; A = (\mathcal{A}_{UV})^{-1} B, \; x = y \tag{II.2.90}$$

The form $\tilde{\omega}$ respects the equivalence relation, thus defining a global $\mathfrak{u}(n)-$valued $1-$form on the unitary frame bundle $\mathcal{F}(E) \to M$. $\tilde{\omega}$ satisfies the compatibility conditions (see [26, p. 314])

(i) $\tilde{\omega}(\underline{\xi}) = \xi \; \forall \xi \in \mathfrak{u}(n)$;

(ii) $\tilde{\omega}(A_* X) = \mathrm{Ad}_{A^{-1}} \tilde{\omega}(X) \; \forall X \in T\mathcal{F}(E), \; \forall A \in U(n)$

These two equations characterize a particular kind of connection on $\mathcal{F}(E)$, as clarified in the next definition.

Definition II.2.5.3. 1. Let G be a compact Lie group, $\pi : \mathcal{E} \to M$ a principal $G-$bundle. An *Ehresmann connection* on $\mathcal{E}$ is a $\mathfrak{g}-$valued one form $\tilde{\omega}$ on E satisfying

(i) $\tilde{\omega}(\underline{\xi}) = \xi \; \forall \xi \in \mathfrak{g}$;

(ii) $(g^* \tilde{\omega})(X) = \mathrm{Ad}_{g^{-1}} \tilde{\omega}(X) \; \forall X \in T\mathcal{E}, \; \forall g \in G$

2. For each $x \in \mathcal{E}$, the subspace $\ker \tilde{\omega}_x \subset T_x \mathcal{E}$ is called the *horizontal subspace at x* determined by $\tilde{\omega}$. Its elements are called *horizontal vectors* at x.

Remark II.2.5.4. 1. It is a priori not clear why, or in which sense, an Ehresmann connection should be seen as a connection. In light of the last remark, we can see it as a generalization of the global, invariant one form on the frame bundle $\mathcal{F}(E) \to M$ arising from a connection on the original bundle $E \to M$;

2. Horizontal subspaces have a series of pleasant properties, as the following lemma shows:

Lemma II.2.5.5. *Let $\pi : \mathcal{E} \to M$ be a principal $G-$bundle, $\tilde{\omega}$ an Ehresmann connection on $\mathcal{E}$. For each $x \in M$, denote by H_x the horizontal subspaces at x determined by $\tilde{\omega}$. Then:*

a) $T_x \mathcal{E} = H_x + V_x \; \forall x \in \mathcal{E}$, where V_x is the inclusion of the tangent space at x of the fiber of $\pi(x)$;

b) $H_{g \cdot x} = g_* H_x \; \forall x \in \mathcal{E}, \; \forall g \in G$;

c) $H : u \mapsto H_u$ is a (smooth) distribution.

Proof. Ad (a). A formal construction of V_x proceeds as follows: we can think of x as a pair (m, h), with $m \in M$ and $h \in G$. We consider the tangent space of h, identify G with the fiber over m, and include the tangent space via the inclusion ι of the fiber in $\mathcal{E}$: $V_x = \iota_*(T_h G)$. It is clear that $\dim V_x = \dim G$.

On the other hand, $\tilde{\omega}_x$ is surjective onto $\mathfrak{g}$ by condition (i) of its definition, so that its kernel has dimension $\dim(\mathcal{E}) - \dim G$. Now, if we identify $T_h G$ with $\mathfrak{g}$ via h_* :

$\mathfrak{g} \to T_h G$, we see that the image of ι_* are exactly the fundamental vector fields. Then $V_x \cap H_x = \{0\}$, and the thesis follows by dimensional considerations.

Ad (b). The spaces have the same dimension. Moreover, for $X \in H_x$,

$$\tilde{\omega}_{g\cdot x}(g_* X) = \mathrm{Ad}_{g^{-1}} \tilde{\omega}(X) = 0 \tag{II.2.91}$$

which defines an injection of H_x into $H_{g\cdot x}$;

Ad (c). For a local frame $\{Y_1, \ldots, Y_{\dim \mathcal{E}}\}$ of sections (vector fields) of $T\mathcal{E}$, we want to find a local frame of sections of H. Consider a basis $\{X_1, \ldots, X_{\dim G}\}$ of $\mathfrak{g}$, then we can write $\omega = \omega^j X_j$, and the (smooth) vector fields

$$\hat{Y}_i = Y_i - \omega^j(Y_i)\underline{X}_j \tag{II.2.92}$$

are in H, and locally generate it. □

This lemma gives a concrete intuition on the object we're working with. It's a way of locally splitting the tangent space into a fiber-part and into a base-part, in a way that globally makes sense. Reinforcing the idea that this is really what a connection *is*, also the converse is true: to such a smooth distribution there corresponds an Ehresmann connection on $\mathcal{E}$ ([26, p. 316]).

The lemma gives way to the definition of curvature for an Ehresmann connection:

Definition II.2.5.6. Let $\pi : \mathcal{E} \to M$ be a principal $G-$bundle, $\tilde{\omega}$ an Ehresmann connection on $\mathcal{E}$. The *curvature* of $\tilde{\omega}$ is a $\mathfrak{g}-$valued $2-$form defined by

$$\tilde{\Omega}(Y_1, Y_2) = (d\tilde{\omega})(hY_1, hY_2) \tag{II.2.93}$$

with $h : T\mathcal{E} \to H$ the projection of X_i onto the horizontal subspace of $T\mathcal{E}$.

It also follows from the lemma that the projection h is smooth, so that the definition is well-posed. It is not clear whether this notion of curvature relates to the usual one - well, it does: the following identities also hold true for $\tilde{\Omega}$

$$\begin{cases} \tilde{\Omega} = d\tilde{\omega} + \omega \wedge \omega \\ d\tilde{\Omega} = \tilde{\omega} \wedge \tilde{\Omega} - \tilde{\Omega} \wedge \tilde{\omega} \end{cases} \tag{II.2.94}$$

The second one is usually expressed in the more palatable form

$$d\tilde{\Omega}(hY_1, hY_2) = 0 \ \forall Y_1, Y_2 \in T\mathcal{F}(E) \tag{II.2.95}$$

Lastly, observe that from the defining property (ii) of $\tilde{\omega}$ we get $A^*\tilde{\Omega} = \mathrm{Ad}_{A^{-1}} \Omega \ \forall A \in U(n)$. These properties will acquire a very important role in the next section.

II.2.6. The Chern-Weil map revisited

Now we are ready to (re)define the Chern-Weil map. We follow three steps:

1. We work on $U(n)-$invariant homogenous polynomials, and define a map which takes such polynomials in input and yields forms in $\mathcal{F}(E)$ as output. Recall that the invariance condition reads $f \cdot Ad_A = f \ \forall A \in U(n)$;

2. We show that these forms can be uniquely associated to closed forms on $M \simeq \mathcal{F}(E)/U(n)$ via π;

3. We extend the map linearly to all polynomials.

Step 1. Pick a homogenous polynomial f of degree k, and regard it as a symmetric tensor over $\mathfrak{g}^*$: the associated form $f(\tilde{\Omega}) \in \Omega^{2k}(\mathcal{F}(E))$ is defined by the equation

$$f(\tilde{\Omega})(X_1, \ldots, X_{2k}) = \frac{1}{2k!} \sum_{\sigma \in \mathcal{P}_{2k}} \epsilon_\sigma f(\tilde{\Omega}(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, \tilde{\Omega}(X_{\sigma(2k-1)}, X_{\sigma(2k)})) \tag{II.2.96}$$

where $X_i \in T_*\mathcal{F}(E) \ \forall i$, $\mathcal{P}_{2k}$ is the group of $2k-$permutation and $\epsilon(\sigma)$ the sign of σ, necessary to get an alternating multilinear map.

Equivalently, pick a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ and the dual basis $X^1, \ldots, X_n$ on $\mathfrak{g}^*$; regard f as a polynomial, we get decompositions

$$\begin{cases} f = a_{i_1,\ldots,i_k} X^{i_i} \ldots X^{i_k} \\ \tilde{\omega} = \tilde{\omega}^j X_j \\ \tilde{\Omega} = \tilde{\Omega}^j X_j \end{cases} \tag{II.2.97}$$

and the expression

$$f(\tilde{\Omega}) = a_{i_1,\ldots,i_l} \tilde{\Omega}^{i_i} \wedge \cdots \wedge \tilde{\Omega}^{i_l} \tag{II.2.98}$$

Step 2. By construction, $\pi_*|_H : H \to TM$ is an isomorphism. Given vector fields $X_1, \ldots, X_{2k} \in TM$, we find some preimages $Y_1, \ldots, Y_{2k}$ in $T\mathcal{F}(E)$: a form $\Lambda \in \Omega^{2k}(M)$ such that $\pi^*\Lambda = f(\tilde{\Omega})$ should satisfy

$$\Lambda(X_1, \ldots, X_{2k}) = f(\tilde{\Omega})(Y_1, \ldots, Y_{2k}) \tag{II.2.99}$$

this equation completely characterizes Λ: if it exists, it's unique.

The point here is that the equation may be used to *define* Λ as well. Look at it pointwise: in choosing a set of preimages of $(X_1)_x, \ldots, (X_{2k})_x$, $x \in M$, we make two choices:

- We pick a point $(x, g) \in \mathcal{E}$;

- We pick vector fields in $T_{(x,g)}$.

We need to show that they do not influence the final output. Start from the second bullet: since $\iota_Y \tilde{\Omega}$ is by definition zero whenever Y is a vertical vector field ($hY = 0$), the quantity $f(\tilde{\Omega})(Y_1, \ldots, Y_{2k})$ does not depend on the set of preimages we choose: this degree of freedom does not make the definition ill-posed.

Now say that we pick another g in the fiber of x, g': for some $l \in G$ it holds $g = l \cdot g'$, and we can choose two related set of preimages:

$$(Y_1, \ldots, Y_{2k}) = (l_* Y'_1, \ldots, l_* Y'_{2k}) \tag{II.2.100}$$

now observe

$$\begin{aligned} f(\Omega)(l_*Y_1',\dots,l_*Y_{2k}') =& a_{i_1,\dots,i_l}\tilde\Omega^{i_i}\wedge\dots\wedge\tilde\Omega^{i_l}(l_*Y_1',\dots,l_*Y_{2k}') \\ =& a_{i_1,\dots,i_l} l^*\tilde\Omega^{i_i}\wedge\dots\wedge l^*\tilde\Omega^{i_l}(Y_1',\dots,Y_{2k}') \\ =& a_{i_1,\dots,i_l}\operatorname{Ad}_{l^{-1}}\tilde\Omega^{i_i}\wedge\dots\wedge l^*\operatorname{Ad}_{l^{-1}}\tilde\Omega^{i_l}(Y_1',\dots,Y_{2k}') \\ =& a_{i_1,\dots,i_l}\tilde\Omega^{i_i}\wedge\dots\wedge l^*\tilde\Omega^{i_l}(Y_1',\dots,Y_{2k}') \\ =& f(\Omega)(Y_1',\dots,Y_{2k}') \end{aligned} \tag{II.2.101}$$

where we used $U(n)-$invariance of f. Then we can use II.2.99 to define Λ.

