

# A CONTACT CYLINDER WITH STANDARD BOUNDARY AND A BOUNDED REEB ORBIT BUT NO PERIODIC REEB ORBIT

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ABSTRACT. We construct an example of a contact form on  $\mathbb{R}^5$  that induces the standard contact structure and coincides with the standard contact form outside a compact subset that satisfies the following property:

Its associated Reeb flow has a compact invariant set but no periodic orbit.

## 1. INTRODUCTION

The Reeb flow on  $\mathbb{R}^5$  with respect to the standard contact form  $\lambda_0 = dz + \sum_{i=1}^2 x_i dy_i - y_i dx_i$  is given by  $t \mapsto (\bar{z} + t, \bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2)$  and hence has no nonempty compact invariant set. We prove the following theorem by giving an explicit example:

**1.1. Theorem.** *There exists a smooth map  $f : \mathbb{R}^5 \rightarrow (0, \infty)$  such that  $f - 1$  has compact support and  $\lambda = f\lambda_0$  is a contact form whose Reeb flow has a compact invariant set but no periodic orbit.*

The one-form  $\lambda$  will be constructed by deforming the standard contact form in a tubular neighborhood of the Clifford torus  $\mathbb{T}^2 = \{(z, x_1, x_2, x_3, x_4) \mid z = 0, x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\}$  in such a way that the  $z$ -component of the Reeb vector field  $R$  associated to the new contact form remains positive on  $\mathbb{R}^5 \setminus \mathbb{T}^2$ . Moreover, on  $\mathbb{T}^2$ , the Reeb flow is an irrational linear flow.

**1.2. Remark.**

- (1) In [1] we constructed an example of a geodesic flow with similar dynamics but different boundary conditions.
- (2) In [2, Theorem 2], Hofer und Eliashberg prove the following: If a contact form  $\beta$  on  $\mathbb{R}^3$  coincides with  $dz + xdy$  outside a compact set and if the associated Reeb flow has no periodic orbit then there exists a diffeomorphism  $\Phi$  such that  $\Phi^*\beta = dz + xdy$ .

Theorem 1.1 provides a counterexample to the generalization of this statement to higher dimensions.

- (3) Helmut Hofer, [4], conjectured that in the situation of Theorem 1.1 the existence of an orbit  $\gamma(t) = (z(t), q(t))$  with the following properties

$$\lim_{t \rightarrow -\infty} z(t) = -\infty, \quad \limsup_{t \rightarrow \infty} z(t) < \infty \quad (\text{T})$$

implies the existence of a periodic orbit.

This can be disproved in the more explicit situation of Theorem 2.1: Choose  $z^- < 0$  such that the contact form is the standard one on  $(-\infty, z^-] \times \mathbb{R}^4$  and a sequence of points  $p_n = (z_n, q_n)$ ,  $n \geq 1$ , converging to a point

on the Clifford torus  $\mathbb{T}^2$  and such that  $z_n < 0$ . By property (T2) we find a positive lower bound for the  $z$ -component of the Reeb vectorfield on  $\{(z, q) \mid z \leq z_n\}$ ,  $n \geq 1$ . Hence the orbits through  $p_n$ ,  $n \geq 1$ , intersect  $z^- \times \mathbb{R}^4$  and the points of intersection form a compact set by property (T1). Now, by continuity every limit point of this set yields an orbit with property (T). But conditions (T2) and (T3) ensure that there exists no periodic orbit in this example.

