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# A conformal invariant from the Dirac operator on noncompact spin manifolds

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# Contents

# Introduction

In geometry conformal invariants are used to examine different properties of a manifold, e.g. to find estimates for eigenvalues. These invariants can be composed by conformally covariant operators like the Laplacian in dimension 2, the conformal Laplacian and the Dirac operator.

The conformal Laplacian or Yamabe operator on an $n$-dimensional Riemannian manifold $(M, g)$ with $n \geq 3$ is defined as

$$L_g := 4\frac{n-1}{n-2}\Delta_g + \mathrm{scal}_g,$$

where $\Delta_g$ denotes the Laplace operator and $\mathrm{scal}_g$ the scalar curvature of $M$. This operator played an important role for the Yamabe problem, the problem of finding in a given conformal class a metric of constant scalar curvature. The solution of this problem crucially involves the investigation of the Sobolev quotient (the Yamabe number)

$$Q(M, g) := \inf \Big\{ \int_M \phi L_g \phi \ \mathrm{dvol}_g \mid \phi \in C_c^\infty(M), \| \phi \|_{L^{\frac{2n}{n-2}}} = 1 \Big\},$$

that is conformally invariant, and its associated partial differential equation. An overview about the main results in the surrounding of the Yamabe problem can be found in [13].
Alternatively, on a compact manifold the Sobolev quotient can be defined with the help of the first eigenvalue $\lambda_1^L(g)$ of the conformal Laplacian $L_g$

$$Q(M, g) = \inf_{g_0 \in [g]} \lambda_1^L(g_0) \mathrm{vol}(M, g_0)^{\frac{2}{n}}.$$

On compact Riemannian spin manifolds this expression can be used to define an analogon for the Dirac operator in terms of the first positive Dirac eigenvalue $\lambda_1^+(g)$

$$\lambda_{min}^+(M, g) := \inf_{g_0 \in [g]} \lambda_1^+(g_0) \mathrm{vol}(M, g_0)^{\frac{1}{n}}.$$

This invariant was studied e.g. in [1, 4, 6]. Like the Sobolev quotient $\lambda_{min}^+$ can be understood as the critical point of a functional. Many results found for the Sobolev quotient have an analogon for the Dirac operator. The main reason for that turns out to be that the transformation law for the Dirac operator under conformal changes of the underlying metric is similar to the one of the conformal Laplacian.

Nevertheless, not all arguments from the Yamabe problem can be taken over since the Dirac operator is unbounded on both sides and for spinors there does not exist a maximum principle.
But most of the considerations can be generalized to noncompact manifolds by using Rayleigh quotients. This will be done in this thesis.
Many results for compact manifolds also hold in the noncompact case, e.g. for every Riemannian spin manifold the constant $\lambda^+_{min}$ is always bounded from above by the appropriate constant of the standard sphere (cf. proposition 2.3.4).

As in the case of the Sobolev quotient it offers new applications such as an obstruction to the conformal compactification of a Riemannian spin manifold. It will be shown that if $\lambda^+_{min}$ of a noncompact complete Riemannian spin manifold of dimension $n \geq 2$ at infinity (cf. definition 3.0.7) does not coincide with the one of the standard sphere, this manifold can not be conformal to a subdomain of any compact Riemannian spin manifold. This is an analogon to a theorem given in [11] that involves Sobolev quotients.

The thesis is structured as follows: In the first chapter the basic concepts of spin manifolds and their Dirac operator are introduced to provide the notations and theorems that are used in the following.
The conformal invariant $\lambda^+_{min}$ will be generalized to noncompact manifolds in the second chapter and some results that hold on compact manifolds will be carried over.
In the last chapter the obstruction to the conformal compactification will be given. Results concerning the development of the metric and the Dirac operator in normal coordinates that are needed in the second chapter are listed in the appendix.

# Chapter 1

# Preliminaries

## 1.1 Spin manifolds and the Dirac operator

In this section basic notions concerning spin manifolds and the Dirac operator that are used in this thesis are shortly listed to fix notations. All this can be found in detail in [8] and [12].

Let $(M, g)$ be an oriented Riemannian manifold with dimension $n \geq 2$ and let $P_{\mathrm{SO}(n)}M_g$ be the $\mathrm{SO}(n)$ principal bundle over $M$ of positively oriented frames. A *spin structure* $\gamma$ of $(M, g)$ is a $\mathrm{Spin}(n)$ principal bundle $P_{\mathrm{Spin}(n)}M_g$ over $M$ with a double covering $\eta : \; P_{\mathrm{Spin}(n)}M_g \to P_{\mathrm{SO}(n)}M_g$ such that the diagram

$$\begin{tikzcd}
\mathrm{Spin}(n)\times P_{\mathrm{Spin}(n)}M_g \arrow[r] \arrow[dd, "\Theta\times\eta"] & P_{\mathrm{Spin}(n)}M_g \arrow[dd, "\eta"] \arrow[dr] & \\
 & & M \\
\mathrm{SO}(n)\times P_{\mathrm{SO}(n)}M_g \arrow[r] & P_{\mathrm{SO}(n)}M_g \arrow[ur] &
\end{tikzcd}$$

commutes, where $\Theta$ is the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ and the horizontal arrows denote the corresponding group actions. A Riemannian manifold that admits such a spin structure is called a *spin manifold*.

**Remark 1.1.1.**
**i)** A spin manifold can allow different spin structures. In the following, when talking about a Riemannian spin manifold $(M, g)$, it will always be assumed that a spin structure is already chosen and fixed.
**ii)** A simply connected Riemannian manifold is spin if and only if its fundamental group $\pi_1(P_{\mathrm{SO}(n)}M_g) = \mathbb{Z}_2$ and then the spin structure is uniquely determined [8, p. 42].

Let further $S_g = P_{\mathrm{Spin}(n)}M_g \times_\rho \Delta_n$ be the associated *spinor bundle*, where $\Delta_n = \mathbb{C}^{2^{[\frac{n}{2}]}}$ and $\rho : \mathrm{Spin}(n) \to End(\Delta_n)$ is the spinor representation. A section of $S_g$ will be called *spinor*. $S_g$ is equipped fibrewise with a hermitian metric $\langle\ .\ ,\ .\ \rangle_m$ that depends smoothly on the base point $m$ and with the Clifford multiplication $TM \otimes S_g \to S_g;\ X \otimes \phi \mapsto X \cdot \phi$ such that for all $m \in M$

$$\langle X_m \cdot \phi_1(m), \phi_2(m)\rangle_m + \langle \phi_1(m), X_m \cdot \phi_2(m)\rangle_m = 0 \quad \forall X \in TM; \phi_1, \phi_2 \in \Gamma(S_g)$$

and

$$X_m \cdot Y_m \cdot \phi(m) + Y_m \cdot X_m \cdot \phi(m) = -2g_m(X_m, Y_m) \cdot \phi(m) \quad \forall X, Y \in TM; \phi \in \Gamma(S_g).$$

Further, with this hermitian metric a $L^2$-scalar product

$$(\phi, \psi)_{M,g} := \int_M \langle \phi(m), \psi(m)\rangle_m \mathrm{dvol}_g$$

is defined for spinors $\phi, \psi$ of $S_g$. Additionally, the Levi-Civita connection on $P_{\mathrm{SO}(n)}M_g$ induces a metric connection $\nabla$ on the spinor bundle that is parallel w.r.t. the Clifford multiplication, that means it fulfills

$$\nabla_X(Y \cdot \phi) = (\nabla_X Y) \cdot \phi + Y \cdot \nabla_X \phi \tag{1.1}$$

for all $X, Y \in \Gamma(TM)$ and $\phi \in \Gamma(S_g)$.