In order to prove closedness of Λ, we use II.2.95:

$$\begin{aligned} d\Lambda(X_1,\dots,X_{2k}) =& d\Lambda(\pi_*Y_1,\dots,\pi_*Y_{2k}) \\ =& d\Lambda(\pi_*hY_1,\dots,\pi_*hY_{2k}) \\ =& \pi^*d\Lambda(hY_1,\dots,hY_{2k}) \\ =& df(\tilde\Omega)(hY_1,\dots,hY_{2k}) \\ =& a_{i_1,\dots,i_l}\tilde\Omega^{i_i}\wedge\dots\wedge\tilde\Omega^{i_l}(hY_1,\dots,hY_{2k}) = 0 \end{aligned} \tag{II.2.102}$$

Step 3. We can induce a map on cohomology. Extending the map linearly to the space of all the polynomials, we obtain an assignment

$$\mathfrak{k} : S(\mathfrak{u}(n)^*)^{U(n)} \to H^*(M) \tag{II.2.103}$$

Despite its very geometric construction, this map turns out to be independent on the chosen connection - as it should: the only information it should track concerns the bundle, not the additional (arbitrary) structure we put over it. Essentially, we need to show:

Lemma II.2.6.1. *The cohomology class of the form Λ obtained in* Step 2 *does not depend on the chosen connection.*

Proof. Consider two Ehresmann connections $\tilde\omega_0$, $\tilde\omega_1$. We consider the pullback of the bundle $\mathcal{E} \to M$ to $M \times [0,1]$:

$$\begin{array}{ccc} p^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ M\times[0,1] & \longrightarrow & M \end{array} \tag{II.2.104}$$

where p is the projection collapsing $[0,1]$ to a point. Then $p^*\mathcal{E} \simeq \mathcal{E}\times[0,1]$ is still a principal $G-$bundle, and we can pull back $\tilde\omega_0$ and $\tilde\omega_1$ to Ehresmann connections over it; we can actually do more, and namely define a new (Ehresmann) connection

$$\tilde\omega_{(x,t)} = tp^*\tilde\omega_1 + (1-t)p^*\tilde\omega_0 \tag{II.2.105}$$

Define $\iota_t : M \to M\times[0,1] : x \to (x,t)$, we obtain the restrictions $\iota_0^*(\tilde\omega) = \tilde\omega_0$, $\iota_1^*(\tilde\omega) = \tilde\omega_1$. Now we can apply the construction of *Step 2* to $\tilde\omega$ to derive a form Λ on $M\times[0,1]$, and obtain accordingly $\iota_0^*(\Lambda) = \Lambda_0$, $\iota_1^*(\Lambda) = \Lambda_1$. But ι_0 and ι_1 are homotopic, and the thesis follows. □

II.2.7. Localizing the first Chern class

Now we want to show how it is possible to apply this machinery in the context of localization. As a toy model, consider a complex line bundle over a manifold M, and the associated frame bundle:

$$\begin{array}{ccc} \mathbb{C} & & S^1 \\ \downarrow & & \downarrow \\ E & \rightsquigarrow & \mathcal{F}(E) \\ \downarrow & & \downarrow \\ M & & M \end{array} \tag{II.2.106}$$

Pick a connection ∇ on E. The compatibility request is empty: we just require $\omega(\xi) \in \mathbb{R}$ for every vector field in M. There are several other simplifications occurring: the relation between connection matrices of different frames reduces to

$$\omega' = \frac{da}{a} + \omega \tag{II.2.107}$$

where the connections are relative to unitary frames s, s' linked by the relation $s = as'$. Observe that $a : M \to S^1$ is nonvanishing: we may as well write $\omega' = d(\ln(a)) + \omega$, so that

$$\Omega' = \Omega \tag{II.2.108}$$

i.e. the curvature matrix defines a $2-$form on the whole manifold.

Now let's have a look at the derived Ehresmann connection. We are particularly interested in its curvature $\tilde{\Omega}$, which carries the information about the invariants: observe that it is invariant under the action of S^1 - the condition is, again, empty because the adjoint action is trivial: then we can proceed as in the second step of the construction of the Chern-Weil map, and associate it to a unique $2-$form $\hat{\Omega}$ on M, defined by

$$\hat{\Omega}(X_1, X_2) = \Omega(Y_1, Y_2)\ \forall X_1, X_2 \in TM \tag{II.2.109}$$

where Y_1, Y_2 are vector fields on $\mathcal{F}(E)$ such that $\pi_* Y_1 = X_1$, $\pi_* Y_2 = X_2$. Now work locally: fix a domain U of a section s of E, and consider the related connection matrix ω. For a point $x \in U$ and an element θ on the fiber, we have

$$\tilde{\omega}_{(x,\theta)} = \pi_U^* \omega + \pi_{S^1}^* \omega_\theta \tag{II.2.110}$$

so that for a pair of fields Y_1, $Y_2 \in T\mathcal{F}(E)$,

$$\tilde{\Omega}(Y_1, Y_2) = d\tilde{\omega}_{(x,\theta)}(hY_1, hY_2) = d\omega(\pi_* Y_1, \pi_* Y_2) = \Omega(\pi_* Y_1, \pi_* Y_2) \tag{II.2.111}$$

which shows $\Omega = \hat{\Omega}$: we can express the curvature of the Ehresmann connection in term of the one of a connection on E.

Look at the first Chern class: it's the trace of Ω, regarded as a complex matrix: but Ω is just a complex number, so that we just have $c_1 = \Omega$.

Now for the equivariant twist. Consider an action of S^1 on E; in analogy with the case of the tangent bundle, we suppose there is a "fundamental vector field" for this action, i.e. an assignment

$$\mathfrak{g} \ni \xi \mapsto \hat{\xi} \in \mathrm{Hom}(E, E) \tag{II.2.112}$$

so that we have obtain an action on sections of E. Again inspired by the tangent bundle, we define a combined action on the tensor product of $E \otimes T^m_n(M)$, the latter being the bundle of tensors of type (m, n): this is expressed by the derivation property

$$\xi(s \otimes A) = \hat{\xi}(s) \otimes A + s \otimes \mathcal{L}_{\underline{\xi}} A, \ \forall A \in \Gamma(T^m_n(M)), \ s \in \Gamma(E), \ \xi \in \mathfrak{g}, \ m, n \geqslant 0 \tag{II.2.113}$$

where $\underline{\xi}$ is the fundamental vector field related to the S^1 action on M, acting on functions as a derivation. To simplify notation, and in view of the compatibility we imposed, in the following we'll just write $\xi = \hat{\xi} = \underline{\xi}$.

Now, we ask the connection ∇ to be equivariant: $\xi\nabla = \nabla\xi$. We can express it in coordinates: given a generating section s, if

$$\begin{cases} \nabla(s) = s \otimes \omega \\ \xi(s) = L \cdot s \end{cases} \tag{II.2.114}$$

where ω is the $\mathbb{R}-$valued connection $1-$form related to s, and $L \in C^\infty(M)$ a smooth function, also depending on the choice of s, then we want

$$\xi(s \otimes \omega) = \nabla(L \cdot s) \tag{II.2.115}$$

we can use the derivation properties of ∇ and ξ to obtain

$$\xi(s) \otimes \omega + s \otimes \mathcal{L}_\xi \omega = s \otimes dL + L\nabla s \tag{II.2.116}$$

substituting once again the terms II.2.114, $\xi(s) \otimes \omega$ and $L\nabla s$ coincide, and we are left with the condition

$$\mathcal{L}_\xi \omega = dL \tag{II.2.117}$$

Now the plan is to find an equivariant extension for c_1 - that is, for the curvature -, in order to be able to apply the localization theorem. Such an extension has the form

$$c_1^{eq} = \Omega + \xi \otimes f \tag{II.2.118}$$

for some function f. The function should be equivariant, and we also need the form to be closed: this amounts to the equations

$$\begin{cases} df(\xi) = 0 \\ \iota_\xi \Omega = df \end{cases} \tag{II.2.119}$$

a closer look at the second one yields

$$df = \iota_\xi \Omega = \iota_\xi d\omega = \mathcal{L}_\xi \omega - d\iota_\xi \omega = d(L - \iota_\xi \omega) \tag{II.2.120}$$

which suggests $f = L - \iota_\xi \omega$. This expression is just local, but it behaves well with respect to a coordinate change: given L', ω' relative to another generating section s', let $a : M \to S^1$ the function such that $s = as'$. Then the transformation laws read

$$\begin{cases} L' = \iota_\xi d(\ln(a)) + L \\ \omega = d(\ln(a)) + \omega' \end{cases} \tag{II.2.121}$$

and we see that f, as a whole, does not depend on the chosen section. Lastly,

$$df(\xi) = (dL - d\iota_\xi \omega)(\xi) = (\iota_\xi d\omega)(\xi) = 0 \tag{II.2.122}$$

and an equivariantly closed extension of the first Chern class is given by

$$c_1^{eq} = \Omega + \xi \otimes (L - \iota_\xi \omega) \tag{II.2.123}$$

Assume that the S^1-action on M has discrete fixed point, let $\dim M = 2n$. Then localization yields

$$\int_M \frac{\Omega^n}{n!} e^{f\xi} = \sum_{p \in \mathcal{P}} \frac{e^{f(p)\xi}}{e_p(\xi)} \tag{II.2.124}$$

the $e_p(\xi)$ are polynomials of degree n over $\mathbb{R}[\xi]$: we may express them as $\epsilon_p \xi^n$, with $\epsilon_p \in \mathbb{R}$. By considering the series expansion of the exponential and equating terms independent by ξ, we obtain a localization formula for the first Chern class:

$$\int_M c_1^n = \sum_{p \in \mathcal{P}} \frac{f^n(p)}{\epsilon_p} \tag{II.2.125}$$

II.2.8. Generalized flag manifolds

In this last section, we develop a formula for computing the volume of generalized flag manifolds; these objects can be used to classify certain kinds of homogeneous Kähler manifolds. We rapidly go through the necessary definitions:

Definition II.2.8.1. A *complex manifold* M of dimension n is a topological space M with a *complex atlas* $\{(U_\alpha, \phi_\alpha)\}_\alpha$, where the U_α's are open sets covering M, and each ϕ_α's is a function from U_α to $\mathbb{C}^n$ such that the transition functions are biholomorphic.

The canonical *almost complex structure* on M is the smooth section J of $\mathrm{Hom}(TM, TM)$ defined as follows: for every $z \in M$, we find local coordinates $(x_1 + iy_1, \dots, x_n + iy_n)$, which induce a basis on $T_z M$; we set

$$\begin{cases} J_z(\frac{\partial}{\partial x_i}|_z) = \frac{\partial}{\partial y_i}|_z \\ J_z(\frac{\partial}{\partial y_i}|_z) = -\frac{\partial}{\partial x_i}|_z \end{cases} \tag{II.2.126}$$

and extend linearly. The section is seen to be well-defined globally applying the Cauchy-Riemann conditions to the transition functions.

The concept of Kähler manifold brings together the structures of Riemannian, complex and symplectic manifold, in a compatible fashion:

Definition II.2.8.2. A *Kähler manifold* (M, ω) is a complex symplectic manifold such that the assignment

$$M \ni x \mapsto \{T_x M \times T_x M \ni (u, v) \mapsto \omega_x(u, J_x v)\} \tag{II.2.127}$$

defines a Riemannian metric on M. ω is called a *Kähler form*.

Denote by G_M the group of isometries of M: we say that M is a *homogenous* Kähler manifold when this action is transitive.

We have the following:

Theorem II.2.8.3. *Let M be a compact, simply connected, homogeneous Kähler manifold, $x \in M$. Then*

(i) G_M is a compact, connected Lie group, and its stabilizer at x is the centralizer of a torus S in G_M;

(ii) Each orbit M' in $\mathfrak{g}_M$ under the adjoint action of G_M admits a canonical, $G-$invariant complex structure, and a compatible Kähler structure.

(iii) M is isomorphic to some orbit M' as a homogeneous complex manifold.

Proof. See [8]. □

Remark II.2.8.4. The theorem hints at the fact that the quotient $G/C(S)$ can always be identified an adjiont orbit of G. This is the case: see [3, p. 95].

These spaces are exactly the generalized flag manifolds we mentioned in the beginning:

Definition II.2.8.5. Let G be a compact, connected Lie group, S a torus of G and $C(S)$ its centralizer. The quotient $G/C(S)$ is called a *generalized flag manifold*, and just a *flag manifold* if S is a maximal torus of G.

The first nice property concerns the fixed point set:

Theorem II.2.8.6. *Let $G/C(S)$ be a generalized flag manifold, T a maximal torus of G. Then the left action of T*

$$t : gC(S) \mapsto tgC(S) \tag{II.2.128}$$

has finitely many fixed points.

Proof. Suppose $[x]$ is a fixed point. Then

$$txC(S) = xC(S), \ \forall t \in T \tag{II.2.129}$$

or equivalently

$$x^{-1}Tx \subseteq C(S) \tag{II.2.130}$$

Observe that $C(S)$ and G have the same rank: in fact, S is contained in some maximal torus, which we may set as T, and clearly $ts = st \ \forall t \in T, \ s \in S$: thus $T \subseteq C(S)$. But clearly $\operatorname{rank} G \geqslant \operatorname{rank} C(S)$, hence the claim.

Observe that if the action of a maximal torus T has finitely many fixed point, so does the action of any other maximal torus T': in fact the two are conjugated by some $g \in G$ (see II.1.6.3), and there is a $1:1$ correspondence between their fixed points:

$$\begin{aligned} txC(S) &= xC(S)\ \forall t \in T \\ &\Updownarrow \\ t'gxC(S) &= gxC(S)\ \forall t' \in T' \end{aligned} \tag{II.2.131}$$

then we may as well show our claim for a torus T containing S.