## 2. THE EXAMPLE

We consider the following coordinate system:

$$F : \mathbb{R} \times ]0, \infty[ \times ]0, \pi/2[ \times \mathbb{R}^2 \longrightarrow \mathbb{R}^5$$

$$(z, r, \psi, \varphi, \theta) \longmapsto \begin{pmatrix} r \cos \psi \cos \varphi \\ r \cos \psi \sin \varphi \\ r \sin \psi \cos \theta \\ r \sin \psi \sin \theta \\ z \end{pmatrix}.$$

These coordinates are well-suited for the construction since the torus  $\mathbb{T}^2$  is given in these coordinates by the set  $\{F(z, r, \psi, \varphi, \theta) \mid (z, r, \psi) = (0, 1, \frac{\pi}{4})\}$ . We will denote  $p_{\mathbb{T}^2} = (0, 1, \frac{\pi}{4})$  and deform the contact form in a tubular neighborhood of this torus that is of the form  $F((z^-, z^+) \times (r^-, r^+) \times (\psi^-, \psi^+) \times \mathbb{R}^2)$  and can be chosen arbitrarily small. We compute that the standard contact form in these coordinates is given by  $\alpha_0 = dz + r^2 \cos^2 \psi d\varphi + r^2 \sin^2 \psi d\theta$ . The following ansatz turns out to be successful:

$$\alpha = f dz + g d\varphi + h d\theta$$

where  $f, g, h \in C^\infty(D, \mathbb{R}_{>0})$  for  $D = \mathbb{R} \times (0, \infty) \times (0, \pi)$  depend only on the variables  $z, r$  and  $\psi$ . Accordingly  $\alpha$  is a contact form if  $\alpha \wedge (d\alpha)^2 \neq 0$ , i.e.

$$f \left( \frac{\partial g}{\partial \psi} \frac{\partial h}{\partial r} - \frac{\partial g}{\partial r} \frac{\partial h}{\partial \psi} \right) + \frac{\partial f}{\partial r} \left( g \frac{\partial h}{\partial \psi} - h \frac{\partial g}{\partial \psi} \right) + \frac{\partial f}{\partial \psi} \left( h \frac{\partial g}{\partial r} - g \frac{\partial h}{\partial r} \right) \neq 0.$$

Hence for the standard contact form  $\alpha_0$  the contact condition is fulfilled if

$$-4r^3 \cos \psi \sin \psi < 0$$

since  $\psi \in (0, \frac{\pi}{2})$ . Thus in order to construct a contact form that induces the same orientation,  $f, g$  and  $h$  need to satisfy the following condition

$$f \left( \frac{\partial g}{\partial \psi} \frac{\partial h}{\partial r} - \frac{\partial g}{\partial r} \frac{\partial h}{\partial \psi} \right) + \frac{\partial f}{\partial r} \left( g \frac{\partial h}{\partial \psi} - h \frac{\partial g}{\partial \psi} \right) + \frac{\partial f}{\partial \psi} \left( h \frac{\partial g}{\partial r} - g \frac{\partial h}{\partial r} \right) < 0. \quad (\text{C})$$

Our main theorem 1.1 follows by Gray stability (cf. [3, Theorem 2.2.2]) from the following:

**2.1. Theorem.** *There exist an open neighborhood  $U$  of  $p_{\mathbb{T}^2}$  with compact closure  $\bar{U} \subseteq D$  and  $f, g, h \in C^\infty(D, \mathbb{R}_{>0})$  such that  $\alpha = f dz + g d\varphi + h d\theta$  meets the following conditions:*

- (T1) *There exists a smooth family of contact forms  $\alpha_t$ ,  $t \in [0, 1]$ , such that  $\alpha_0$  is the standard contact form,  $\alpha_1 = \alpha$  and each  $\alpha_t$  coincides outside  $U$  with  $\alpha_0$ .*
- (T2) *The  $z$ -component of the Reeb vector field  $R$  of  $\alpha$  is nonnegative and vanishes exactly on the Clifford torus  $\mathbb{T}^2$ .*

(T3) On the Clifford torus the Reeb vector field is given by  $R = \partial_\varphi + \frac{1}{\chi}\partial_\theta$  for some  $\chi \in \mathbb{R} \setminus \mathbb{Q}$ .