By the composition of the connection and the Clifford multiplication the Dirac operator is defined:

**Definition 1.1.2.** [8, p. 75]
The operator $D_g = \mu \circ \nabla :\ \Gamma(S_g) \to \Gamma(T^*M \otimes S_g) \cong \Gamma(TM \otimes S_g) \to \Gamma(S_g)$ is called *Dirac operator*, where $\mu$ denotes the Clifford multiplication. Its local form w.r.t. an orthonormal Repère $e = (e_1, \ldots, e_n)$ on the manifold $(M^n, g)$ is given by

$$D_g\psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi.$$

**Remark 1.1.3.**
**i)** The local form of the Dirac operator is independent of the choice of the frame $(e_1, \ldots, e_n)$ [10, p. 144].
**ii)** The Dirac operator is a first order elliptic differential operator [12, p. 113] and using the $L^2$-product introduced above it is defined as an operator over $L^2(S_g)$.
**iii)** The Dirac operator is formally self-adjoint [12, p. 115], i.e. for all spinors $\phi, \psi$, at least one of them being compactly supported on $M$, it is $(\phi, D\psi)_{M,g} = (\psi, D\phi)_{M,g}$. Thus, due to this self-adjointness and the hermiticity of the scalar product, $(\phi, D\phi)$ is real for all compactly supported spinors $\phi$.

## 1.2 Conformal transformation of the Dirac operator

The spin structure, the spinor bundle and hence the Dirac operator depend on the metric $g$ of $M$. Below, the Dirac operators belonging to two conformal equivalent metrics and the corresponding isomorphic spinor bundles are compared.

Let $g$ and $\tilde{g}$ be conformal equivalent metrics, i.e. there is a function $f \in C^\infty(M)$ with $f > 0$ and $\tilde{g} = f^2 g$. The appropriate equivalence class is denoted by $[g]$. Having the fixed spin structure $\gamma$ on $(M, g)$ there always exists a spin structure $\tilde{\gamma}$ (and the corresponding spinor bundle $S_{\tilde{g}}$) on $(M, \tilde{g})$ and a vector bundle isomorphism $A: S_g \to S_{\tilde{g}}$ that is fibrewise an isometry [9]. Using this isometry it is possible to compare the Dirac operators $D_g$ and $D_{\tilde{g}}$.

**Proposition 1.2.1.** [9, prop. 4.3.1.] Let $(M, g)$ be an $n$-dimensional Riemannian spin manifold. The Dirac operators $D_g$ and $D_{\tilde{g}}$, respectively, satisfy

$$D_{\tilde{g}}(A(f^{-\frac{n-1}{2}}\psi)) = A(f^{-\frac{n+1}{2}} D_g \psi) \tag{1.2}$$

for any spinor $\psi$ of $S_g$.

From now on, the explicit notation for the identification of spinors of the different spinor bundles will be omitted and it is set $\tilde{\psi} := A(f^{-\frac{n-1}{2}}\psi) \equiv f^{-\frac{n-1}{2}}\psi$ and $S = S_g = S_{\tilde{g}}$.

Using the transformation law (1.2), the identification of spinors $\tilde{\psi} = f^{-\frac{n-1}{2}}\psi$ and $\mathrm{dvol}_{\tilde{g}} = f^n \mathrm{dvol}_g$, conformal invariants can be found:

$$\begin{aligned}
\text{I.}\quad (\phi, D_g\phi)_g &= (f^{\frac{n-1}{2}}\tilde{\phi}, D_g(f^{\frac{n-1}{2}}\tilde{\phi}))_g = (f^{\frac{n-1}{2}}\tilde{\phi}, f^{\frac{n+1}{2}} D_{\tilde{g}}\tilde{\phi})_g \\
&= (f^n\tilde{\phi}, D_{\tilde{g}}\tilde{\phi})_g = (\tilde{\phi}, D_{\tilde{g}}\tilde{\phi})_{\tilde{g}} \\
\text{II.}\quad \| \phi \|_{L^p,g} &= \| f^{\frac{n-1}{2}}\tilde{\phi} \|_{L^p,g} = \| f^{\frac{n}{p}}\tilde{\phi} \|_{L^p,g} = \| \tilde{\phi} \|_{L^p,\tilde{g}} \\
\text{III.}\quad \| D_g\phi \|_{L^q,g} &= \| D_g(f^{\frac{n-1}{2}}\tilde{\phi}) \|_{L^q,g} = \| f^{\frac{n}{q}} D_{\tilde{g}}\tilde{\phi} \|_{L^q,g} = \| D_{\tilde{g}}\tilde{\phi} \|_{L^q,\tilde{g}},
\end{aligned}$$

where $p = \frac{2n}{n-1}$ and $q = \frac{2n}{n+1}$.

## 1.3 Spectrum

Next, some general properties of the spectrum of the Dirac operator and its behaviour under conformal transformations are given.
In general, the Dirac operator can possess all types of spectrum: point spectrum, continuous spectrum and residual spectrum. But in special cases some of these spectra vanish. For the following, the most important one will be the case of a compact manifold:

**Theorem 1.3.1.** [8, pp. 108, 111],[2, prop. 4.30] Let $(M, g)$ be a closed Riemannian spin manifold. Then the Dirac operator $D_g$ has pure real point spectrum and it exists an orthonormal basis $\psi_i$ of $L^2(S_g)$ $(i \in \mathbb{N})$ such that $D_g\psi_i = \lambda_i\psi_i$ with $\lambda_i \in \mathbb{R}$. Further, as well $+\infty$ as $-\infty$ are accumulation points of the spectrum.

For noncompact manifolds it can also occur that the Dirac operator has no point spectrum. This is for example the case for $(\mathbb{R}, g_E)$. Thus, in chapter 2 a generalization will be used to make statements for noncompact manifolds similar to those given in [4] for compact ones.

## 1.4 Elliptic regularity

In this section the Sobolev embedding theorems for the Dirac operator and the definition of the spaces needed, as Sobolev and Hölder spaces, will be provided.