If $[x]$ is a fixed point, both T and $x^{-1}Tx$ are maximal tori in $C(S)$, so that for some $u \in C(S)$ it holds $(xu)^{-1}T(xu) = T$: that is, xu belongs to $N_G(T)$, the normalizer of T in G; on the other hand, if $xC(S)$ contains an $x' \in N_G(T)$ we obtain

$$(x')^{-1}Tx'C(S) \subseteq TC(S) \subseteq C(S) \tag{II.2.132}$$

so that $[x]$ is a fixed point of the action. We showed:

$$[x] \text{ is a fixed point } \iff \exists x' \in N_G(T):\ x' \in [x] \tag{II.2.133}$$

We want to refine this result. Suppose that $[x]$ contains two elements $a, b \in N_G(T)$: since $aC(S) = bC(S)$, we have $b^{-1}a \in C(S)$, and

$$(b^{-1}a)^{-1}tb^{-1}a = a^{-1}btb^{-1}a = t' \in T,\ \forall t \in T \tag{II.2.134}$$

that is, $b^{-1}a \in N_{C(S)}(T)$. For the converse, observe that $N_{C(S)}T$ is contained in $C(S)$: if $a, b \in N_G(T)$ satisfy $b^{-1}a \in N_{C(S)}(T)$, it must be $[a] = [b] \in G/C(S)$. Since T is a subgroup of both, we have an isomorphism

$$N_G(T)/N_{C(S)}(T) \simeq W(G)/W(C(S)) \tag{II.2.135}$$

where $W(G)$, $W(C(S))$ denote the Weyl groups of G and $C(S)$. Observe that $C(S)$ is connected, since we may realize it as the union of all maximal tori containing S: we already showed that all such tori are contained in $C(S)$, moreover every element of $C(S)$ must be contained in one of its maximal tori, all of which contain S. It is clear that $C(S)$ is compact, thus $W(G)$ and $W(C(S))$ are finite: see II.1.6.4.

We obtained a bijection

$$\{\text{fixed points of the } T\text{−action}\} \leftrightarrow W(G)/W(C(S)) \tag{II.2.136}$$

where the set on the right hand side is finite as coset space of a finite group. □

We have an immediate corollary:

Corollary II.2.8.7. *Generalized flag manifolds are even dimensional.*

Proof. Thanks to the theorem, the action has at least one fixed point x_0. The action of T at the tangent space of x_0 doesn't fix any vector: then the tangent spaces splits as the sum of $T-$invariant $2-$dimensional vector spaces. □

In view of the theorem, we give the following definition:

Definition II.2.8.8. Let $M = G/C(S)$ be a generalized flag manifold with fixed points $\{x_0, \dots, x_n\}$. The *Weyl set* $\mathcal{F}(M) = \{e, g_1, \dots, g_n\}$ of M is the coset of the Weyl groups of $C(S)$ in the one of G:

$$\mathcal{F}(M) = W(G)/W(C(S)) \tag{II.2.137}$$

with $x_i = g_i \cdot x_0$.

Remark II.2.8.9. (i) $\mathcal{F}(M)$ does not inherit, in general, a group structure from $W(G)$: $W(C(S))$ is not a normal subgroup. The assignment $g_i \mapsto g_i \cdot x_0$ mentioned in the definition is, however, well defined;

(ii) If we regard M as an adjoint orbit $\mathrm{Ad}_G\, x_0$ on $\mathfrak{g}$, we get an assignment

$$\mathcal{F}(M) \to M : \ [g] \mapsto \mathrm{Ad}_g\, x_0 \tag{II.2.138}$$

so that $x_i = \mathrm{Ad}_{g_i}\, x_0$;

(iii) Consider the case of a flag manifold, $S = T$. Maximality of T implies $C(T) = T$, so that $M = G/T$, and $\mathcal{F}(M) = W(G)$: in this case, the theorem immediately yields the number of fixed points and their position in the manifold!

Our ultimate goal is to apply the localization theorem to these generalized flag manifolds, simplifying computations as much as possible: it is therefore natural to ask whether the previous result tell us something in term of the equivariant Euler classes e_i at the fixed points x_i, $i = 0, \dots, n$.

Regard M as an adjoint orbit of G, $M = \mathrm{Ad}_G\, x_0$ for $x_0 \in \mathfrak{g}$: remember that these classes are given - up to a factor $(2\pi)^n$ - by the determinant of the map

$$\mathfrak{t} \to \mathrm{End}(T_{x_i}M) \tag{II.2.139}$$

Explicitly, for a path γ with at x_i:

$$X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_0 \frac{\mathrm{d}}{\mathrm{d}\tau}|_0 \,\mathrm{Ad}_{e^{tX}}\, \gamma(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\, \mathrm{ad}_X\, \gamma(\tau) \tag{II.2.140}$$

that is

$$X \mapsto [X, \cdot] \tag{II.2.141}$$

For $g_i \in \mathcal{F}(M)$, $x_i = \mathrm{Ad}_{g_i}\, x_0$, we have a commuting diagram:

$$\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathrm{End}(T_{x_0}M) \\ \downarrow & & \downarrow \\ \mathrm{Ad}_g\, \mathfrak{t} & \longrightarrow & \mathrm{End}(T_{x_i}M) \end{array} \tag{II.2.142}$$

in fact the adjoint action is a Lie algebra homomorphism, so that

$$[\mathrm{Ad}_g\, X, \cdot] = \mathrm{Ad}_g [X, \mathrm{Ad}_g^{-1}(\cdot)] \tag{II.2.143}$$

the equivariant class e_i of x_i is then given by the determinant of

$$\mathfrak{t} \to \mathrm{End}(T_{x_0}M) : \ X \mapsto [\mathrm{Ad}_{g_i}\, X, \cdot] \tag{II.2.144}$$

Lastly, the tangent space at x_0 may be identified with $\mathfrak{g}/\mathfrak{g}_{x_0}$, $\mathfrak{g}_{x_0}$ denoting the Lie algebra of the stabilizer (see proof of II.2.9.4): this is particularly useful in computations, which boil down to the action of T on $\mathfrak{g}$. We provide an example at the end of the section, computing the volume of $\mathbb{C}P^n$.

All in all, we obtain a polynomial over $\mathfrak{t}$, which we denote by $\det\left([\mathrm{Ad}_{g_i}^{-1}\mathfrak{t},\cdot]\right)$. We proved:

Lemma II.2.8.10. *Let $G/C(S) \simeq \mathrm{Ad}_G\, x_0$ be a $2n-$dimensional, orientable generalized flag manifold, T a maximal torus of G. Let $x_0, \dots, x_n$ be the fixed points of the $T-$action and denote by e_i the equivariant Euler class of*

$$p : T_{x_i}(G/C(S)) \to \{x_i\} \tag{II.2.145}$$

Then

$$e_i = \frac{\det\left([\mathrm{Ad}_{g_i}^{-1}\mathfrak{t},\cdot]\right)}{(2\pi)^n} \tag{II.2.146}$$

for the corresponding g_i's in $\mathcal{F}(G/C(S))$.

II.2.9. The semisimple case

In the end we want to be able to compute the volume of $G/C(S)$ from the localization theorem. We are working on symplectic manifold, then we must be sure that the torus action we are considering is not only symplectic, but actually Hamiltonian: only then (see II.2.2.1) we can find an equivariant extension of the symplectic form.

Let's work in this direction: when discussing existence and uniqueness of moment maps, we saw that groups having $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$ were always granted a unique moment map. Although it will result in a loss of generality, we will see that this is quite a convenient choice to make: start by defining the *Killing form* of a Lie algebra, and the related concept of *semisimple Lie group*:

Definition II.2.9.1. (i) The *Killing form* of a Lie algebra $\mathfrak{g}$ is the bilinear form

$$K : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R} :\ (X,Y) \mapsto \mathrm{tr}(\mathrm{ad}_X \cdot \mathrm{ad}_Y) \tag{II.2.147}$$

(ii) A Lie group G is called *semisimple* if the Killing form of its Lie algebra is non-degenerate.

Remark II.2.9.2. (i) The Killing form is invariant under the adjoint representation:

$$K(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = K(X,Y),\ \forall g \in G \tag{II.2.148}$$

To see this, observe that

$$\mathrm{Ad}_g(\mathrm{ad}_X(Y)) = \mathrm{Ad}_g[X,Y] = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] = \mathrm{ad}_{\mathrm{Ad}_g X}(\mathrm{Ad}_g(Y)) \tag{II.2.149}$$

and apply the definition of Killing form:

$$\begin{aligned} K(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) &= \mathrm{tr}(\mathrm{ad}_{\mathrm{Ad}_g X}\cdot \mathrm{ad}_{\mathrm{Ad}_g Y}) \\ &= \mathrm{tr}(\mathrm{Ad}_g \cdot \mathrm{ad}_X \cdot \mathrm{Ad}_{g^{-1}}\, \mathrm{Ad}_g\, \mathrm{ad}_Y \cdot \mathrm{Ad}_{g^{-1}}) \\ &= \mathrm{tr}(\mathrm{Ad}_{g^{-1}}\cdot \mathrm{Ad}_g \cdot \mathrm{ad}_X \cdot \mathrm{ad}_Y) = \mathrm{tr}(\mathrm{ad}_X\cdot \mathrm{ad}_Y) = K(X,Y) \end{aligned} \tag{II.2.150}$$

(ii) For a semisimple Lie group, this gives an isomorphism Φ between $\mathfrak{g}$ and $\mathfrak{g}^*$, via

$$\mathfrak{g} \ni X \mapsto K(X,\cdot) \in \mathfrak{g}^* \tag{II.2.151}$$

This isomorphism respects the $G-$action, so that the adjoint and coadjoint representation are isomorphic:

$$\Phi(\mathrm{Ad}_g X) = K(\mathrm{Ad}_g X,\cdot) = K(X,\mathrm{Ad}_{g^{-1}}(\cdot)) = \mathrm{Ad}_g^*(K(X,\cdot)) = \mathrm{Ad}_g^* \Phi(X) \tag{II.2.152}$$

There is an alternative way of characterizing semisimple Lie groups (see [13, p. 167]):

Theorem II.2.9.3 (Whitehead Lemmas). *Let G be a compact Lie group. Then G is semisimple if and only if $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$.*

Then our criterion for existence and uniqueness of a moment map yields equivalence of adjoint and coadjoint representation for free. Now, we saw before that the generalized flag manifold M we're interested in can be identified with orbits of the adjoint representation of G - but orbits of the coadjoint representation happen to have a lot of nice properties, which we'll now describe, and at the end of the day we will transfer everything back to M. Advantages will become apparent.

Proposition II.2.9.4. *Let $M \subset \mathfrak{g}^*$ be an orbit of the coadjoint representation of G. Then the $2-$form ω defined by*

$$\omega(\xi)(\underline{X},\underline{Y}) = \xi([X,Y]),\ \forall \xi \in M,\ \underline{X},\underline{Y} \in T_\xi(M) \tag{II.2.153}$$

where $\underline{X},\underline{Y}$ are the fundamental vector fields associated to X and Y, is a symplectic form on M.

Proof. It is clear that ω is bilinear and alternating; we need to show that it is non-degenerate and closed.

Before we start, let us have a closer look at the tangent space of M at ξ. Consider the differential in 0 of the map from G to M which sends g to $g\cdot\xi$; if we denote by G_ξ the stabilizer of ξ, we obtain an isomorphism differentiating in 0 the induced map on the quotient:

$$\mathfrak{g}/\mathfrak{g}_\xi \simeq T_\xi M \tag{II.2.154}$$

this map sends an element $X + \mathfrak{g}_\xi$ to $\mathrm{ad}_X^*(\xi)$, and we will see $\mathrm{ad}_X^*(\xi) = \underline{X}_\xi$. Thus, equation II.2.153 uniquely defines ω.

Non-degeneracy. Differentiating the identity $\mathrm{Ad}^*_{\exp(-tX)}\xi(Y) = \xi(\mathrm{Ad}_{\exp(tX)} Y)$ in zero we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}|_0 \mathrm{Ad}^*_{\exp(-tX)}\xi\right)(Y) = \xi(\mathrm{ad}_X Y) = \xi([X,Y]) = \omega(\xi)(\underline{X},\underline{Y}) \tag{II.2.155}$$

then we can identify the kernel of $\omega(\xi)$ with the Lie algebra $\mathfrak{g}_\xi \subset \mathfrak{g}$ of the stabilizer of ξ. But we saw $T_\xi M \simeq \mathfrak{g}/\mathfrak{g}_\xi$, hence the form is non-degenerate.

Closedness. The tangent space at any point is generated by the fundamental vector fields of the action. These have the form

$$\underline{X}_\xi = \frac{\mathrm{d}}{\mathrm{d}t}|_0 \mathrm{Ad}^*_{\exp(tX)}\xi = \mathrm{ad}^*_X(\xi) \tag{II.2.156}$$

by linearity, it is sufficient to check closedness of the fundamental vector fields:

$$\begin{aligned}(d\omega)(\xi)(\underline{X},\underline{Y},\underline{Z}) =& -\underline{X}(\xi[Y,Z]) + \underline{Y}(\xi[X,Z]) - \underline{Z}(\xi[X,Y]) \\ &+ \xi([[X,Y],Z]) - \xi([[X,Z],Y]) + \xi([[Y,Z],X]) \\ =& -\xi([[Y,Z],X]) + \xi([[X,Z],Y]) - \xi([[X,Y],Z]) \\ &+ \xi([[X,Y],Z]) - \xi([[X,Z],Y]) + \xi([[Y,Z],X]) = 0\end{aligned} \tag{II.2.157}$$

where we used $[\mathrm{ad}^*_X, \mathrm{ad}^*_Y] = \mathrm{ad}^*_{[X,Y]}$. □

Remark II.2.9.5. The construction of ω only relies on the Lie bracket on $\mathfrak{g}$: we will refer to ω as the canonical symplectic form on M.