We start with the following technical:

**2.2. Lemma.** *There exist open neighborhoods  $V \subseteq U \subseteq D$  of  $p_{\mathbb{T}^2}$  with compact closure  $\bar{V} \subseteq U$ ,  $\bar{U} \subseteq D$ ,  $\chi \in \mathbb{R} \setminus \mathbb{Q}$  and functions  $f$ ,  $g$  and  $h \in C^\infty(D, \mathbb{R}_{>0})$  with the following properties:*

$$(F_B) \quad f|_{D \setminus U} \equiv 1$$

$$(F_r) \quad \frac{\partial f}{\partial r}|_V < 0$$

$$(F_C) \quad \left(2f - r \frac{\partial f}{\partial r}\right)|_{D \setminus V} > 0$$

$$(F_\psi) \quad \frac{\partial f}{\partial \psi}|_V \equiv 0$$

$$(G_B) \quad g(z, r, \psi)|_{D \setminus V} = r^2 \cos^2 \psi \quad \text{and} \quad g(p_{\mathbb{T}^2}) = \cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$(G_r) \quad \frac{\partial g}{\partial r}|_{D \setminus \{p_{\mathbb{T}^2}\}} > 0 \quad \text{and} \quad \frac{\partial g}{\partial r}(p_{\mathbb{T}^2}) = 0$$

$$(G_z) \quad \frac{\partial g}{\partial z}(p_{\mathbb{T}^2}) = 0$$

$$(G_\psi) \quad \frac{\partial g}{\partial \psi} < 0 \quad \text{and} \quad \frac{\partial g}{\partial \psi}(p_{\mathbb{T}^2}) = -2 \cos\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) = -1$$

$$(H_B) \quad h(z, r, \psi)|_{D \setminus V} = r^2 \sin^2 \psi \quad \text{and} \quad h(p_{\mathbb{T}^2}) = \chi \sin^2\left(\frac{\pi}{4}\right) = \frac{\chi}{2}$$

$$(H_r) \quad \frac{\partial h}{\partial r}|_{D \setminus \{p_{\mathbb{T}^2}\}} > 0 \quad \text{and} \quad \frac{\partial h}{\partial r}(p_{\mathbb{T}^2}) = 0$$

$$(H_z) \quad \frac{\partial h}{\partial z}(p_{\mathbb{T}^2}) = 0$$

$$(H_\psi) \quad \frac{\partial h}{\partial \psi} > 0 \quad \text{and} \quad \frac{\partial h}{\partial \psi}(p_{\mathbb{T}^2}) = 2\chi \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) = \chi$$

These functions can be constructed explicitly. The exact proof will be given in the appendix.

**2.3. Remark.** We will need the interplay of the properties of all three functions for the proof of each condition (T1)-(T3). But one should consider  $g$  and  $h$  responsible for the dynamics of the Reeb flow, and  $f$  chosen afterwards to make sure that  $\alpha$  is of contact type.

*Proof of theorem 2.1.*

Step 1:  $\alpha_t$ ,  $t \in [0, 1]$ , is a family of contact forms as required in (T1):

We denote  $\alpha_t =: f_t dz + g_t d\varphi + h_t d\theta$ . Then  $f_t$ ,  $g_t$  and  $h_t$  are convex combinations of  $f$ ,  $g$  and  $h$  and the coefficient functions of the standard case. It is easy to see that  $f_t$ ,  $g_t$  and  $h_t$  meet the first part of every condition of Lemma 2.2 for all  $t \in (0, 1)$ .

Now we show that this implies condition (C) for  $t \in (0, 1]$ : On  $V$  we reduce condition (C) by applying property  $(F_\psi)$  to the following form

$$\underbrace{f_t}_{>0} \left( \underbrace{\frac{\partial g_t}{\partial \psi}}_{<0} \underbrace{\frac{\partial h_t}{\partial r}}_{\geq 0} - \underbrace{\frac{\partial g_t}{\partial r}}_{\geq 0} \underbrace{\frac{\partial h_t}{\partial \psi}}_{>0} \right) + \underbrace{\frac{\partial f_t}{\partial r}}_{<0} \underbrace{\left( g_t \frac{\partial h_t}{\partial \psi} - h_t \frac{\partial g_t}{\partial \psi} \right)}_{>0} < 0.$$