The Sobolev spaces $H_k^q$ with $q \in (1, \infty)$ and $k \in \mathbb{N}$ defined as the completion of $C^\infty(M, S)$ by the $H_k^q$-norm

$$\| \phi \|_{H_k^q} = \| \nabla^k \phi \|_{L^q}$$

can be generalized on compact manifolds to $k \in \mathbb{R}$:
Let now $(M, g)$ be compact and define for a spinor $\phi$ the norm

$$\| \phi \|_{\tilde{H}_k^q} = \| \, |D|^k \phi \|_{L^q} + \| \pi\phi \|_1,$$

where $\pi$ denotes the projection on the kernel of $D$ and $\| \,.\, \|_1$ an arbitrary norm on it. The operator $|D|^k$ is defined on compact manifolds for every $k \in \mathbb{R}$ (see [4]) and acts by

$$|D|^k \sum \beta_i\phi_i := \sum_{\lambda_i \neq 0} \beta_i|\lambda_i|^k\phi_i$$

where $\phi_i$ is an orthonormal basis of $L^2(S)$ consisting of Dirac eigenspinors with the corresponding eigenvalues $\lambda_i$ (cf. theorem 1.3.1). For $k \in \mathbb{N}$ the norms $H_k^q$ and $\tilde{H}_k^q$ are equivalent [4]. Thus, in both cases the corresponding Sobolev spaces will be denoted by $H_k^q$.

The next type of spaces needed are the Hölder spaces.

**Definition 1.4.1.** The Hölder spaces $C^{0,\alpha}(M, S)$ and $C^{1,\alpha}(M, S)$, respectively, for $\alpha \in (0, 1]$ are the completions of $C^\infty(M, S)$ w.r.t. the Hölder norm

$$\| \phi \|_{C^{0,\alpha}} := \text{höl}_\alpha(\phi) \text{ and } \| \phi \|_{C^{1,\alpha}} := \| \phi \|_{C^0} + \text{höl}_\alpha(\nabla\phi),$$

respectively, where

$$\text{höl}_\alpha(\phi) := \sup\left\{ \frac{|\phi(x) - P_\gamma\phi(y)|}{d(x, y)^\alpha} \,\middle|\, x, y \in M, x \neq y \right\}$$

and $P_\gamma$ being the parallel transport along a shortest geodesic $\gamma$ from $x$ to $y$.

With these definitions the Sobolev embedding theorems on spin manifolds can be formulated.

**Theorem 1.4.2** (Sobolev embedding I, [4] thm. 3.4.1)**.** Let $k, s \in \mathbb{R}$, $k \geq s$ and $q, r \in (1, \infty)$ with

$$\frac{1}{r} - \frac{s}{n} \geq \frac{1}{q} - \frac{k}{n}, \tag{1.3}$$

then $H_k^q(M, S)$ is continuously embedded into $H_s^r(M, S)$.

**Theorem 1.4.3** (Rellich-Kondrakov, [4] thm. 3.4.3)**.** Under the same conditions as in theorem 1.4.2, but with strict inequality (1.3) the inclusion $H_k^q(M, S) \to H_s^r(M, S)$ is a compact operator.

**Theorem 1.4.4** (Sobolev embedding II, [4] thm. 3.4.4)**.** Suppose $0 < \alpha < 1$, $m \in \{0, 1\}$ and

$$\frac{1}{q} \leq \frac{k - m - \alpha}{n}. \tag{1.4}$$

Then $H_k^q(M, S)$ is continuously embedded into $C^{m,\alpha}(M, S)$.

# Chapter 2

# The conformal invariant

## 2.1 Generalization of the first positive Dirac eigenvalue

As stated in section 1.3, on noncompact spin manifolds Dirac eigenvalues do not have to exist and in general the spectrum does not only consist of eigenvalues. In analogy to the first positive Dirac eigenvalue $\lambda_1^+(M,g)$ of a closed Riemannian spin manifold $(M,g)$ and the corresponding first conformal positive Dirac eigenvalue $\lambda_{min}^+(M,[g])$ [1, p. 4], a similar notion shall be also defined on open manifolds:

**Definition 2.1.1.** Let $(M,g)$ be a Riemannian spin manifold without boundary. Then,

$$\lambda_1^+(M,g) := \inf\left\{ \frac{(D_g\phi, D_g\phi)_{M,g}}{(D_g\phi,\phi)_{M,g}} \,\middle|\, 0 < (D_g\phi,\phi)_{M,g} < \infty, \phi \in C_c^\infty(M,S) \right\} \tag{2.1}$$

Further, the conformal invariant $\lambda_{min}^+(M,g) := \inf_{g'\in\mathcal{M}(g)} \lambda_1^+(g')$ is defined, where $\mathcal{M}(g)$ denotes the set of metrics in $[g]$ with unit volume.

**Remark 2.1.2.**
**i)** In general, the volume of a noncompact manifold $(M,g)$, denoted by $\mathrm{vol}(M,g)$, is not finite. But $\mathcal{M}(g)$ is always non-empty.
**ii)** If $M$ is closed, $\lambda_1^+(M,g)$ really gives the first positive Dirac eigenvalue. This can be seen by using an orthonormal basis of eigenspinors of $L^2(S)$ that always exists due to theorem 1.3.1. Thus, $\phi = \sum_i \beta_i\phi_i$ with $D_g\phi_i = \lambda_i\phi_i$ and $(\phi_i,\phi_j)_{M,g} = \delta_{ij}$ for all $i,j$. Let $\lambda$ be the non-zero eigenvalue with the lowest magnitude. Then it holds

$$\frac{(D_g\phi, D_g\phi)_{M,g}}{(D_g\phi,\phi)_{M,g}} = \frac{\sum \lambda_i^2|\beta_i|^2}{\sum \lambda_i|\beta_i|^2} \geq \lambda \frac{\sum_{\lambda_i>0} \frac{\lambda_i^2}{\lambda}|\beta_i|^2}{\sum_{\lambda_i>0}\lambda_i|\beta_i|^2} \geq \lambda \frac{\sum_{\lambda_i>0}\lambda_i|\beta_i|^2}{\sum_{\lambda_i>0}\lambda_i|\beta_i|^2} = \lambda.$$

Furthermore, $\lambda$ can be attained by inserting for $\phi$ an eigenspinor of $\lambda$. Thus, $\lambda = \lambda_1^+$.

Firstly, some properties of $\lambda_{min}^+$ shall be presented in analogy of some properties of the Sobolev quotient of the Yamabe problem (see e.g. [14] lemma 2.1).

**Lemma 2.1.3.**
**i)** If $\Omega_1 \subseteq \Omega_2$ are two open subsets of a manifold $(M, g)$, it holds

$$\lambda^+_{min}(\Omega_1, g) \geq \lambda^+_{min}(\Omega_2, g).$$

**ii)** Let $(M, g)$ be a Riemannian spin manifold of dimension $n > 2$ and fix a point $p \in M$. Then

$$\lim_{\epsilon \to 0} \lambda^+_{min}(M \setminus B_\epsilon(p), g) = \lambda^+_{min}(M, g)$$

with $B_\epsilon(p)$ being a ball around $p$ of radius $\epsilon$ w.r.t. $g$.
**iii)** If $\Omega$ is an open subset of $S^n$ with $n > 2$ equipped with its standard metric $g_{st}$, then

$$\lambda^+_{min}(\Omega, g_{st}) = \lambda^+_{min}(S^n, g_{st}).$$

**Remark 2.1.4.** Since the sphere $S^n$ for $n \geq 2$ is simply connected it possesses only one spin structure (see remark 1.1.1.ii).