Observe that this symplectic form is $G-$invariant by construction, so that we get a unique moment map for free. What does it look like? And: can we derive one for the torus action?

Proposition II.2.9.6. *Let (M,ω) be an orbit of the coadjiont representation, equipped with its canonical symplectic form. Then the action of T on (M,ω) is Hamiltonian, with moment map*

$$\varphi : M \hookrightarrow \mathfrak{g}^* \twoheadrightarrow \mathfrak{t}^* \tag{II.2.158}$$

Proof. First, we show that the orbit inclusion $\phi : M \hookrightarrow \mathfrak{g}^*$ is the moment map for the $G-$action. Indeed, the related comoment map $\hat{\phi}$ should be a Lie algebra homomorphism making the diagram commute:

$$\begin{array}{ccc} C^\infty(M) & \xleftarrow{\hat{\phi}} & \mathfrak{g} \\ \downarrow & & \downarrow \\ \mathcal{H}_{loc}(M) & \longrightarrow & \mathcal{H}(M) \end{array} \tag{II.2.159}$$

the vertical arrow on the right associates X with its fundamental vector field ad^*_X; the image of X along the other path can be computed in two steps:

(i) $X \mapsto \hat{\phi}(X) = \{\xi \mapsto \phi(\xi)(X) = \xi(X)\}$, the evaluation map at X;

(ii) Solve the equation $d(\hat{\phi}(X)) = \iota_Y\omega$: as before, it suffices to evaluate the expression on fundamental vector fields, and we may as well write Y as a fundamental vector field $\underline{Y}$. We get:

$$\xi([Z,X]) = \mathrm{ad}^*_Z(\xi)(X) = d\hat{\phi}(\xi)(\underline{Z}) = \omega(\underline{Z},\underline{Y}) = \xi([Z,Y]) \tag{II.2.160}$$

so that $Y = \mathrm{ad}^*_X(\xi)$.

Then the diagram commutes. It is also clear that $\hat{\phi}$ is a Lie algebra homomorphism:

$$\hat{\phi}([X,Y])(\xi) = \xi([X,Y]) = \omega_\xi(\underline{X},\underline{Y}) = \{\hat{\phi}(X),\hat{\phi}(Y)\} \tag{II.2.161}$$

The thesis follows observing that a comoment map for the $T-$action is given by

$$\begin{array}{ccccc} C^\infty(M) & \xleftarrow{\hat{\phi}} & \mathfrak{g} & \xleftarrow{\iota} & \mathfrak{t} \\ \downarrow & & \downarrow & \swarrow & \\ \mathcal{H}_{loc}(M) & \longrightarrow & \mathcal{H}(M) & & \end{array} \tag{II.2.162}$$

where ι is induced from an inclusion $T \to G$ and the diagonal map is defined by commutativity. It is then clear that $\hat{\varphi} = \hat{\phi} \cdot \iota$ is a comoment map; the related moment map is exactly the one in the statement of the proposition. □

Look at the image of the fixed point set through the moment map:

Corollary II.2.9.7. *Let (M, ω) be an orbit of the coadjiont representation, equipped with its canonical symplectic form. Consider the Hamiltonian action of a maximal torus T with the moment map φ described in II.2.9.6 and fixed point set F. Then*

(i) $\varphi|_F = \mathrm{id}$;

(ii) $\varphi(M)$ *is the convex hull of* F *in* $\mathfrak{t}^*$.

Proof. Ad (i). We just need to show that $F \subseteq \mathfrak{t}^*$. Denote the Killing form on $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle$, recall there is an isomorphism

$$\mathfrak{g} \ni X \mapsto \langle X, \cdot \rangle \in \mathfrak{g}^* \tag{II.2.163}$$

Pick $\langle x_0, \cdot \rangle \in F$. We have the relation

$$\mathrm{Ad}^*_{e^{tX}} \langle x_0, \cdot \rangle = \langle x_0, \cdot \rangle \;\; \forall X \in \mathfrak{t} \tag{II.2.164}$$

which we can differentiate to obtain

$$\langle x_0, [\cdot, X] \rangle = 0 \; \forall X \in \mathfrak{t} \tag{II.2.165}$$

invariance of the Killing form implies $[x_0, X] = 0 \; \forall X \in \mathfrak{t}$; by maximality, $x_0 \in \mathfrak{t}$. Then $\langle x_0, \cdot \rangle \in \mathfrak{t}^*$, and the thesis follows.

Ad (ii). We make use of the following theorem from Konstant (as quoted in [4]):

Theorem II.2.9.8. *The orthogonal projection of an adjoint orbit of G in $\mathfrak{t}$ is given by the convex hull of the corresponding $W(G)-$orbit in $\mathfrak{t}$.*

We can refine the result by applying II.2.8.6; given an adjoint orbit $\mathrm{ad}_G\, x_0$, we know that x_0 is a fixed point of the $T-$action and that the Weyl group of G acts transitively on the fixed point set. Then the $W(G)-$orbit coincides with the fixed point set, and identification of adjoint and coadjoint representation proves the claim. □

Putting everything together, we can state our main result. A definition:

Definition II.2.9.9. Let $M = G/C(S)$ be a generalized flag manifold. We call M *semisimple* if G is.

Theorem II.2.9.10. *Let $(M, \omega) \simeq \mathrm{Ad}_G\, x_0$ be a semisimple, orientable generalized flag manifold of dimension $2n$ together with its canonical symplectic form, $\mathcal{F}(M)$ its Weyl set. Then*

$$\mathrm{vol}_\omega(M) = \frac{(-2\pi)^n}{n!} \sum_{g_i \in \mathcal{F}(M)} \frac{x_i^n}{\det\left([\mathrm{Ad}_{g_i} \mathfrak{t}, \cdot]\right)} \tag{II.2.166}$$

where $x_i = g_i \cdot x_0$.

Proof. Let us clarify the meaning of x_i^n. Recall that we obtain an equivariant extension of ω by subtracting the moment map,

$$\omega_{eq} = \omega - \varphi \tag{II.2.167}$$

more formally: set $l = \mathrm{rank}(G)$, we consider the components φ^j of $\varphi : M \to \mathfrak{t}^* \simeq R^l$ and the related basis $\theta_1, \ldots, \theta_l$ of $\mathfrak{t}^*$; we then have

$$\omega_{eq} = 1 \otimes \omega - \sum_{j=1}^{l} \theta_j \otimes \varphi^j \tag{II.2.168}$$

In our case the moment map restricts to the identity on the fixed point set. Let $x_i = (x_i^1, \ldots, x_i^l)$, then we denote

$$x_i^n = \left(\sum_{j=1}^{l} \theta_j \otimes x_i^j \right)^n \tag{II.2.169}$$

Now, localize with respect to the action of a maximal torus. We have:

$$\pi_*^M \omega_{eq}^n = \sum_i \pi_*^i \left(\frac{\iota_i^* \omega_{eq}^n}{e_i} \right) \tag{II.2.170}$$

where the sum on the right ranges over the connected components of the fixed point set. As we already saw in the proof of II.2.2.3, π_*^M only sees the $2n-$form component, whereas when the fixed point set is discrete ι_i^* only saves the $0-$form components; moreover, the fixed point set coincides with the Weyl set of M. We immediately obtain

$$\int_M \frac{\omega^n}{n!} = \frac{1}{n!} \sum_{x \in \mathcal{F}(M)} \frac{(-x_i)^n}{e_{x_i}} \tag{II.2.171}$$

apply II.2.8.10 to conclude the proof. □

The theorem can be generalized to homogeneous Kähler manifolds. We have the following:

Proposition II.2.9.11. *Let (M, ω) be a homogeneous Kähler manifold on which the isometry group G acts in Hamiltonian fashion, $x \in M$, $\varphi : M \to \mathfrak{g}^*$ a moment map for the action.*

Then $\varphi(M) = \mathrm{Ad}_G^* \varphi(x)$, and φ is a symplectomorphism with respect to the canonical symplectic form $\hat{\omega}$ on $\mathrm{Ad}_G^* \varphi(x)$.

Proof. The moment map is equivariant with respect to the coadjiont action on $\mathfrak{g}^*$ (see [6, p. 54]), and it is injective (see [20, p. 40]), so that we can identify $\varphi(M)$ with an orbit $\mathrm{Ad}_G^* \varphi(x) \subset \mathfrak{g}^*$. Let us show that it respects the symplectic forms: denote by $\psi : \mathfrak{g} \to C^\infty(M)$ the comoment map of the action derived from φ, and let ξ_f be the Hamiltonian vector field related to $f \in C^\infty(M)$, i.e.

$$\iota_{\xi_f} \omega = df \tag{II.2.172}$$

then on the one hand

$$\begin{aligned} \omega(\underline{X}, \underline{Y}) &= \omega(\xi_{\psi(X)}, \xi_{\psi(Y)}) \\ &= \{\psi(X), \psi(Y)\} \\ &= \psi[X, Y] = \varphi(\cdot)([X, Y]), \ \forall X, Y \in \mathfrak{g} \end{aligned} \tag{II.2.173}$$

on the other hand

$$(\varphi^*\hat{\omega})(\underline{X}, \underline{Y}) = \varphi(\cdot)([\varphi_*\underline{X}, \varphi_*\underline{Y}]),\ \forall X, Y \in \mathfrak{g} \tag{II.2.174}$$

so it suffices to show $\varphi(\cdot)[X, Y] = \varphi(\cdot)[\varphi_*\underline{X}, \varphi_*\underline{Y}],\ \forall X, Y \in \mathfrak{g}$. Recall that equation II.2.174 only makes sense under the identification

$$\mathfrak{g}/\mathfrak{g}_{\varphi(x)} \xrightarrow{\simeq} T_{\varphi(x)}\varphi(M) :\ X \mapsto \underline{X} \tag{II.2.175}$$

with $\mathfrak{g}_{\varphi(x)}$ the Lie algebra of the stabilizer of $\varphi(x)$; similarly, $T_xM \simeq \mathfrak{g}/\mathfrak{g}_x$, and equivariance of φ implies $G_x \simeq G_{\varphi(x)} =: H$. We get a commutative diagram

$$\begin{array}{ccc} & \mathfrak{g}/\mathfrak{h} & \\ \swarrow & & \searrow \\ T_xM & \xrightarrow[\varphi_*]{} & T_{\varphi(x)}\varphi(M) \end{array} \tag{II.2.176}$$

in other words, $[\varphi_*\underline{X}, \varphi_*\underline{Y}] = \left[\underline{\underline{X}}, \underline{\underline{Y}}\right]$, where $\underline{\underline{X}}$ is the fundamental vector field related to X in $\varphi(M)$. Then, taking into account II.2.175,

$$\varphi(\cdot)([\varphi_*\underline{X}, \varphi_*\underline{Y}]) = \varphi(\cdot)\left[\underline{\underline{X}}, \underline{\underline{Y}}\right] = \varphi(\cdot)[X, Y] \tag{II.2.177}$$

which concludes the proof. □

This yields a generalization of II.2.9.10:

Corollary II.2.9.12. *Let (M, ω) be a simply connected, homogeneous $2n-$dimensional Kähler manifold on which the semisimple isometry group G acts in Hamiltonian fashion, $\varphi : M \to \mathfrak{g}^*$ a moment map for the action.*

Then

$$\mathrm{vol}_\omega(M) = \frac{(-2\pi)^n}{n!} \sum_{g_i \in \mathcal{F}(\varphi(M))} \frac{x_i^n}{\det\left([\mathrm{Ad}_{g_i}\mathfrak{t}, \cdot]\right)} \tag{II.2.178}$$

Proof. Since φ is a symplectomorphism,

$$\mathrm{vol}_\omega(M) = \int_M \frac{\omega}{n!} = \int_M \frac{\varphi^*(\hat{\omega})}{n!} = \int_{\varphi(M)} \frac{\hat{\omega}}{n!} = \mathrm{vol}_{\hat{\omega}}(\varphi(M)) \tag{II.2.179}$$

where $\hat{\omega}$ is the canonical symplectic form on $\varphi(M)$. From II.2.8.3, we know $M \simeq G/C(S)$ as a complex manifold, hence $\varphi(M) \simeq G/C(S)$ and we can apply II.2.8.6 to conclude that the action of a maximal torus has a finite number of fixed points $\mathcal{F}(\varphi(M))$; the action of the torus is Hamiltonian with respect to the canonical symplectic form, and has the moment map of II.2.9.6.

Now we can proceed as in the proof of II.2.9.10 to localize the volume of $\varphi(M)$ and obtain the thesis. □

II.2.10. The volume of $\mathbb{C}P^n$

We apply the theory we developed to compute the volume of the complex projective space in any dimension.

The first step is to identify the complex projective space with a generalized flag manifold. We take a little detour, and consider the adjoint action of $U(n+1)$ on $\mathfrak{u}(n+1)$: the Lie algebra may be identified with the space $i\mathcal{H}$ of skew-hermitian matrices, so that the orbits are just labelled by a set $(\lambda_1, \dots, \lambda_{n+1})$ of eigenvalues; every matrix in the orbit splits $\mathbb{C}^{n+1}$ into mutually orthogonal $\mathbb{C}-$vector spaces according to their multiplicity.