On the complement  $D \setminus V$  we use that  $g$  and  $h$  are standard, so (C) simplifies to

$$-2r^3 \cos \psi \sin \psi \left( 2f_t - r \frac{\partial f_t}{\partial r} \right) < 0.$$

Step 2: The Reeb vector field  $R$  associated to  $\alpha$  has properties (T2) and (T3):

The Reeb vector field is defined by  $i_R d\alpha = 0$  and  $\alpha(R) = 1$ . This is equivalent to the following system of linear equations for the coefficients of  $R = R_z \partial_z + R_r \partial_r + R_\psi \partial_\psi + R_\varphi \partial_\varphi + R_\theta \partial_\theta$ :

$$\frac{\partial f}{\partial r} R_r + \frac{\partial f}{\partial \psi} R_\psi - \frac{\partial g}{\partial z} R_\varphi - \frac{\partial h}{\partial z} R_\theta = 0 \quad (\text{I})$$

$$\frac{\partial f}{\partial r} R_z + \frac{\partial g}{\partial r} R_\varphi + \frac{\partial h}{\partial r} R_\theta = 0 \quad (\text{II})$$

$$\frac{\partial f}{\partial \psi} R_z + \frac{\partial g}{\partial \psi} R_\varphi + \frac{\partial h}{\partial \psi} R_\theta = 0 \quad (\text{III})$$

$$\frac{\partial g}{\partial z} R_z + \frac{\partial g}{\partial r} R_r + \frac{\partial g}{\partial \psi} R_\psi = 0 \quad (\text{IV})$$

$$\frac{\partial h}{\partial z} R_z + \frac{\partial h}{\partial r} R_r + \frac{\partial h}{\partial \psi} R_\psi = 0 \quad (\text{V})$$

$$f R_z + g R_\varphi + h R_\theta = 1 \quad (\text{VI})$$

It is obvious that the Reeb vector field  $R$  does not depend on the variables  $\varphi$  and  $\theta$ . Hence slightly abusing the notation we will write  $R = R(z, r, \psi)$  for the remainder of the proof.

First we prove (T2) and assume for this purpose that for a point  $p \in D$  the  $z$ -component of  $R$  vanishes. Then (II), (III) and (VI) imply

$$\begin{aligned} \frac{\partial g}{\partial r}(p) R_\varphi(p) + \frac{\partial h}{\partial r}(p) R_\theta(p) &= 0 \\ \frac{\partial g}{\partial \psi}(p) R_\varphi(p) + \frac{\partial h}{\partial \psi}(p) R_\theta(p) &= 0 \\ g(p) R_\varphi(p) + h(p) R_\theta(p) &= 1 \end{aligned}$$

If  $p \neq p_{T^2}$ , conditions  $(G_r)$ ,  $(G_\psi)$ ,  $(H_r)$ , and  $(H_\psi)$  applied to the first two equations imply  $R_\varphi(p) = R_\theta(p) = 0$ . This contradicts the last of the three

equations. So  $R_z$  vanishes only on  $\mathbb{T}^2$  and is equal to 1 outside of  $U$ . Since  $\mathbb{R}^5 \setminus \mathbb{T}^2$  is connected, we conclude that  $R_z$  is positive on this set.

By properties  $(G_r)$  and  $(H_r)$ , on the other hand, equation (II) yields

$$\frac{\partial f}{\partial r}(p_{\mathbb{T}^2})R_z(p_{\mathbb{T}^2}) = 0.$$

Thus  $(F_r)$  implies  $R_z(p_{\mathbb{T}^2}) = 0$ . This finishes the proof of condition (T2).

Applying  $(G_r)$ ,  $(G_z)$ , and  $(G_\psi)$  to equation (IV) we analogously deduce  $R_\psi(p_{\mathbb{T}^2}) = 0$ . Using this result as well as the conditions  $(G_z)$ ,  $(H_z)$ ,  $(F_r)$ , and equation (I) we see that  $R_r(p_{\mathbb{T}^2}) = 0$ .