*Proof of lemma 2.1.3:*
i) That follows since every spinor $\phi \in C^\infty_c(\Omega_1, S)$ can be extended by zero to a spinor $\phi \in C^\infty_c(\Omega_2, S)$.
ii) From i) it is immediately seen that the limes has to exist and fulfills

$$\lim_{\epsilon \to 0} \lambda^+_{min}(M \setminus B_\epsilon(p), g) \geq \lambda^+_{min}(M, g).$$

Let now $\phi \in C^\infty_c(M, S)$ be a spinor on $M$ and let $\rho : [0, \infty) \to [0, 1]$ be a smooth function with $\rho \equiv 0$ on a neighbourhood of 0, $\rho \equiv 1$ on a neighbourhood of $[1, \infty)$ and $0 \leq \rho' \leq 2$. For $0 \leq \delta < \epsilon$ and for $x \in M$ such that $d(x, p) \geq \delta$ it is

$$\rho_{\delta,\epsilon}(x) := \rho\left(\frac{d(x,p) - \delta}{\epsilon - \delta}\right)$$

and $|\text{grad}\rho_{\delta,\epsilon}| \leq \frac{2}{\epsilon - \delta}$. Setting $\phi_\alpha := \rho_{\frac{\alpha}{2},\alpha}\phi$ for all $x$ with $d(x, p) \geq \alpha/2$ and else $\phi_\alpha := 0$, the spinors obtained are elements of $C^\infty_c(M \setminus B_\epsilon(p), S)$ for $\epsilon < \alpha/2$. It holds

$$\| \phi_\alpha - \phi \|_{L^2(M,g)} = \| (\rho_{\frac{\alpha}{2},\alpha} - 1)\phi \|_{L^2(B_\alpha(p),g)} \leq F_\alpha \text{vol}(B_\alpha(p), g)^{\frac{1}{2}},$$

where $F_\alpha := \max\{|\phi(x)| \mid x \in B_\alpha(p)\}$. Now an estimation of $F_\alpha$ shall be obtained: Let $x \in B_\alpha(p)$. Then

$$|\phi(x)| \leq |\phi(x) - \phi(p)| + |\phi(p)| \leq C_\alpha \alpha + |\phi(p)|, \tag{2.2}$$

where $C_\alpha := \max\{|\nabla\phi(x)| \bigm| d(x, p) < \alpha\}$ and where the last inequality is obtained by applying the mean value theorem. Hence, for $\alpha \leq 1$ it is

$$\| \phi_\alpha - \phi \|_{L^2(M,g)} \leq (C_1\alpha + |\phi(p)|)\text{vol}(B_\alpha(p), g)^{\frac{1}{2}}$$

and thus, $\phi_\alpha \to \phi$ in $L^2$ for $\alpha \to 0$. Due to definition of $\lambda_1^+$ it is for $\epsilon \leq \alpha/2$

$$\lambda_1^+(M \setminus B_\epsilon(p), g) \leq \frac{\| D_g\phi_\alpha \|^2_{L^2(M\setminus B_\epsilon(p),g}}{(D_g\phi_\alpha, \phi_\alpha)_{M\setminus B_\epsilon(p),g}}.$$

Thus, it suffices to show that

$$\lim_{\epsilon\to 0} \lambda_1^+(M \setminus B_\epsilon(p), g) \leq \lim_{\alpha\to 0} \frac{\| D_g\phi_\alpha \|^2_{L^2(M,g)}}{(D_g\phi_\alpha, \phi_\alpha)_{M,g}} = \frac{\| D_g\phi \|^2_{L^2(M,g)}}{(D_g\phi, \phi)_{M,g}},$$

since if this inequality holds for arbitrary $\phi \in C_c^\infty(M, S)$, it is

$$\lim_{\epsilon\to 0} \lambda_1^+(M \setminus B_\epsilon(p), g) \leq \lambda_1^+(M, g).$$

With $D_g\phi_\alpha = \rho_{\frac{\alpha}{2},\alpha} D_g\phi + \mathrm{grad}\rho_{\frac{\alpha}{2},\alpha} \cdot \phi$ it converges

$$\begin{aligned}\| D_g\phi_\alpha - D_g\phi \|_{L^2} &\leq \| (\rho_{\frac{\alpha}{2},\alpha} - 1) D_g\phi \|_{L^2} + \| \mathrm{grad}\rho_{\frac{\alpha}{2},\alpha} \cdot \phi \|_{L^2} \\ &\leq \max_{d(x,p)\leq\alpha} |D_g\phi(x)| \ \mathrm{vol}B_\alpha(p)^{\frac{1}{2}} + \tfrac{4}{\alpha} F_\alpha \mathrm{vol}B_\alpha(p)^{\frac{1}{2}} \\ &\leq (\max_{d(x,p)\leq\alpha} |D_g\phi(x)| + \tfrac{4}{\alpha}(C_\alpha\alpha + |\phi(p)|))\mathrm{vol}B_\alpha(p)^{\frac{1}{2}} \\ &\to 0 \quad \text{for } \alpha \to 0,\end{aligned}$$

since $\mathrm{vol}(B_\alpha(p))^{\frac{1}{2}}/\alpha \to 0$ for $\alpha \to 0$. Thus, the limes

$$\begin{aligned}|(D_g\phi_\alpha, \phi_\alpha) - (D_g\phi, \phi)| &\leq |(D_g\phi_\alpha, \phi_\alpha - \phi)| + |(D_g\phi - D_g\phi_\alpha, \phi)| \\ &\leq \| D_g\phi_\alpha \|_{L^2} \| \phi_\alpha - \phi \|_{L^2} + \| D_g\phi - D_g\phi_\alpha \|_{L^2} \| \phi \|_{L^2} \\ &\to 0 \quad \text{for } \alpha \to 0\end{aligned}$$

is obtained.
Thus, $\lambda_1^+(M, g) = \lim_{\epsilon\to 0} \lambda_1^+(M \setminus B_\epsilon, g)$. Thus, there have to exist $\delta_\epsilon^g \in \mathbb{R}_{\geq 0}$ with $\lambda_1^+(M, g) = \lambda_1^+(M \setminus B_\epsilon, g) - \delta_\epsilon^g$ and $\lim_{\epsilon\to 0} \delta_\epsilon^g = 0$. Hence, with

$$\lambda^+_{min}(M, g) = \inf_{\tilde{g}\in[g]} (\lambda_1^+(M \setminus B_\epsilon, \tilde{g}) - \delta_\epsilon^{\tilde{g}})$$

the claim is shown.
iii) Due to i) it is $\lambda^+_{min}(\Omega) \geq \lambda^+_{min}(S^n)$. Further, with ii) for all $\epsilon > 0$ it holds

$$\lim_{\epsilon\to 0} \lambda^+_{min}(S^n \setminus B_\epsilon) = \lambda^+_{min}(S^n),$$

where $B_\epsilon$ denotes the ball with radius $\epsilon$ around a fixed point of the sphere w.r.t. the standard metric. Since $\Omega$ is a domain of the sphere, for a fixed $\epsilon > 0$ there exists a conformal map $\Phi: \ S^n \to S^n$ such that $\Phi(\Omega) \subseteq S^n \setminus B_\epsilon$[1]. Due to the conformal invariance of $\lambda^+_{min}$ it holds

$$\lambda^+_{min}(\Omega) = \lambda^+_{min}(\Phi(\Omega)) \leq \lambda^+_{min}(S^n \setminus B_\epsilon).$$

Carrying out the limiting process $\epsilon \to 0$ yields $\lambda^+_{min}(\Omega) \leq \lambda^+_{min}(S^n)$. □

[1]E.g. such a map can be obtained by a composition of stereographic projection, multiplication by a constant and the inverse stereographic projection.