The idea is to pick the orbit of the matrix

$$\mathscr{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{II.2.180}$$

denoted by $\mathcal{H}_{\mathscr{I}}$. Each of the matrices in this orbit have the effect of singling out a $\mathbb{C}-$line in $\mathbb{C}^{n+1}$: given homogeneous coordinates $\ell = [z_1 : \dots : z_{n+1}]$ for a line with $z_1, \dots, z_{n+1} \neq 0$, the related matrix reads

$$M_\ell = \frac{1}{n+1} \begin{bmatrix} 1 & z_2/z_1 & z_3/z_1 & \dots & z_{n+1}/z_1 \\ z_1/z_2 & 1 & z_3/z_2 & \dots & z_{n+1}/z_2 \\ z_1/z_3 & z_2/z_3 & 1 & \dots & z_{n+1}/z_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1/z_{n+1} & z_2/z_{n+1} & z_3/z_{n+1} & \dots & 1 \end{bmatrix} \tag{II.2.181}$$

if some of the z_i's are zero, we get M_ℓ by setting to zero all the components involving the related z_i's and the 1 in the (i,i) position. The prefactor should be changed to $1/m$, where m is the number of non-zero z_i's; notice that this number is always strictly positive, so that everything is well defined. We can identify the orbit with $\mathbb{C}P^n$, seen as the space of $\mathbb{C}-$lines in $\mathbb{C}^{n+1}$.

Now, if $U(n+1)$ were semisimple, we could identify this orbit with an orbit of the coadjoint action, and get a symplectic structure on $\mathbb{C}P^n$: this is not the case - the Killing form is degenerate - but there is *another* form which is not. Since $\mathfrak{u}(n+1)$ is already made up of matrices, we can naïvely just pick

$$(X, Y) \mapsto \operatorname{tr}(XY) \tag{II.2.182}$$

this brings us back in business: this pairing is non-degenerate, and clearly adioint-invariant. We get an isomorphism $\mathfrak{u}(n+1) \simeq \mathfrak{u}(n+1)^*$, with which we regard elements of $\mathfrak{u}(n+1)^*$ as maps $\operatorname{tr}((\cdot)X)$, with $X \in \mathfrak{u}(n+1)$. We use II.2.9.4 and this identification to obtain a symplectic form on the adioint orbit:

$$\omega(Z)(\underline{X}, \underline{Y}) = \operatorname{tr}([X, Y]Z), \ \ X, Y \in T_Z\mathscr{I} \tag{II.2.183}$$

where $\underline{X} = \mathrm{ad}_X$, $\underline{Y} = \mathrm{ad}_Y$ are the fundamental vector fields related to X and Y (see II.2.156). Now consider the action of a maximal torus: we get automatically a moment map from II.2.9.6:

$$\varphi : \mathscr{I} \hookrightarrow \mathfrak{u}(n+1)^* \twoheadrightarrow \mathfrak{t}^* \tag{II.2.184}$$

we may pick the maximal torus to be

$$T = \{\mathrm{diag}[e^{i\theta_1}, \dots, e^{i\theta_{n+1}}]\} \subset U(n+1) \tag{II.2.185}$$

Then the moment map just projects a matrix to its diagonal elements. So far we obtained an Hamiltonian action of T on $\mathbb{C}P^n$, explicitly describing its moment map; now let's have a look at its fixed points.

We can realize $\mathcal{H}_{\mathscr{I}}$ as a coset $U(n+1)/C(S)$ for a 1−dimensional torus. Indeed, the stabilizer of $\mathscr{I}$ is given by

$$\{A \in U(n) : \mathrm{Ad}_A \mathscr{I} = \mathscr{I}\} \tag{II.2.186}$$

which we can exponentiate to

$$\{A \in U(n) : Ae^{t\mathscr{I}} = e^{t\mathscr{I}}A,\ t \in \mathbb{R}\} = C(\{e^{t\mathscr{I}} :\ t \in \mathbb{R}\}) \tag{II.2.187}$$

Set $S_{\mathscr{I}} = \{e^{t\mathscr{I}} :\ t \in \mathbb{R}\}$. Elements of this torus have the form

$$\mathscr{K} = \begin{bmatrix} e^{i\phi} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{II.2.188}$$

for some $\phi \in \mathbb{R}$. We'll compute $C(S_{\mathscr{I}})$ as the union of the maximal tori including $S_{\mathscr{I}}$ after determining the set of fixed points.

We know that the fixed points are in bijection with $W(U(n+1))/W(C(S_{\mathscr{I}}))$, in particular they can be no more than $W(U(n+1))$. The elements of $W(U(n+1)) = S_{n+1}$, $n+1$ are symmetric permutations: when acting on $\mathscr{I}$, all they do is sending the 1 somewhere else along the diagonal, so that different elements of $W(U(n+1))$ sending the 1 in $\mathscr{I}$ to the same point are identified. This is the effect of the $W(C(S_{\mathscr{I}}))$−quotient.

The fixed points of the action correspond in homogeneous coordinates to the lines

$$[1:0:0:\dots:0:0],\ [0:1:0:\dots:0:0],\ \dots,\ [0:0:0:\dots:0:1] \tag{II.2.189}$$

We have everything we need to apply theorem II.2.9.10 and compute the volume of $\mathbb{C}P^n$. As a warm-up, we work out the cases $n = 1$ and $n = 2$:

$(n = 1)$ The fixed points are $[1:0]$ and $[0:1]$. If θ_1 and θ_2 are coordinates on the maximal torus II.2.185, the equivariant extension of the given symplectic form is

$$\omega_{eq} = \omega - \theta_1 \otimes \varphi^1 - \theta_2 \otimes \varphi^2 \tag{II.2.190}$$

So that the image of the first fixed point is θ_1, that of the second θ_2.

The equivariant Euler classes are computed by considering the action of the torus on the bundle of frames over the fixed points. In this case, it suffices to consider the action on the tangent spaces; we saw in the proof of II.2.9.4 that

$$T_\xi \mathcal{H}_{\mathscr{I}} \simeq \mathfrak{u}(2)/\mathfrak{u}(2)_\xi \tag{II.2.191}$$

where $\mathfrak{u}(2)_\xi$ is the Lie algebra of the stabilizer of ξ. For the first fixed point, $\xi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the stabilizer is $C(S_{\mathscr{I}})$. Since

$$S_{\mathscr{I}} = \{\mathrm{diag}\,[e^{i\phi}, 1],\ \phi \in \mathbb{R}\} \tag{II.2.192}$$

The union of maximal tori containing it is just the one we are already considering, and elements of $\mathfrak{u}(2)/\mathfrak{u}(2)_\xi$ are represented by equivalence classes $\begin{bmatrix} 0 & z \\ \bar{z} & 0 \end{bmatrix}$ The adjoint action of the torus on these elements can be computed:

$$\mathrm{Ad}_{\mathrm{diag}\,[\theta_1,\theta_2]} \begin{bmatrix} 0 & z \\ \bar{z} & 0 \end{bmatrix} = \begin{bmatrix} 0 & ze^{i(\theta_2-\theta_1)} \\ \bar{z}e^{i(\theta_1-\theta_2)} & 0 \end{bmatrix} \tag{II.2.193}$$

if we identify a frame with an element $z \neq 0$, the action of T on frames is just

$$z \mapsto ze^{i(\theta_2-\theta_1)} \tag{II.2.194}$$

The bundle is $1-$dimensional, and it is acted upon by $U(1)$: this means that the inclusion of T in $U(1)$ is given by

$$T \to U(1) : (e^{i\theta_1}, e^{i\theta_2}) \mapsto e^{i(\theta_2-\theta_1)} \tag{II.2.195}$$

on the level of Lie algebras this is just $(\theta_1, \theta_2) \mapsto \theta_2 - \theta_1$, the equivariant Euler class is the determinant of this number - the number itself! - times $1/2\pi$.

The second fixed point is linked to the first by the action of an element of the Weyl group:

$$\eta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \mathrm{Ad}_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{II.2.196}$$

So we can compute the action as prescribed from II.2.8.10:

$$\begin{aligned} \mathrm{Ad}_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \mathrm{Ad}_{\mathrm{diag}\,[\theta_1,\theta_2]} \left(\begin{bmatrix} 0 & z \\ \bar{z} & 0 \end{bmatrix} \right) &= \mathrm{Ad}_{\mathrm{diag}\,[\theta_2,\theta_1]} \left(\begin{bmatrix} 0 & z \\ \bar{z} & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & ze^{i(\theta_1-\theta_2)} \\ \bar{z}e^{i(\theta_2-\theta_1)} & 0 \end{bmatrix} \end{aligned} \tag{II.2.197}$$

and get the equivariant Euler class $\theta_1 - \theta_2$. Applying theorem II.2.9.10 now yields

$$\mathrm{vol}(\mathbb{C}P^1) = -2\pi \left(\frac{\theta_1}{\theta_2 - \theta_1} + \frac{\theta_2}{\theta_1 - \theta_2} \right) = 2\pi \tag{II.2.198}$$

$(n = 2)$ The fixed points are $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$, their images under the moment map respectively θ_1, θ_2, θ_3.

The bundle of frames over the fixed points here is more complicated, since the tangent space has complex dimension 2. Look at $\mathfrak{g}/\mathfrak{g}_\xi$ for

$$\xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{II.2.199}$$

The stabilizer is given by the centralizer of the space

$$S = \left\{ \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ t \in \mathbb{R} \right\} \tag{II.2.200}$$

The centralizer is the union of maximal tori containing S. There are two of them:

$$T_1 = \left\{ \begin{bmatrix} e^{it_1} & 0 & 0 \\ 0 & e^{it_2} & 0 \\ 0 & 0 & e^{it_3} \end{bmatrix}, \ t_i \in \mathbb{R} \right\}, \ T_2 = \left\{ \begin{bmatrix} e^{it_1} & 0 & 0 \\ 0 & e^{it_2} & e^{it_3} \\ 0 & e^{it_3} & e^{it_2} \end{bmatrix}, \ t \in \mathbb{R} \right\} \tag{II.2.201}$$

so that elements of $\mathfrak{u}(2)/\mathfrak{u}(2)_\xi$ are represented by equivalence classes

$$\begin{bmatrix} 0 & z_1 & z_2 \\ \bar{z}_1 & 0 & 0 \\ \bar{z}_2 & 0 & 0 \end{bmatrix} \tag{II.2.202}$$

The adjoint action of the torus on these elements can be computed, similarly as before:

$$\mathrm{Ad}_{\mathrm{diag}\,[\theta_1,\theta_2,\theta_3]} \begin{bmatrix} 0 & z_1 & z_2 \\ \bar{z}_1 & 0 & 0 \\ \bar{z}_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & z_1 e^{i(\theta_2-\theta_1)} & z_2 e^{i(\theta_3-\theta_1)} \\ \bar{z}_1 e^{i(\theta_1-\theta_2)} & 0 & 0 \\ \bar{z}_2 e^{i(\theta_1-\theta_3)} & 0 & 0 \end{bmatrix} \tag{II.2.203}$$

if we identify a frame with a pair $(z_1, z_2) \neq 0$, the action of T on frames is given by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 e^{i(\theta_2-\theta_1)} \\ z_2 e^{i(\theta_3-\theta_1)} \end{bmatrix} \tag{II.2.204}$$

and this means that the inclusion of T in $U(2)$ is given by

$$T \to U(2) : (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mapsto \begin{bmatrix} e^{i(\theta_2-\theta_1)} & 0 \\ 0 & e^{i(\theta_3-\theta_1)} \end{bmatrix} \tag{II.2.205}$$

on the level of Lie algebras this is

$$(\theta_1, \theta_2, \theta_3) \mapsto \begin{bmatrix} \theta_2 - \theta_1 & 0 \\ 0 & \theta_3 - \theta_1 \end{bmatrix} \tag{II.2.206}$$

the equivariant Euler class is the determinant of this matrix times $(2\pi)^{-2}$:

$$e_{[1:0:0]} = \frac{(\theta_2 - \theta_1)(\theta_3 - \theta_1)}{4\pi^2} \tag{II.2.207}$$

The strategy for computing the action on the other fixed points is similar as before. The point $[0:1:0]$ is linked to $[1:0:0]$ by

$$\eta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Ad}_{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{II.2.208}$$

And we can compute the action as in II.2.197, obtaining

$$\mathrm{Ad}_{\mathrm{diag}\,[\theta_1,\theta_2,\theta_3]} \begin{bmatrix} 0 & z_1 & z_2 \\ \bar{z}_1 & 0 & 0 \\ \bar{z}_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & z_1 e^{i(\theta_1-\theta_2)} & z_2 e^{i(\theta_3-\theta_2)} \\ \bar{z}_1 e^{i(\theta_2-\theta_1)} & 0 & 0 \\ \bar{z}_2 e^{i(\theta_2-\theta_3)} & 0 & 0 \end{bmatrix} \tag{II.2.209}$$

and get the equivariant Euler class

$$e_{[0:1:0]} = \frac{(\theta_1-\theta_2)(\theta_3-\theta_2)}{4\pi^2} \tag{II.2.210}$$