Now we insert  $(G_\psi)$ ,  $(H_\psi)$ , and  $R_z(p_{\mathbb{T}^2}) = 0$  into (III) and obtain  $R_\varphi(p_{\mathbb{T}^2}) = \chi R_\theta(p_{\mathbb{T}^2})$ . By scaling with respect to (VI) we conclude that  $R(p_{\mathbb{T}^2}) = \partial_\varphi|_{p_{\mathbb{T}^2}} + \frac{1}{\chi}\partial_\theta|_{p_{\mathbb{T}^2}}$ . □

#### APPENDIX A. PROOF OF LEMMA 2.2

We start with some notation: For real numbers  $t_1 < t_2 \leq t_3 < t_4$  let  $a = a_{t_1, t_2, t_3, t_4} : \mathbb{R} \rightarrow [0, 1]$  be a smooth bump function such that

$$a|_{\mathbb{R} \setminus (t_1, t_4)} \equiv 0, \quad a|_{[t_2, t_3]} \equiv 1 \quad \text{and} \quad a'(t) \begin{cases} > 0, & \text{if } t \in (t_1, t_2) \\ < 0, & \text{if } t \in (t_3, t_4). \end{cases} \quad (1)$$

Now we choose open neighborhoods of  $p_{\mathbb{T}^2}$

$$\begin{aligned} U &= (z_1^-, z_1^+) \times (r_1^-, r_1^+) \times (\psi_1^-, \psi_1^+), \\ V &= (z_2^-, z_2^+) \times (r_2^-, r_2^+) \times (\psi_2^-, \psi_2^+), \\ W &= (z_3^-, z_3^+) \times (r_3^-, r_3^+) \times (\psi_3^-, \psi_3^+), \\ W_\delta &= (z_3^-, z_3^+) \times (1 - \delta, 1 + \delta) \times (\psi_3^-, \psi_3^+), \\ X_\delta &= (z_4^-, z_4^+) \times (1 - \delta, 1 + \delta) \times (\psi_4^-, \psi_4^+) \end{aligned}$$

such that  $\overline{X_\delta} \subseteq W \subseteq \overline{W} \subseteq V \subseteq \overline{V} \subseteq U \subseteq \overline{U} \subseteq D$ .

THE CONSTRUCTION OF  $f$ :

Set  $a_{\overline{z}} = a_{z_1^-, z_2^-, z_2^+, z_1^+}$ ,  $a_{\overline{r}} = a_{r_2^-, r_2^+, r_2^+, r_1^+}$  (mind the choice of parameters!),  $a_{\overline{\psi}} = a_{\psi_1^-, \psi_2^-, \psi_2^+, \psi_1^+}$  and  $C := \sup_{r \in [r_2^+, r_1^+]} -a'_{\overline{r}}(r) < \infty$ . Then for  $0 < \varepsilon < \min\{\frac{1}{2}, \frac{1}{C \cdot r_1^+}\}$  the map

$$f(z, r, \psi) = 1 - \varepsilon \cdot a_{\overline{z}}(z) \cdot a_{\overline{r}}(r) \cdot a_{\overline{\psi}}(\psi)$$

clearly meets the conditions  $(F_B)$ ,  $(F_r)$ , and  $(F_\psi)$  by the choice of parameters. To verify property  $(F_C)$  we consider two cases: By definition of  $\varepsilon$  it holds for  $r \in [r_2^+, r_1^+]$

$$\begin{aligned} \left(2f(z, r, \psi) - r \frac{\partial f}{\partial r}(z, r, \psi)\right) &= 2 - a_{\overline{z}}(z) \cdot a_{\overline{\psi}}(\psi) \cdot (2\varepsilon a_{\overline{r}}(r) - \varepsilon r a'_{\overline{r}}(r)) \\ &> 2 - (1 + 1) = 0. \end{aligned}$$

On the other hand, this estimate follows from  $f > 0$  and  $\frac{\partial f}{\partial r}(z, r, \psi) \leq 0$  if  $r \in \mathbb{R} \setminus [r_2^+, r_1^+]$ . So  $(F_C)$  is fulfilled.