**Remark 2.1.5.** The proof even shows that the first and second statement of lemma 2.1.3 already hold when replacing $\lambda^+_{min}$ by $\lambda^+_1$.

For a compact manifold the condition for the spinors in (2.1) of being compactly supported is always fulfilled. Similarly, in the case of an open smoothly bounded subset $\Omega$ the infimum can be taken over all smooth spinors defined on the closure of $\Omega$.

**Lemma 2.1.6.** If $\Omega$ is an open smoothly bounded subset of a Riemannian spin manifold $(M, g)$, then

$$\lambda_1^+(\Omega, g) = \inf_{\phi \in C^\infty(\overline{\Omega}, S)} \left\{ \frac{\| D\phi \|^2_{L^2(\Omega,g)}}{(D\phi, \phi)_{\Omega,g}} \,\middle|\, 0 < (D\phi, \phi)_{\Omega,g} < \infty \right\}.$$

*Proof:* The proof is done analogously to the one of lemma 2.1.3.ii, but in the definition of $\rho_{\delta,\epsilon}$ the distance $d(x, p)$ is replaced by $d(x, \partial\Omega)$ and thus $B_\alpha(p)$ by $B_\alpha(\partial\Omega) := \{x \in M | \ d(x, \partial\Omega) < \alpha\}$. Further, the estimation (2.2) turns to

$$|\phi(x)| \leq |\phi(x) - \phi(p)| + |\phi(p)| \leq C_\alpha \alpha + |\phi(P)|,$$

where $p \in \partial\Omega$ such that $d(x, p) < \alpha$ and $P \in \partial\Omega$ such that $|\phi(y)| \leq |\phi(P)|$ for all $y \in \partial\Omega$. □

## 2.2 The corresponding variational problem

For $q \in [q_D = \frac{2n}{n+1}, \infty)$ define

$$\mathcal{F}_q^M : \ H_q := \{\psi \in C_c^\infty(M, S) | \ \| D_g\psi \|_{L^q(M,g)} = 1\} \to \mathbb{R}, \ \psi \mapsto (\psi, D_g\psi)_{M,g} \quad (2.3)$$

$$\mu_q := \mu_q(M, g, \sigma) := \sup_{\psi \in H_q} \mathcal{F}_q^M(\psi) \quad (2.4)$$

If only one particular manifold $M$ or one metric $g$ is under consideration, they will be omitted in the notation.

**Remark 2.2.1.**
**i)** For $q = q_D$ the functional is conformally invariant (see section 1.2), thus $\mu_{q_D}$ will be the major object of interest.
**ii)** On a compact manifold the Dirac operator $D_g$ has positive eigenvalues. Hence, $\mu_q > 0$. It also holds $\mu_q < \infty$ (see [4] and lemma 2.3.2).
**iii)** $\mu_2 = \lambda_1^+(g)^{-1}$

**Lemma 2.2.2.** The Euler-Lagrange equation of the variational problem given by $\mathcal{F}_q^M$ reads (with $D = D_g$)

$$D(\psi - \mu_q |D\psi|^{q-2} D\psi) = 0, \ \| D\psi \|_{L^q} = 1. \quad (2.5)$$

*Proof:* Let $\psi, \eta \in C_c^\infty(M, S)$. A variation $\psi + \epsilon\eta$ of the functional $\mathcal{F}_q^M(\psi)$ restricted to $H_q$, where $\epsilon$ is the variation parameter, gives

$$\begin{aligned} \mathcal{F}_q^M(\psi+\epsilon\eta) &= \int_M \langle D(\psi+\epsilon\eta), \psi+\epsilon\eta\rangle + \lambda\langle D(\psi+\epsilon\eta), D(\psi+\epsilon\eta)\rangle^{\frac{q}{2}}\mathrm{dvol}_g \\ 0 \overset{!}{=} \tfrac{\partial}{\partial\epsilon}\mathcal{F}_q^M(\psi+\epsilon\eta)|_{\epsilon=0} &= 2\ \mathrm{Re}\int_M \langle\psi, D\eta\rangle + \lambda\tfrac{q}{2}|D\psi|^{q-2}\langle D\psi, D\eta\rangle\mathrm{dvol}_g \\ &= 2\ \mathrm{Re}\int_M \langle\psi + \lambda\tfrac{q}{2}|D\psi|^{q-2}D\psi, D\eta\rangle\mathrm{dvol}_g \\ &= 2\ \mathrm{Re}\int_M \langle D(\psi + \lambda\tfrac{q}{2}|D\psi|^{q-2}D\psi), \eta\rangle\mathrm{dvol}_g \end{aligned}$$

Then, by the fundamental lemma of variational calculus

$$\psi + \lambda\frac{q}{2}|D\psi|^{q-2}D\psi \in \ker D.$$

Rescaling to a solution $\psi$ with $\| D\psi \|_{L^q} = 1$ yields (2.5). $\square$

The Euler-Lagrange equation will be further considered in section 2.4.

**Remark 2.2.3.**
**i)** Since $q-2 > -1$ for all $q \in [2n/(n+1), \infty)$ the expression $|D_g\psi|^{q-2}D_g\psi$ converges to 0 as $D_g\psi \to 0$. Setting $|D_g\psi|^{q-2}D_g\psi = 0$ for $D_g\psi = 0$ the obtained spinor $|D_g\psi|^{q-2}D_g\psi$ is smooth if $\psi$ and $|D_g\psi|$ is so.
**ii)** The map $\mathcal{F}_q^M$ introduced in [4] maps $\phi \in \mathrm{im}_{C^\infty} D$ with $\| \phi \|_{L^q} = 1$ to $(\phi, D^{-1}\phi)$, where $D^{-1}\phi$ is the preimage under $D$ orthogonal to its kernel and $\mathrm{im}_{C^\infty} D$ is the image of the Dirac operator on $(M, g)$. In this case the Euler-Lagrange equation reads

$$D^{-1}\phi - \mu_q|\phi|^{q-2}\phi \in \ker D, \quad \| \phi \|_{L^q} = 1.$$

On closed manifolds, with $\phi = D\psi$ and the decomposition $\psi = \psi_0 + \psi^\perp$ with $\psi_0$ belonging to the kernel of $D$ and $\psi^\perp$ being orthogonal to it, there holds

$$(\psi, D\psi) = (\psi_0 + \psi^\perp, D\psi^\perp) = \underbrace{(D\psi_0, \psi^\perp)}_{=0} + (\psi^\perp, D\psi^\perp) = (D^{-1}\phi, \phi).$$

Thus, both maps give the same constant $\mu_q$.