The spirit of the computation should have now become clearer: we use the isomorphism from the tangent space of the base point to $\mathfrak{g}/\mathfrak{g}_\xi$ to compute the adjoint action of T, we express it on frames and then we move it to the other points; this is equivalent to permuting the coordinates θ_i's of T.The equivariant Euler class for the last fixed point is obtain by permuting θ_1 with θ_3:

$$e_{[0:0:1]} = \frac{(\theta_2-\theta_3)(\theta_1-\theta_3)}{4\pi^2} \tag{II.2.211}$$

We can apply our formula again:

$$\begin{aligned} \mathrm{vol}(\mathbb{C}P^2) &= \frac{4\pi^2}{2}\left(\frac{\theta_1^2}{(\theta_2-\theta_1)(\theta_3-\theta_1)} + \frac{\theta_2^2}{(\theta_1-\theta_2)(\theta_3-\theta_2)} + \frac{\theta_3^2}{(\theta_1-\theta_3)(\theta_2-\theta_3)}\right) \\ &= 2\pi^2\left(\frac{\theta_1^2(\theta_2-\theta_3) - \theta_2^2(\theta_1-\theta_3) + \theta_2^3(\theta_1-\theta_2)}{(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_1-\theta_3)}\right) = 2\pi^2 \end{aligned} \tag{II.2.212}$$

Now we can describe the computation for the general case. Pick the base point

$$\mathscr{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{II.2.213}$$

its tangent space can be identified with $\mathfrak{u}(n+1)/\mathfrak{u}(n+1)_\xi$, and the equivalence classes have representatives

$$\begin{bmatrix} 0 & z_1 & z_2 & \dots & z_n \\ \bar{z}_1 & 0 & 0 & \dots & 0 \\ \bar{z}_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{z}_n & 0 & 0 & \dots & 0 \end{bmatrix} \tag{II.2.214}$$

The action of an element diag $[\theta_1, \dots, \theta_{n+1}]$ brings elements of this form to

$$\begin{bmatrix} 0 & z_1 e^{i(\theta_2-\theta_1)} & z_2 e^{i(\theta_3-\theta_1)} & \dots & z_n e^{i(\theta_{n+1}-\theta_1)} \\ \bar{z}_1 e^{-i(\theta_2-\theta_1)} & 0 & 0 & \dots & 0 \\ \bar{z}_2 e^{-i(\theta_3-\theta_1)} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{z}_n e^{-i(\theta_{n+1}-\theta_1)} & 0 & 0 & \dots & 0 \end{bmatrix} \tag{II.2.215}$$

Then the action on frames is again diagonal. This leads to the equivariant Euler class

$$e_{[1:0:\dots:0]} = \frac{(\theta_2-\theta_1)(\theta_3-\theta_1)\dots(\theta_{n+1}-\theta_1)}{(2\pi)^n} \tag{II.2.216}$$

To compute the other equivariant Euler classes, we just exchange θ_1 with θ_n, where n is the position of the 1 in the coordinates of the fixed point $[0 : \dots : 0 : 1 : 0 : \dots : 0]$. If we label the fixed points by x_i, $i = 1, \dots, n+1$, according to the position of the 1,we may write

$$e_i = \frac{\Pi_{j=1, j\neq i}^{n+1}(\theta_j - \theta_i)}{(2\pi)^n} \tag{II.2.217}$$

and applying theorem II.2.9.10 and taking the LCD yields

$$\mathrm{vol}(\mathbb{C}P^n) = \frac{(-2\pi)^n}{n!} \frac{\sum_{k=1}^{n+1}(-1)^{n+1-k}\theta_k^n(\Pi_{i,j=1,\ i<j,\ i,j\neq k}^{n+1}(\theta_j-\theta_i))}{\Pi_{i,j=1,\ i<j}^{n+1}(\theta_j-\theta_i)} \tag{II.2.218}$$

Look at the denominator. The term with coefficient θ_k^n, for any given k, is

$$(-1)^{k-1}\theta_k^n \Pi_{i,j=1,\ i<j,\ i,j\neq k}^{n+1}(\theta_j-\theta_i) \tag{II.2.219}$$

then the ratio simplifies to $(-1)^n$, and we get

$$\mathrm{vol}(\mathbb{C}P^n) = \frac{(-1)^{2n}}{n!}(2\pi)^n = \frac{(2\pi)^n}{n!} \tag{II.2.220}$$

Remark II.2.10.1. Taking $\omega' = \frac{\omega}{2}$ we obtain $\mathrm{vol}(\mathbb{C}P^n) = \frac{\pi^n}{n!}$, the volume according to the Fubini-Study metric.

II.2.11. Root decomposition and localization

In this section we will use the root decomposition of semisimple Lie groups to describe their generalized flag manifolds and compute their volume.

Before starting to introduce the necessary notions, we explain the spirit of the venture. As we saw in the previous section, it is in general not easy to provide an accessible description of the centralizer $C(S)$ of a torus. The data we need to extract is:

1. its Weyl group, in order to determine precisely the Weyl set;
2. its Lie algebra, in order to compute the equivariant Euler classes.

the second token of information is just local, and we'll see that also the Weyl group can be described in a Lie algebra setting. The bottom line is that a geometric description is not only more convolute than a local description, it's also more than what we need: in this section we analyze the problem from the local point of view, where semisimple Lie groups give their best.

The concept of *root* traces back essentially to the linear algebra construction of eigenvalues and eigenvectors; we want to generalize the idea of splitting a vector space into the sum of eigenspaces of a given operator to Lie algebras: as we will see, everything boils down to diagonalizing matrices, and that's why the preferred setting is that of *complex* Lie groups - diagonalizing over $\mathbb{C}$ is way easier.

Definition II.2.11.1. Let G be a semisimple Lie group, T a maximal torus of G, $\mathfrak{g}$, $\mathfrak{t}$ the respective Lie algebras. Then

(i) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is the *complexification* of $\mathfrak{g}$;

(ii) $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathfrak{g}_{\mathbb{C}}$ is the *Cartan subalgebra* of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{t}$.

In the following, we denote $\mathfrak{k} = \mathfrak{g}_{\mathbb{C}}$, $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$.

We have a family of skew self-adjoint operators on $\mathfrak{k}$:

$$\{\mathrm{ad}_h : \mathfrak{k} \to \mathfrak{k} : \ h \in \mathfrak{h}\} \tag{II.2.221}$$

each of them can be diagonalized, and the commutator on $\mathfrak{h}$ vanishes by construction, so that they can be simultaneously diagonalized. We obtain a set of eigenvectors E_α, and for each E_α an assignment $\alpha : \mathfrak{h} \to \mathbb{C}$ which sends $h \in \mathfrak{h}$ into the eigenvalue of ad_h for E_α.

Definition II.2.11.2. Let G be a semisimple Lie group, T a maximal torus of G, $\mathfrak{g}$, $\mathfrak{t}$ the respective Lie algebras.

We call $E_\alpha \in \mathfrak{k}$ a *root vector* of $\mathfrak{k}$ with respect to $\mathfrak{h}$ if

$$\mathrm{ad}_h(E_\alpha) = i\alpha(h)E_\alpha \ \forall h \in \mathfrak{h} \tag{II.2.222}$$

for some $\alpha \in \mathfrak{h}^*$. If $\alpha \neq 0$, we call it a *root*.

Remark II.2.11.3. (i) Consider the case $\alpha = 0$ the equation

$$[h, X] = \mathrm{ad}_h(X) = 0, \ \forall h \in \mathfrak{h} \tag{II.2.223}$$

characterizes the $X \in \mathfrak{h}$, so that we immediately obtain the related eigenspace; $\mathfrak{k}$ splits into a direct sum of vector spaces, an expression known as *Cartan decomposition*:

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \in R} V_\alpha \tag{II.2.224}$$

where R is the set of roots, and $V_\alpha = \mathrm{span}_{\mathbb{C}} E_\alpha$;

(ii) Write $\mathfrak{h}^* = \mathfrak{t}^* + i\mathfrak{t}^*$, then $R \subseteq \mathfrak{t}^*$. In fact, the fact that ad_H is a family of skew self-adjoint operators implies that the eigenvalues be real:

$$|\alpha(h)|^2 E_\alpha = \mathrm{ad}_h^* \mathrm{ad}_h(E_\alpha) = -\mathrm{ad}_h \mathrm{ad}_h(E_\alpha) = (\alpha(h))^2 E_\alpha \tag{II.2.225}$$

and the pairing is given by the Killing form $\langle\cdot,\cdot\rangle$, which respects the splitting $\mathfrak{t}^* + i\mathfrak{t}^*$.

(iii) We obtain another characterization of the set of roots:

$$\hat{R} = \{\alpha \in \mathfrak{t} : \langle \alpha, \cdot\rangle \in R\} \tag{II.2.226}$$

we won't make distinctions between $\hat{R}$ and R, choosing the most convenient expression in each setting;

(iv) We started from a real Lie algebra to get a complex one; one may wonder whether all complex semisimple Lie algebras arise in this way. Indeed, given a complex semisimple Lie algebra $\mathfrak{k}$ with the Cartan decomposition II.2.224, we can consider the real Lie algebra

$$\mathfrak{g} = \mathrm{span}_\mathbb{R}\{R\} + \sum_{\alpha\in R^+} \mathrm{span}_\mathbb{R}\{E_\alpha + E_{-\alpha} + i(E_\alpha - E_{-\alpha})\} \tag{II.2.227}$$

where $R^+ \subset R$ is constructed by considering the hyperplanes $W_\alpha,\ \alpha \in R$ orthogonal to each of the roots in R, and picking any vector h in its complement; then

$$R^+ = \{\alpha \in R : \ \alpha(h) > 0\} \tag{II.2.228}$$

We have $\mathrm{span}_\mathbb{C}\{R\} = \mathfrak{h}$, $\mathfrak{g}_\mathbb{C} = \mathfrak{k}$ (see [1, p. 8]); $\mathfrak{g}$ is called the *standard compact real form* of $\mathfrak{k}$.

The action of the Weyl group on the Lie algebra, which we use to compute the position of the fixed points, has a nice expression in terms of the root set:

Theorem II.2.11.4. *Let G be a compact, simply connected Lie group. Then G is semisimple, and the action of its Weyl group on the Lie algebra $\mathfrak{t}$ of a maximal torus is generated by the elements*

$$s_\alpha : \mathfrak{t} \to \mathfrak{t} : \ X \mapsto X - 2\frac{\langle X, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \tag{II.2.229}$$

Sketch of the proof. By the Whitehead Lemmas (see II.2.9.3), G is semisimple if and only if its first and second cohomology groups vanish. Simply connectedness immediately implies $H^1(G;\mathbb{R}) = 0$, while $H^2(G;\mathbb{R}) = 0$ follows from a theorem of Hopf (see [17]) which states that the rational homology ring of a closed Lie group is isomorphic to that of a product of odd-dimensional spheres.

The proof of the second statement can be found in [15, p. 335] □

Remark II.2.11.5. (i) The generators s_α have a geometric interpretation in terms of reflections through the hyperplane W_α orthogonal to α: it is immediate to check that $s_\alpha(\alpha) = -\alpha$, and $s_\alpha(X) = 0$ whenever $\langle X, \alpha\rangle = 0$.

(ii) The elements of

$$\mathfrak{t}_{reg} = \mathfrak{t} - \cup_{\alpha \in R} W_\alpha \tag{II.2.230}$$

are called *regular*. The connected components of $\mathfrak{t}_{reg}$ are the *Weyl chambers* of R.

The reason why regular elements are so important for us is that they are in the orbits corresponding to the flag manifold of G, G/T. Before stating a result in this direction, we want to extend these concepts to bring generalized flag manifolds in the picture.

We explain the idea, following [1]: given a semisimple generalized flag manifold $G/C(S)$, consider the related Lie algebras $\mathfrak{g}$ and $\mathfrak{s}$. After complexifying, we can find a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_\mathbb{C}$ which is also a Cartan subalgebra for $\mathfrak{s}_\mathbb{C}$, that is, the complexification of the Lie algebra of a common maximal torus.