THE CONSTRUCTION OF  $h$ :

The construction of  $h$  consists of two parts: Moving from the outside to the inside we deform in a first step  $h_0 = r^2 \sin^2 \psi$  to a function that equals  $\tilde{\chi} \cdot h_0$  on  $W$

for some adequate irrational  $\tilde{\chi}$ . This function meets all constraints except for the conditions on the partial derivatives with respect to  $r$  at the point  $p_{\mathbb{T}^2}$ . This failure can be remedied in a second step by adding a small function chosen in such a way that the other properties remain unmodified.

We start with an auxiliary map  $u_\chi \in C^\infty(D, \mathbb{R}_{>0})$  for positive  $\chi \in \mathbb{R}$ :

$$u_\chi(z, r, \psi) = (1 - b(z, r, \psi))r^2 \sin^2 \psi + b(z, r, \psi)\chi r^2 \sin^2 \psi.$$

Here  $b : D \rightarrow [0, 1]$  is a bump function given by

$$b(z, r, \psi) = a_{z_2^-, z_3^-, z_3^+, z_2^+}(z) \cdot a_{r_2^-, r_3^-, r_3^+, r_2^+}(r) \cdot a_{\psi_2^-, \psi_3^-, \psi_3^+, \psi_2^+}(\psi).$$

Then  $u_1 = r^2 \sin^2 \psi$  and therefore its partial derivatives with respect to  $r$  and  $\psi$  are positive. By continuity in  $\chi$  and compactness of  $\bar{V}$  we can choose  $\tilde{\chi} \in \mathbb{R} \setminus \mathbb{Q}$  near to 1 such that for  $u = u_{\tilde{\chi}}$  the following holds

$$u|_{D \setminus V} = h_0, \quad u|_W = \tilde{\chi} h_0, \quad \frac{\partial u}{\partial r} > 0, \quad \frac{\partial u}{\partial \psi} > 0, \quad \text{and} \quad \frac{\partial u}{\partial z}(p_{\mathbb{T}^2}) = 0. \quad (2)$$

Now, in a second step, we change  $u$  in a small neighborhood of  $p_{\mathbb{T}^2}$  in order to obtain a map that fulfills in addition property  $(H_r)$ . For this purpose set

$$\bar{b}_\delta(z, r, \psi) = a_{z_3^-, z_4^-, z_4^+, z_3^+}(z) \cdot a_{1-\delta, 1-\frac{\delta}{2}, 1+\frac{\delta}{2}, 1+\delta}(r) \cdot a_{\psi_3^-, \psi_4^-, \psi_4^+, \psi_3^+}(\psi).$$

Then  $\bar{b}_\delta$  has the following properties for some positive constant  $C_1 < \infty$ :

$$\begin{aligned} \bar{b}_\delta|_{\mathbb{R}^3 \setminus W_\delta} &\equiv 0, \quad \bar{b}_\delta|_{X_{\frac{\delta}{2}}} \equiv 1 \quad \text{and} \quad \forall \delta > 0 : \left| \frac{\partial \bar{b}_\delta}{\partial \psi} \right| < C_1 \\ \text{for } p = (z, r, \psi) \in W : \frac{\partial \bar{b}_\delta}{\partial r}(p) &\begin{cases} > 0, & \text{if } 1 - \delta < r < 1 - \frac{1}{2}\delta \\ = 0, & \text{if } 1 - \frac{1}{2}\delta \leq r \leq 1 + \frac{1}{2}\delta \\ < 0, & \text{if } 1 + \frac{1}{2}\delta < r < 1 + \delta \end{cases} \quad (3) \end{aligned}$$

Furthermore, we define for  $\eta \in \mathbb{R}$

$$v_\eta(z, r, \psi) = (1 - r^2) \tilde{\chi} \sin^2 \psi + \eta \left[ \left( z^2 + \left( \psi - \frac{\pi}{4} \right)^2 \right) r + (r - 1)^3 \right].$$