## 2.3 Properties of $\mu_q$

In this section some properties of $\mu_q$ and especially of $\mu_{q_D}$ are stated. In particular, the interest is directed towards the connection of $\mu_{q_D}$ with the previous conformal invariant $\lambda^+_{min}$ and the comparison of $\mu_{q_D}$ of an arbitrary manifold with $\mu_{q_D}$ of the standard sphere.

**Lemma 2.3.1.** For any Riemannian spin manifold $(M, g)$ it is $\lambda^+_{min} = \mu_{q_D}^{-1}$, where the case $\lambda^+_{min} = 0$ corresponds to $\mu_{q_D} = \infty$.

*Proof:* For compact spin manifolds the claim was proved in [1, prop. 5.5.], the same arguments work here.
Let $g_0 \in \mathcal{M}(g)$. Then the function $q \to \mu_q(M, g_0)$ is nondecreasing in q, since for $q_2 \leq q_1$ the Hölder inequality gives:

$$\| \varphi \|_{L^{q_2}} = \Big(\int |\varphi|^{q_2} \mathrm{dvol}_{g_0}\Big)^{\frac{1}{q_2}} \leq \Big(\Big(\int |\varphi|^{q_1} \mathrm{dvol}_{g_0}\Big)^{\frac{q_2}{q_1}} \mathrm{vol}(M, g_0)^{\frac{1}{q_2} - \frac{1}{q_1}}\Big)^{\frac{1}{q_2}} = \| \varphi \|_{L^{q_1}} .$$

With remark 2.2.1.iii it holds $\lambda_1^+(g_0) = \mu_2(g_0)^{-1} \geq \mu_{q_D}(g_0)^{-1}$ and hence $\lambda_{min}^+ \geq \mu_{q_D}^{-1}$. Thus, the inverse relation remains to show. If $\lambda_{min}^+ = 0$, then $\mu_{q_D} = \infty$ follows directly.
Assume now $\lambda_{min}^+ \neq 0$. Choose spinors $\psi_\epsilon$ such that $\mathcal{F}_{q_D}(\psi_\epsilon) \to \mu_{q_D}$ for $\epsilon \to 0$ and $\| D_{g_0}\psi_\epsilon \|_{L^{q_D}} = 1$. By using a small perturbation of $\psi_\epsilon$ it can be assumed that $|D_{g_0}\psi_\epsilon|$ is smooth and thus, it can be used to define the conformal equivalent metric $g_\epsilon = (|D_{g_0}\psi_\epsilon| + \delta)^{\frac{4}{n+1}} g_0$ and $\psi = (|D_{g_0}\psi_\epsilon| + \delta)^{\frac{n-1}{n+1}} \psi_\epsilon$. Then

$$\begin{aligned} D_{g_\epsilon}\psi &= (|D_{g_0}\psi_\epsilon| + \delta)^{-1} D_{g_0}\psi_\epsilon \quad \text{i.e. } |D_{g_\epsilon}\psi| \to 1 \text{ if } \delta \to 0 \\ \| D_{g_\epsilon}\psi \|^2_{L^2(M,g_\epsilon)} &\to 1 \text{ and } \mathrm{vol}(M, g_\epsilon) \to 1. \end{aligned}$$

Thus, it holds

$$\mathcal{F}_{q_D}(\psi_\epsilon) = \frac{(D_{g_0}\psi_\epsilon, \psi_\epsilon)_{g_0}}{\| D_{g_0}\psi_\epsilon \|^2_{L^{q_D}}} = (D_{g_\epsilon}\psi, \psi)_{g_\epsilon}$$

For $\delta \to 0$ $(D_{g_\epsilon}\psi, \psi)_{g_\epsilon}$ approaches $\frac{(D_{g_\epsilon}\psi,\psi)_{g_\epsilon}}{\|D_{g_\epsilon}\psi\|^2_{L^2}} \leq \lambda_1^+(g_\epsilon)^{-1}$.
Hence, $\mu_{q_D} \leq {\lambda_{min}^+}^{-1} = \sup_{g \in \mathcal{M}(g_0)} \lambda_1^+(g)^{-1}$. □

As mentioned in remark 2.2.1.ii $\mu_{q_D} < \infty$ holds for every compact manifold. The same is obtained for every bounded open subset of a Riemannian spin manifold.

**Lemma 2.3.2.** Let $(\Omega, g)$ be as above an open smoothly bounded subset of a Riemannian spin manifold $(M, g)$. Then $\mu_q(\Omega, g)$ is finite for every $q \in [q_D, \infty)$.

*Proof:* $\overline{\Omega}$ is a compact manifold with boundary. By gluing two manifolds of this kind together (at their boundary), a closed manifold $\Omega_2$ is obtained. Then, due to lemma 2.1.3.i it holds $\mu_q(\Omega) \leq \mu_q(\Omega_2)$. Thus, due to the finiteness of $\mu_q$ on closed manifolds, $\mu_q(\Omega)$ is finite, too.
For the sake of completeness the proof for $\mu_q < \infty$ on closed manifolds shall be sketched below:
Setting $\phi = D^{-1}\psi$ (see remark 2.2.3.ii) it follows by the definition of $H^2_{-1/2}$ (cf. section 1.4) and due to theorem 1.4.2 that there exists a constant $C$ with

$$\frac{(\psi, D\psi)}{\| D\psi \|_{L^q}} = \frac{(D^{-1}\phi, \phi)}{\| \phi \|^2_{L^q}} \leq \frac{(|D|^{-1}\phi, \phi)}{\| \phi \|^2_{L^q}} = \frac{\| \, |D|^{-\frac{1}{2}}\phi \|^2_{L^2}}{\| \phi \|^2_{L^q}} \leq \frac{\| \phi \|^2_{H^2_{-1/2}}}{\| \phi \|^2_{L^q}} \leq C.$$

□

As in the case of the Sobolev quotient the constant $\mu_{q_D}$ of a manifold can be compared with the one of the standard sphere $\mu_{q_D}(S^n, g_{st})$, namely $\mu_{q_D}(M) \geq \mu_{q_D}(S^n)$ (For compact manifolds this was shown in [3]). Before proving this estimate, it will be verified that in the limit of a point the value of $\mu_{q_D}$ coincides with $\mu_{q_D}(S^n)$.