$C(S)$ is itself a compact Lie group, and as such, its Lie algebra is *reductive*, the direct sum of an Abelian and a semisimple Lie algebra:

$$\mathfrak{s} = \mathfrak{a} + \mathfrak{l} \tag{II.2.231}$$

Decompose the semisimple component $\mathfrak{l}$ as $\mathfrak{h}' + \sum_{\alpha \in R_0} V_\alpha$. We can pick $\mathfrak{h}' \subseteq \mathfrak{h}$: the remainder is exactly the *center* of $\mathfrak{s}_\mathbb{C}$, which we identify with $\mathfrak{a}$. We have

$$\mathfrak{s}_\mathbb{C} = \mathfrak{h} + \sum_{\alpha \in R_0} V_\alpha \tag{II.2.232}$$

Explicitly, $\mathfrak{a}$ has the form

$$\mathfrak{a} = \{X \in \mathfrak{s}_\mathbb{C} : \ [X, \mathfrak{s}_\mathbb{C}] = 0\} \subseteq \mathfrak{h} \tag{II.2.233}$$

furthermore, observe that the quotient $\mathfrak{g}_\mathbb{C}/\mathfrak{s}_\mathbb{C}$ can be expressed as

$$\mathfrak{g}_\mathbb{C}/\mathfrak{s}_\mathbb{C} = \sum_{\alpha \in R - R_0} V_\alpha \tag{II.2.234}$$

these are the roots which identify the generalized flag manifold on the Lie algebra level, that is, its tangent space at $eC(S)$ - here in the complexified version.

We excluded the information coming from $C(S)$ from the Cartan decomposition of $\mathfrak{g}_\mathbb{C}$; we want to do something similar for the Weyl group and the Weyl chambers. From the relation

$$[\mathfrak{h}, V_\alpha] = i\alpha(\mathfrak{h}) V_\alpha \ \forall \alpha \in R \tag{II.2.235}$$

together with $[\mathfrak{a}, V_\alpha] = 0 \ \forall \alpha \in R_0$, which holds by construction, we obtain $\mathfrak{a} \subseteq \ker \alpha$ for all the roots in R_0. Since $\alpha \in \mathfrak{t}^*$, the meaningful information is contained in $\tau = \mathfrak{t} \cap \mathfrak{a}$: by considering $R - R_0$ we pick the roots excluded from the Cartan decomposition of $\mathfrak{s}_\mathbb{C}$, and now we consider the portion of domain that the roots in R_0 do not see. This leads to the following definition:

Definition II.2.11.6. Let $G/C(S)$ be a semisimple generalized flag manifold. Following our notation from above, we call $\alpha' \neq 0$ a $T-$*root* if $\alpha' = \alpha|_\tau$ for $\alpha \in R$.

Observe that by construction α restricts to a $T-$root only if $\alpha \in R - R_0$: thus, $T-$roots are obtained by considering the roots which are left out from the quotient restricted to the portion of domain left out from from the roots in the quotient. We obtain a restricted version of the Weyl group and of the Weyl chambers:

Definition II.2.11.7. Let R' be the set of $T-$roots of $G/C(S)$. Then

(i) The elements of

$$\tau_{reg} = \tau - \cup_{\alpha \in R'} W_\alpha \tag{II.2.236}$$

where W_α is the hyperplane in τ orthogonal to the $T-$root α, are called $T-$*regular*. The connected components of τ_{reg} are the $T-$ *Weyl chambers* of R'.

(ii) The group generated by

$$s_\alpha : \tau \to \tau : \; X \mapsto X - 2\frac{\langle X, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha, \; \alpha \in R' \tag{II.2.237}$$

is called the $T-$ *Weyl group* of $G/C(S)$.

Remark II.2.11.8. (i) Observe that for the case of a flag manifold G/T, the definitions above collapse to the usual ones of roots, Weyl chambers and Weyl group;

(ii) If we regard τ as a subset of $\mathfrak{t}$, the reflections

$$s_\alpha : \tau \to \tau : \; X \mapsto X - 2\frac{\langle X, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha, \; \alpha \in R' \tag{II.2.238}$$

fix the set $V_0 = \cup_{\alpha \in R_0} V_\alpha$, and vice versa, a root inducing a reflection which fixes V_0 restricts to a $T-$root.
We can then regard the $T-$Weyl group as the stabilizer of V_0 in the Weyl group $W(G)$ of G:

$$W_T = \{r \in W(G) : \; r|_{V_0} = \mathrm{id}\} \tag{II.2.239}$$

We have a result indicating exactly which elements of the Lie algebra generate a given generalized flag manifold:

Proposition II.2.11.9. *Let $G/C(S)$ be a semisimple generalized flag manifold. Then $G/C(S) \simeq \mathrm{ad}_G\, t$ if and only if $t \in \tau_{reg}$.*

Proof. See [1]. □

A further advantage of the Cartan decomposition is that it diagonalizes the adjoint action of $\mathfrak{t}$:

Lemma II.2.11.10. *Let $\mathfrak{k}$ be a complex semisimple Lie algebra with Cartan decomposition*

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \in R} V_\alpha \tag{II.2.240}$$

and the related standard compact real form

$$\mathfrak{g} = \mathfrak{t} + \mathfrak{m} \tag{II.2.241}$$

where $\mathfrak{t} = \text{span}_{\mathbb{R}}\{R\}$, $\mathfrak{m} = \sum_{\alpha \in R^+} \text{span}_{\mathbb{R}}\{E_\alpha + E_{-\alpha} + i(E_\alpha - E_{-\alpha})\}$. *Then*

$$\det[\mathfrak{t}, \cdot] = \prod_{\alpha \in R^+} \alpha \tag{II.2.242}$$

for $[\mathfrak{t}, \cdot] : \mathfrak{m} \to \mathfrak{m}$.

Proof. We know $[\mathfrak{h}, E_\alpha] = i\alpha(h)E_\alpha \ \forall h \in \mathfrak{h}$, and $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. Then

$$\begin{aligned}[\mathfrak{t}, E_\alpha + E_{-\alpha} + i(E_\alpha - E_{-\alpha})] &= [\mathfrak{t}, E_\alpha(1+i)] + [\mathfrak{t}, E_{-\alpha}(1-i)] \\ &= (1-i)i\alpha E_\alpha + (1+i)\alpha E_{-\alpha} \\ &= \alpha(E_\alpha + E_{-\alpha} + i(E_\alpha - E_{-\alpha})\end{aligned} \tag{II.2.243}$$

the claim follows. □

Remark II.2.11.11. Starting from a semisimple, real Lie algebra, passing to the complexification and then considering the standard real form yields a real Lie algebra which is isomorphic to the one we began with (see [24, p. 55]).

Recall that the complexified tangent space of a generalized flag manifold $G/C(S)$ takes the form

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{s}_{\mathbb{C}} = \sum_{\alpha \in R_0 - R} V_\alpha \tag{II.2.244}$$

with R_0 given by the roots generating the Lie algebra of $C(S)$. Passing to the compact real forms yields an expression for the (non-complexified) tangent space

$$\mathfrak{g}/\mathfrak{s} = \sum_{\alpha \in (R - R_0) \cap R^+} \text{span}_{\mathbb{R}}\{E_\alpha + E_{-\alpha}, +i(E_\alpha - E_{-\alpha})\} \tag{II.2.245}$$

Denote $R' = (R - R_0) \cap R^+$. We can then simplify further Theorem II.2.9.10:

Theorem II.2.11.12. *Let* $(M, \omega) \simeq \text{Ad}_G\, x_0$ *be a semisimple, orientable generalized flag manifold of dimension* $2n$ *together with its canonical symplectic form,* $\mathcal{F}(M)$ *its Weyl set. Then*

$$\text{vol}_\omega(M) = \frac{(-2\pi)^n}{n!} \sum_{g_i \in \mathcal{F}(M)} g_i \cdot \frac{x_0^n}{\prod_{\alpha \in R'} \alpha} \tag{II.2.246}$$

where $x_i = g_i \cdot x_0$, R *is the root space of* $\mathfrak{g}_{\mathbb{C}}$, R_0 *the roots generating* $\mathfrak{s}_{\mathbb{C}}$, $R' = (R - R_0) \cap R^+$.

Proof. Let us clarify the notation. We denoted

$$g_i \cdot \frac{x_0^n}{\prod_{\alpha \in R'} \alpha} = \frac{(g \cdot x_0)^n}{\prod_{\alpha \in R'} \text{Ad}_{g_i}^{-1} \alpha} \tag{II.2.247}$$

Now, we know from II.2.9.10

$$\text{vol}_\omega(M) = \frac{(-2\pi)^n}{n!} \sum_{g_i \in \mathcal{F}(M)} \frac{x_i^n}{\det([\text{Ad}_{g_i} \mathfrak{t}, \cdot])} \tag{II.2.248}$$

then we need to show

$$\det([\text{Ad}_{g_i} \mathfrak{t}, \cdot]) = \prod_{\alpha \in R'} \text{Ad}_{g_i}^{-1} \alpha \tag{II.2.249}$$

for $g_i \in \mathcal{F}(M)$. For $g_i = e$, we can apply II.2.11.10 and restrict $[\mathfrak{t}, \cdot]$ to $\mathfrak{g}/\mathfrak{s}$: this yields exactly $\prod_{\alpha \in R'} \alpha$. For $g_i \neq e$, observe

$$[\mathrm{Ad}_{g_i} t, E_\alpha] = \alpha(\mathrm{Ad}_{g_i} t) E_\alpha \tag{II.2.250}$$

and writing $\alpha = \langle \alpha, \cdot \rangle$ we get

$$\alpha(\mathrm{Ad}_{g_i} t) = \langle \alpha, \mathrm{Ad}_{g_i} t \rangle = \langle \mathrm{Ad}_{g_i}^{-1} \alpha, t \rangle = (\mathrm{Ad}_{g_i}^{-1} \alpha)(t) \tag{II.2.251}$$

by invariance of the Killing form. The thesis follows. □

II.2.9.12 is generalized accordingly:

Corollary II.2.11.13. *Let (M, ω) be a simply connected, homogeneous $2n-$dimensional Kähler manifold on which the semisimple isometry group G acts in Hamiltonian fashion, $\varphi : M \to \mathfrak{g}^*$ a moment map for the action, $x \in M$.*

Then

$$\mathrm{vol}_\omega(M) = \frac{(-2\pi)^n}{n!} \sum_{g_i \in \mathcal{F}(\varphi(M))} g_i \cdot \frac{x_0^n}{\prod_{\alpha \in R'} \alpha} \tag{II.2.252}$$

where $x_i = g_i \cdot x_0$, R is the root space of $\mathfrak{g}_\mathbb{C}$, R_0 the roots generating the complexified Lie algebra $\mathfrak{s}_\mathbb{C}$ of the stabilizer of $\varphi(x)$, $R' = (R - R_0) \cap R^+$.

Proof. See the proof of II.2.9.12. □

Example II.2.11.14. We compute the volume of some generalized flag manifolds of $SU(n)$. If we pick the maximal torus given by diagonal matrices in $SU(n)$, the related Lie algebra $\mathfrak{t}$ is given by traceless matrices; since every generalized flag manifold is realizable as the orbit of an element in $\mathfrak{t}$, we restrict our attention to these elements. We can represent any $t \in \mathfrak{t}$, choosing a suitable coordinate system, as a matrix

$$t = \mathrm{diag}(a_1, \ldots, a_1, a_2, \ldots, a_2, a_3, \ldots, a_{m-1}, a_m, \ldots, a_m), \;\; a_i \neq a_j \text{ for } i \neq j \tag{II.2.253}$$

each of the a_i's having first occurrence in position r_i.

The stabilizer of the orbit is a centralizer $C(S)$, with Lie algebra $\mathfrak{s}$. The roots orthogonal to t satisfy

$$0 = \langle e_{ij}, t \rangle = t_i - t_j \tag{II.2.254}$$

Notice that the actual value of the a_i's does not play any role. The orthogonal set is

$$R_0 = \{e_{ij} : \; \{i, j\} \subseteq \{r_k, \ldots, r_{k+1} - 1\} \text{ for a } k = 1, \ldots, l\} \tag{II.2.255}$$

and the roots left out are those given by vectors e_{ij} whose summands belong to different "clusters" in t.

Before we begin, recall (see II.2.8.4) that we can relate the adjoint orbit to a quotient $SU(n)/C(S)$ by exponentiating the element whose orbit we're considering, and setting

$$S = \overline{\{e^{\mathbb{R}t}\}} \tag{II.2.256}$$

then $\mathrm{Ad}_{SU(n)} t \simeq SU(n)/C(S)$. The isomorphism holds for any semisimple group G.

For $n = 2$ there are only two possible clusters, with representatives

$$t_1 = \mathrm{diag}(1, -1),\ t_2 = \mathrm{diag}(0, 0) \tag{II.2.257}$$

Let's work on t_1. All roots (e_{12}, e_{21}) are orthogonal: its orbit expresses a flag manifold, and the fixed points are obtained by considering the action of the Weyl group. In this case we just need to consider

$$s_{e_{21}}(t_1) = t_1 - 2\frac{\langle(-1,1),(1,-1)\rangle}{\langle(-1,1),(-1,1)\rangle}(-1,1) = (-1,1) \tag{II.2.258}$$

the equivariant Euler class at t_1 can be computed with the help of II.2.11.12, and it's just $\theta_2 - \theta_1$; the equivariant Euler class at the other fixed point can be computed by applying the same permutation we used to get there, so that it equals $\theta_1 - \theta_2$. The volume is

$$-2\pi\left(\frac{\theta_1}{\theta_2 - \theta_1} + \frac{\theta_2}{\theta_1 - \theta_2}\right) = 2\pi \tag{II.2.259}$$

This was just $\mathbb{C}P^1$!