Note that  $v_\eta(p_{\mathbb{T}^2}) = 0$  for every  $\eta \in \mathbb{R}$ . Additionally, the partial derivative with respect to  $r$  of  $u$  and the first summand of this map cancel at the point  $p_{\mathbb{T}^2}$ . If  $\eta > 0$ , the second summand will ensure that this partial derivative remains positive outside of  $p_{\mathbb{T}^2}$ . Moreover, we see that for  $p = (z, r, \psi) \in W$

$$\begin{aligned} \frac{\partial}{\partial \psi} (u + \bar{b}_\delta v_0)(p) &= 2\tilde{\chi} \sin \psi \cos \psi \cdot r^2 + \left( \frac{\partial \bar{b}_\delta}{\partial \psi} \tilde{\chi} \sin^2 \psi + 2\bar{b}_\delta \tilde{\chi} \sin \psi \cos \psi \right) (1 - r^2) \\ &\geq C_2 r^2 - C_3 \cdot |1 - r^2|. \end{aligned}$$

with a positive constant  $C_2 := \min_{\psi \in [\psi_3^-, \psi_3^+]} 2\tilde{\chi} \sin \psi \cos \psi$  and a finite constant  $C_3 := \max_{\psi \in [\psi_3^-, \psi_3^+]} \tilde{\chi} (C_1 \sin^2 \psi + 2 \sin \psi \cos \psi)$  that is by (3) independent of  $\delta$ . Hence we can choose  $\tilde{\delta} > 0$  small enough, so that for  $p \in W_{\tilde{\delta}} \subseteq W$

$$\frac{\partial}{\partial \psi} (u + \bar{b}_{\tilde{\delta}} v_0)(p) > 0.$$

By definition holds

$$v_0(z, r, \psi) = (1 - r^2) \tilde{\chi} \sin^2 \psi \begin{cases} > 0, & \text{if } r < 1 \\ < 0, & \text{if } r > 1. \end{cases}$$

Again, we use continuity and compactness, here of  $\overline{W_\delta}$ , to find an irrational  $\tilde{\eta} > 0$  such that for  $v = v_{\tilde{\eta}}$  and every  $p = (z, r, \psi) \in W_{\tilde{\delta}}$

$$v(z, r, \psi) \begin{cases} > 0, & \text{if } 1 - \delta < r < 1 - \frac{1}{2}\delta \\ < 0, & \text{if } 1 + \frac{1}{2}\delta < r < 1 + \delta \end{cases} \quad \text{and} \quad \frac{\partial}{\partial \psi}(u + \bar{b}_\delta v)(p) > 0. \quad (4)$$

We define

$$h = u + \bar{b}_\delta v.$$

It is then easy to check that this function meets conditions  $(H_B)$ ,  $(H_z)$ , and  $(H_\psi)$ .

For  $(H_r)$  it suffices to consider  $\frac{\partial h}{\partial r}$  for  $p \in W_{\tilde{\delta}}$ :

$$\begin{aligned} \frac{\partial h}{\partial r}(p) &= \frac{\partial u}{\partial r} + \frac{\partial \bar{b}_\delta}{\partial r} v + \bar{b}_\delta \frac{\partial v}{\partial r} \\ &\stackrel{(3), (4)}{\geq} 2r\tilde{\chi} \sin^2 \psi + \bar{b}_\delta \left[ -2r\tilde{\chi} \sin^2 \psi + \eta \left( z^2 + \left( \psi - \frac{\pi}{4} \right)^2 + 3(r-1)^2 \right) \right] \\ &= (1 - \bar{b}_\delta) 2r\tilde{\chi} \sin^2 \psi + \bar{b}_\delta \left[ \eta \left( z^2 + \left( \psi - \frac{\pi}{4} \right)^2 + 3(r-1)^2 \right) \right] \geq 0. \end{aligned}$$

Hence  $\frac{\partial h}{\partial r}|_{D \setminus \{p_{\mathbb{T}^2}\}} > 0$  and  $\frac{\partial h}{\partial r}(p_{\mathbb{T}^2}) = 0$ . This yields  $(H_r)$ .

The construction of  $g$  can be realized in a similar way while skipping the first part of the deformation.

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