**Lemma 2.3.3.** Let $(M, g)$ a Riemannian spin manifold of dimension $n > 2$ and let $p \in M$ be fixed. For any $\epsilon > 0$ let $U_\epsilon$ be a sequence of nested neighbourhoods around $p$ such that diameter$(U_\epsilon) \to 0$ as $\epsilon \to 0$. Then it holds

$$\lim_{\epsilon \to 0} \mu_{q_D}(U_\epsilon, g) = \mu_{q_D}(S^n, g_{st}).$$

*Proof:* Introduce rescaled geodesic normal coordinates:

$$\sigma_\epsilon : T_pM \cong \mathbb{R}^n \to M, \quad \sigma_\epsilon(x) = \exp_p(\epsilon x).$$

Let $B \subset T_pM$ a ball around 0 w.r.t. $g_E$ such that the exponential map $\exp_p$ restricted to $B$ is a diffeomorphism. Then, $B_\epsilon := \sigma_\epsilon(B)$ defines a sequence with $B_\epsilon \to \{p\}$ as $\epsilon \to 0$. W.l.o.g. it can be assumed $U_\epsilon = B_\epsilon$ for all $\epsilon$.
Defining $\psi_\epsilon := \epsilon^{-\frac{n-1}{2}} \psi \circ \sigma_\epsilon$, where $\psi \in C_c^\infty(B, S)$ and $g_\epsilon = \epsilon^{-2} \sigma_\epsilon^*(g)$ it holds by conformal invariance

$$\frac{(D_g \psi_\epsilon, \psi_\epsilon)_{B_\epsilon, g}}{\| D_g \psi_\epsilon \|^2_{L^{q_D}(B_\epsilon, g)}} = \frac{(D_\epsilon \psi, \psi)_{B, g_\epsilon}}{\| D_\epsilon \psi \|^2_{L^{q_D}(B, g_\epsilon)}},$$

where $D_\epsilon$ is the Dirac operator w.r.t. $g_\epsilon$. Further, it is

$$\mu_{q_D}(S^n) = \mu_{q_D}(\hat{B}) \geq \frac{(D^s \hat{\psi}, \hat{\psi})_{\hat{B}, g_s}}{\| D^s \hat{\psi} \|^2_{L^{q_D}(\hat{B}, g_s)}} = \frac{(D^{flat} \psi, \psi)_{B, g_E}}{\| D^{flat} \psi \|^2_{L^{q_D}(B, g_E)}},$$

where $\hat{B} \subset S^n$ corresponds to $B \subset \mathbb{R}^n$ under the stereographic projection, $D^s$ is the Dirac operator on the sphere and $\hat{\psi}$ is the spinor corresponding to $\psi$ w.r.t. the identification of the bundles. The first equality holds due to lemma 2.1.3.ii, the second inequality is just due to definition and the last equality is given by the conformal invariance of the quotient $\mu_{q_D}$.
In the next step, it will be shown that

$$\left| \frac{(D_\epsilon \psi, \psi)_{B, g_\epsilon}}{\| D_\epsilon \psi \|^2_{L^{q_D}(B, g_\epsilon)}} - \frac{(D^{flat} \psi, \psi)_{B, g_E}}{\| D^{flat} \psi \|^2_{L^{q_D}(B, g_E)}} \right| \to 0 \text{ as } \epsilon \to 0, \tag{2.6}$$

The reason turns out to be that on a fixed ball $B$ the metrics $g_\epsilon$ tend to the Euclidean metric $g_E$ for $\epsilon \to 0$, the same is true for the corresponding Dirac operators $D_\epsilon, D^{flat}$. This can be obtained by the development of $g$ in the rescaled geodesic normal coordinates and is done in appendix A.
Thus, it follows from (A.4)

$$\left| \| D_\epsilon \psi \|^{q_D}_{L^{q_D}(B, g_\epsilon)} - \| D_\epsilon \psi \|^{q_D}_{L^{q_D}(B, g_E)} \right| \leq \int_B |D_\epsilon \psi|^{q_D} (\mathrm{dvol}_{g_\epsilon} - \mathrm{dvol}_{g_E}) \to 0 \text{ for } \epsilon \to 0$$

and from (A.5)

$$|D_\epsilon\psi|^{q_D} - |D^{flat}\psi|^{q_D} \to 0 \text{ for } \epsilon \to 0.$$

Hence,

$$\begin{aligned}
&| \parallel D_\epsilon\psi \parallel^2_{L^q(B,g_\epsilon)} - \parallel D^{flat}\psi \parallel^2_{L^q(B,g_E)} | \\
&\le | \parallel D_\epsilon\psi \parallel^2_{L^q(B,g_\epsilon)} - \parallel D_\epsilon\psi \parallel^2_{L^q(B,g_E)} | + | \parallel D_\epsilon\psi \parallel^2_{L^q(B,g_E)} - \parallel D^{flat}\psi \parallel^2_{L^q(B,g_E)} | \\
&\to 0 \text{ for } \epsilon \to 0.
\end{aligned}$$

Analogously, it is obtained that

$$|(D_\epsilon\psi,\psi)_{B,g_\epsilon} - (D^{flat}\psi,\psi)_{B,g_E}| \to 0 \text{ for } \epsilon \to 0$$

and thus (2.6) is shown.
Having now a sequence of spinors $\psi^i \in C_c^\infty(B,S)$ and a sequence $\delta_i \in \mathbb{R}_{>0}$ with $\delta_i \to 0$ for $i \to \infty$ such that (2.3) becomes

$$\mu_{q_D}(S^n) - \delta_i = \frac{(D^{flat}\psi^i,\psi^i)_{B,g_E}}{\parallel D^{flat}\psi^i \parallel^2_{L^{q_D}(B,g_E)}} = \lim_{\epsilon\to 0} \frac{(D_\epsilon\psi^i,\psi^i)_{B,g_E}}{\parallel D_\epsilon\psi^i \parallel^2_{L^{q_D}(B,g_E)}} \le \lim_{\epsilon\to 0} \mu_{q_D}(B_\epsilon, g)$$

and so $\mu_{q_D}(S^n) \le \lim_{\epsilon\to 0}\mu_{q_D}(B_\epsilon,g)$. Analogously, the converse relation is obtained when choosing a sequence of spinors $\psi^i \in C_c^\infty(B,S)$ and a sequence $\delta_i \in \mathbb{R}_{>0}$ with $\delta_\epsilon \to 0$ for $\epsilon \to 0$ such that

$$\lim_{\epsilon\to 0}\mu_{q_D}(B_\epsilon,g) - \delta_i = \lim_{\epsilon\to 0} \frac{(D_\epsilon\psi^i,\psi^i)_{B,g_E}}{\parallel D_\epsilon\psi^i \parallel^2_{L^{q_D}(B,g_E)}} = \frac{(D^{flat}\psi^i,\psi^i)_{B,g_E}}{\parallel D^{flat}\psi^i \parallel^2_{L^{q_D}(B,g_E)}} \le \mu_{q_D}(S^n).$$

Hence, the claim is shown. □

**Proposition 2.3.4.** For any Riemannian spin manifold $(M,g)$ it holds

$$\mu_{q_D}(M,g) \ge \mu_{q_D}(S^n, g_{st}).$$

*Proof:* Let $p \in M$ be fixed and choose balls $B_\epsilon$ around $p$ with radius $\epsilon$ w.r.t. $g$. Then, this sequence fulfills the assumptions of lemma 2.3.3. Thus, $\mu_{q_D}(M,g) \ge \mu_{q_D}(B_\epsilon,g) \to \mu_{q_D}(S^n,g_{st})$ for $\epsilon \to 0$. □

**Remark 2.3.5.**
**i)** $\mu_{q_D}$ of the standard sphere is given by $\frac{2}{n}\mathrm{vol}(S^n,g_{st}))^{-\frac{1}{n}}$ [3]. This result is obtained by using the Hijazi inequality ([9] and for $n=2$ [7])

$$(\lambda^+_{min})^2 \ge \frac{n}{4(n-1)}\lambda_Y,$$

where $\lambda_Y$ is the Yamabe invariant that, in case of the standard sphere, equals $n(n-1)\mathrm{vol}(S^n,g_{st}))^{2/n}$ . Further, this bound is really attained by an eigenvector to the eigenvalue $\frac{n}{2}$.
**ii)** The question arises when a manifold fulfills $\mu_{q_D}(M,g) < \mu_{q_D}(S^n,g_{st})$. In [6] it is e.g. shown that this condition is fulfilled if $(M,g)$ is a compact but not conformally flat manifold of dimension $n \ge 7$.