For $n = 3$ we have nontrivial representatives

$$t_1 = \mathrm{diag}(1, -1, 0),\ t_2 = \mathrm{diag}(1, 1, -2) \tag{II.2.260}$$

there aren't roots orthogonal to t_1, whose orbit realizes the flag manifold M_1. The fixed points are computed considering (combinations of) reflections across the planes

$$\begin{cases} E_1 = \{(x, x, -2x)\} \\ E_2 = \{(x, -2x, x)\} \\ E_3 = \{(-2x, x, x)\} \end{cases} \tag{II.2.261}$$

identified by e_{12}, e_{13}, e_{23}. Call the reflections r_1, r_2 and r_3, we have

$$\begin{cases} r_1(t_1) = (-1, 1, 0) = a_1 \\ r_2(t_1) = (0, -1, 1) = a_2 \\ r_3(t_1) = (1, 0, -1) = a_3 \end{cases} \tag{II.2.262}$$

i.e. r_i exchanges the i^{th} and $(i+1)^{th}$ term. Then there are other two points reachable:

$$\begin{cases} r_2 r_1(t_1) = (-1, 0, 1) = a_4 \\ r_2 r_2(t_1) = (0, 1, -1) = a_5 \end{cases} \tag{II.2.263}$$

These are the fixed points of the action. $SU(3)$ has dimension 8 and the maximal torus has dimension 2, so that we obtain a manifold of dimension 6; the equivariant Euler class $e(t_1)$ at t_1 is the polynomial $(\theta_2 - \theta_1)(\theta_3 - \theta_1)(\theta_3 - \theta_2)$, again according to II.2.11.12, and we can compute the Euler class at a_i by considering the adjoint action of the elements of $\mathcal{F}(M)$ on the roots. We obtain

$$\begin{cases} e(a_1) = (\theta_1 - \theta_2)(\theta_3 - \theta_2)(\theta_3 - \theta_1) \\ e(a_2) = (\theta_2 - \theta_3)(\theta_1 - \theta_3)(\theta_1 - \theta_2) \\ e(a_3) = (\theta_3 - \theta_1)(\theta_2 - \theta_1)(\theta_2 - \theta_3) \\ e(a_4) = (\theta_1 - \theta_3)(\theta_2 - \theta_3)(\theta_2 - \theta_1) \\ e(a_5) = (\theta_3 - \theta_2)(\theta_1 - \theta_2)(\theta_1 - \theta_3) \end{cases} \tag{II.2.264}$$

that is

$$e(t_1) = e(a_5) = e(a_4) = -e(a_3) = -e(a_2) = -e(a_1) \tag{II.2.265}$$

Compute the volume of the flag manifold M_1: we identify via the moment map the element $\mathrm{diag}(a, b, c) \in \mathfrak{t}$ with the polynomial $a\theta_1 + b\theta_2 + c\theta_3$, so that the fixed points have image

$$\begin{cases} t_1 \mapsto \theta_1 - \theta_2 \\ a_1 \mapsto -\theta_1 + \theta_2 \\ a_2 \mapsto -\theta_2 + \theta_3 \\ a_3 \mapsto \theta_1 - \theta_3 \\ a_4 \mapsto -\theta_1 + \theta_3 \\ a_5 \mapsto \theta_2 - \theta_3 \end{cases} \tag{II.2.266}$$

We need to raise these polynomials to the $6/2 = 3$. Finally:

$$\begin{aligned} \mathrm{vol}(M_1) &= \frac{(2\pi)^3/3!}{(\theta_2 - \theta_1)(\theta_3 - \theta_1)(\theta_3 - \theta_2)}(2(\theta_1 - \theta_2)^3 + 2(\theta_2 - \theta_3)^3 + 2(\theta_3 - \theta_1)^3) \\ &= \frac{(2\pi)^3}{(\theta_2 - \theta_1)(\theta_3 - \theta_1)(\theta_3 - \theta_2)}(\theta_1\theta_2^2 - \theta_1^2\theta_2 + \theta_2\theta_3^2 - \theta_2^2\theta_3 + \theta_3\theta_1^2 - \theta_3^2\theta_1) \\ &= (2\pi)^3 \end{aligned} \tag{II.2.267}$$

For $n = 4$ we have clusters

$$t_1 = \mathrm{diag}(1, -1, 2, -2),\ t_2 = \mathrm{diag}(1, -1, 0, 0),\ t_3 = \mathrm{diag}(1, 1, 1, -3),\ t_4 = \mathrm{diag}(1, 1, -1, -1) \tag{II.2.268}$$

The orbit of t_1 realizes the flag manifold and t_3 is a complex projective space. As a last example, we compute the volume of the "intermediate" generalized flag manifold M_2 realized by t_2: the only orthogonal root is e_{34}, so that the equivariant Euler class $e(t_2)$ of the point is

$$(\theta_2 - \theta_1)(\theta_3 - \theta_1)(\theta_3 - \theta_2)(\theta_4 - \theta_1)(\theta_4 - \theta_2) \tag{II.2.269}$$

while t_2 gets sent to the polynomial $\theta_1 - \theta_2$. Now we need to apply all the meaningful permutations and sum everything up:

t_2	a_0	a_1	a_2	a_3	a_4
$\theta_1 - \theta_2$	$\theta_2 - \theta_1$	$\theta_1 - \theta_3$	$\theta_3 - \theta_1$	$\theta_1 - \theta_4$	$\theta_4 - \theta_1$

a_5	a_6	a_7	a_8	a_9	a_{10}
$\theta_2 - \theta_3$	$\theta_3 - \theta_2$	$\theta_2 - \theta_4$	$\theta_4 - \theta_2$	$\theta_3 - \theta_4$	$\theta_4 - \theta_3$

Table II.1.: Polynomials relative to the fixed points

We get the corresponding equivariant Euler classes by permuting the equivariant Euler class of t_2:

$e(t_2)$	$e(a_0)$
$(\theta_2-\theta_1)(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_4-\theta_1)(\theta_4-\theta_2)$	$(\theta_1-\theta_2)(\theta_3-\theta_2)(\theta_3-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_1)$
$e(a_1)$	$e(a_2)$
$(\theta_3-\theta_1)(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_4-\theta_1)(\theta_4-\theta_3)$	$(\theta_1-\theta_3)(\theta_2-\theta_3)(\theta_2-\theta_1)(\theta_4-\theta_3)(\theta_4-\theta_1)$
$e(a_3)$	$e(a_4)$
$(\theta_4-\theta_1)(\theta_3-\theta_1)(\theta_3-\theta_4)(\theta_2-\theta_1)(\theta_2-\theta_4)$	$(\theta_1-\theta_4)(\theta_3-\theta_4)(\theta_3-\theta_1)(\theta_2-\theta_4)(\theta_2-\theta_1)$
$e(a_5)$	$e(a_6)$
$(\theta_3-\theta_1)(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_4-\theta_1)(\theta_4-\theta_3)$	$(\theta_3-\theta_2)(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_4-\theta_2)(\theta_4-\theta_3)$
$e(a_7)$	$e(a_8)$
$(\theta_2-\theta_4)(\theta_3-\theta_4)(\theta_3-\theta_2)(\theta_1-\theta_4)(\theta_1-\theta_2)$	$(\theta_1-\theta_4)(\theta_3-\theta_4)(\theta_3-\theta_1)(\theta_2-\theta_4)(\theta_2-\theta_1)$
$e(a_9)$	$e(a_{10})$
$(\theta_4-\theta_3)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_2-\theta_3)(\theta_2-\theta_4)$	$(\theta_4-\theta_3)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_1-\theta_3)(\theta_1-\theta_4)$

Table II.2.: Equivariant Euler classes of the fixed points

Now to localize we need to compute

$$\frac{(-2\pi)^5}{5!}\left(\frac{t_2^5}{e(t_2)}+\sum_{i=0}^{10}\frac{a_i^5}{e(a_i)}\right) \tag{II.2.270}$$

The computation is easily carried out with the help of e.g. Mathematica, and yields

$$\mathrm{vol}(M_2)=\frac{(-2\pi)^5}{5!}(-20)=\frac{16}{3}\pi^5 \tag{II.2.271}$$

M_2 corresponds to the coset space $SU(4)/C(S)$ with

$$S=\left\{\begin{bmatrix} e^{it} & 0 & 0 & 0\\ 0 & e^{-it} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\ t\in\mathbb{R}\right\} \tag{II.2.272}$$

Example II.2.11.15. As a further example, we consider the *symplectic group* $Sp(n)$ (see [25] for details). It is constructed as follows: if

$$M(2n;\mathbb{C})\ni J=\begin{bmatrix} 0 & -I_n\\ I_n & 0\end{bmatrix} \tag{II.2.273}$$

we first define

$$Sp(n;\mathbb{C})=\{A\in GL(2n,\mathbb{C}):\ A^tJA=J\} \tag{II.2.274}$$

and subsequently

$$Sp(n)=U(2n)\cap Sp(n;\mathbb{C}) \tag{II.2.275}$$

The Lie algebra $\mathfrak{sp}(n)$ has the following characterization:

$$\mathfrak{sp}(n)=\{X\in\mathfrak{gl}(2n;\mathbb{C}):\ JX^tJ=X=-X^*\} \tag{II.2.276}$$

If A, $B \in \mathfrak{gl}(2n;\mathbb{C})$, a typical element of $\mathfrak{sp}(n)$ has the form

$$X = \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}, \text{ with } A^* = -A,\ B^t = B \tag{II.2.277}$$

and a maximal torus T is given by

$$T = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n}) :\ \theta_i \in \mathbb{R}\} \tag{II.2.278}$$

$\mathfrak{sp}(n)$ is a simple Lie algebra, with roots

$$\begin{aligned} &\{\varepsilon_i - \varepsilon_j,\ i \neq j \in \{1, \ldots, n\}\} \\ &\{\pm(\varepsilon_i + \varepsilon_j),\ i, j \in \{1, \ldots, n\}\} \end{aligned} \tag{II.2.279}$$

and relative root vectors

$$\{E_{i,j} - E_{j+n,i+n},\ E_{i,j+n} + E_{j,i+n},\ E_{i+n,j} + E_{j+n,i},\ i, j \in \{1, \ldots, n\}\} \tag{II.2.280}$$

One then checks that the orbit of an element in $\mathfrak{sp}(n)$ via the Weyl group is given by all possible permutations and sign changes of its coordinates.

Set $n = 2$. We obtain a 10−dimensional vector space $\mathfrak{sp}(2)$ with a maximal torus of rank 2. Let us compute, as an example the volume of the adjoint orbit M of

$$t = \mathrm{diag}(1, -1) \tag{II.2.281}$$

this vector is orthogonal to the roots $\pm(\epsilon_1 + \epsilon_2)$ and gets sent to the polynomial $\theta_1 - \theta_2$; applying the Weyl group we obtain the set of fixed points:

t	a_0	a_1	a_2
$\theta_1 - \theta_2$	$\theta_2 - \theta_1$	$\theta_1 + \theta_2$	$-\theta_1 - \theta_2$

The generalized flag manifold has dimension 6, and the equivariant Euler class at t is the polynomial $4\theta_1\theta_2(\theta_1 - \theta_2)$, given by the product of the non-orthogonal roots $2\varepsilon_1,\ 2\varepsilon_2, \epsilon_1 - \epsilon_2$; we obtain the equivariant Euler classes at the other fixed point again by permutations and sign changes:

$e(t)$	$e(a_0)$	$e(a_1)$	$e(a_2)$
$4\theta_1\theta_2(\theta_1 - \theta_2)$	$4\theta_2\theta_1(\theta_2 - \theta_1)$	$-4\theta_1\theta_2(\theta_1 + \theta_2)$	$4\theta_1\theta_2(\theta_1 + \theta_2)$

We don't need Mathematica's help to compute the volume:

$$\begin{aligned} \mathrm{vol}(M) &= \frac{(-2\pi)^3}{3!} \frac{1}{4\theta_1\theta_2(\theta_1^2 - \theta_2^2)} (2(\theta_1 - \theta_2)^3(\theta_1 + \theta_2) - 2(\theta_1 + \theta_2)^3(\theta_1 - \theta_2)) \\ &= \frac{(-2\pi)^3}{3!} \frac{-4\theta_1\theta_2(\theta_1^2 - \theta_2^2)}{4\theta_1\theta_2(\theta_1^2 - \theta_2^2)} = \frac{4}{3}\pi^3 \end{aligned} \tag{II.2.282}$$

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Erklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig verfasst zu haben und keine anderen als die in der Arbeit angegebenen Quellen und Hilfsmittel benutzt zu haben.

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