## 2.4 The Euler-Lagrange equation

Now the Euler-Lagrange equation (2.5) of the variational problem corresponding to $\mu_q$ shall be examined. For compact manifolds all this can be found again in [4] and here the analoga for an open smoothly bounded subset $\Omega$ of $(M, g)$ are stated.

The first lemma shows the duality between the differential equation (2.5) and another one that would arise as the Euler-Lagrange equation of the map $\mathcal{F}_q^\Omega$ under a different constraint, namely $\| \phi \|_{L^p(\Omega,g)} = 1$. Unfortunately, even on compact manifolds the supremum $\mu_q$ would then always be infinite since the Dirac operator is unbounded on both sides (see theorem 1.3.1).

**Lemma 2.4.1** (duality principle, [1] lem. 2.2.)**.** Let $p, q > 1, \lambda, \mu \in \mathbb{R}^+$ with $p^{-1} + q^{-1} = 1$ and $\lambda\mu = 1$ and $D = D_g$.
**i)** If $\phi$ satisfies

$$D\phi = \lambda|\phi|^{p-2}\phi \text{ on } \Omega, \quad \| \phi \|_{L^p(\Omega,g)} = 1, \tag{2.7}$$

then $\psi = \mu\phi$ satisfies (2.5).
**ii)** If $\psi$ satisfies (2.5), then $\phi = |D\psi|^{q-2}D\psi$ satisfies (2.7).

*Proof:*
i)

$$\begin{aligned} D(\psi - \mu|D\psi|^{q-2}D\psi) &= D(\mu\phi - \mu^q|D\phi|^{q-2}D\phi) \\ &= D(\mu\phi - \mu^q\lambda^{q-1}|\phi|^{(p-1)(q-2)}|\phi|^{p-2}\phi) \\ &= D(\mu\phi - \mu\phi) = 0 \end{aligned}$$

$$\| D\psi \|_{L^q} = \| \mu D\phi \|_{L^q} = \| |\phi|^{p-2}\phi \|_{L^q} = \| \phi \|_{L^p} = 1$$

ii)

$$\begin{aligned} D\phi - \lambda|\phi|^{p-2}\phi &= D(|D\psi|^{q-2}D\psi) - \lambda|D\psi|^{(q-1)(p-2)}|D\psi|^{q-2}D\psi \\ &= D(|D\psi|^{q-2}D\psi) - \lambda D\psi \\ &= \lambda D(\mu|D\psi|^{q-2}D\psi - \psi) = 0 \end{aligned}$$

$$\| \phi \|_{L^p} = \| |D\psi|^{q-2}D\psi \|_{L^p} = \| D\psi \|_{L^q} = 1$$

□

The next step shall be to examine whether there exists a spinor that fulfills the differential equation if the inequality in proposition 2.3.4 is strict.

**Proposition 2.4.2** (analogon to [4] thm. 4.2.2.)**.** Let $(M, g)$ be a Riemannian spin manifold of dimension $n \geq 2$ with a fixed conformal class $[g]$ and a spin structure $\sigma$ and $\Omega$ an open and smoothly bounded subset. Assume that

$$\lambda_{min}^+ = \lambda_{min}^+(\Omega, g_0, \sigma) < \lambda_{min}^+(S^n) \tag{2.8}$$

Then, there is a spinor field $\phi \in C^{(1,\alpha)}(\Omega, S)$ that is continuous on $\overline{\Omega}$ and smooth on $\Omega \setminus \phi^{-1}(0)$ such that

$$D_g\phi = \lambda^+_{min}|\phi|^{\frac{2}{n-1}}\phi \text{ on } \Omega, \quad \| \phi \|_{L^{\frac{2n}{n-1}}} = 1. \tag{2.9}$$

*Proof:* The proof follows mainly the proof of the analog statement for compact $M = \Omega$. This can be done by gluing two copies of $\Omega$ together at their boundaries to obtain a closed manifold $\Omega_2$. Now, $\overline{\Omega}$ is viewed as a compact subset of $\Omega_2$ and all the embedding theorems and regularity theorems for a compact manifold can be applied.
To proof this statement the plan goes as follows: Firstly, in lemma 2.4.3 it is shown that for $q > q_D$ the subcritical problem of $\mu_q(\Omega)$ has a weak solution $\phi_q \in C^{1,\alpha}(\Omega_2, S)$ of (2.5) such that $\phi_q = D^{-1}\psi_q$ for a $\psi_q \in C^{0,\alpha}(\Omega_2, S)$. It is convenient to consider at first the subcritical problem since $q_D$ turns out to be the critical exponent of the Rellich-Kondrakov lemma 1.4.3 such that the embedding $L^q \hookrightarrow H^2_{-1/2}$ is no more compact.
Then theorem 2.4.5 shows that the $\psi_q$ are uniformly bounded in $L^\infty$.
Let now $q \in (q_D, 2]$ be close enough to $q_D$ that $\mu_q$ is bounded by a positive constant $K$. This can always be achieved, since $\mu_2^{-1} = \lambda_1^+ < \infty$ (see remark 2.2.1.iii) and $q \mapsto \mu_q$ is non-increasing and continuous from the right (due to the continuity of $q \mapsto \| \psi \|_q$ for fixed $\psi$). Applying the regularity theorem 2.4.4 the $\psi_q$ and the $\phi_q$, respectively, are uniformly bounded in $C^{0,\alpha}(\Omega_2, g)$ and in $C^{1,\alpha}(\Omega_2, g)$, respectively, by a $q$-independent constant. Thus, there exists a subsequence $\phi_{q_i}$ with $q_i \to q_D$ for $i \to \infty$ such that $\phi_{q_i}$ converges in $C^1$ to a spinor field $\phi_{q_D}$. Then, $\phi_{q_D}$ fulfills 2.5. To conclude that $\phi_{q_D}$ is smooth on $\Omega \setminus \phi^{-1}_{q_D}(0)$ interior Schauder estimates can be applied inductively as in the closed case (see [4] thm. 3.1.16 and the remark on page 46). □

With the above notations the lemmata needed in the previous proof will be formulated:

**Lemma 2.4.3.** Let $q > q_D$. Then the supremum $\mu_q(\Omega)$ is attained by a spinor field $\phi \in C^{1,\alpha}(\Omega_2, S)$ which is a solution of (2.5).

*Proof:* The proof follows the arguments of the case of a closed manifold in [4, prop 7.4.]: Let $\phi_i \in C^\infty_c(\Omega, S)$ be a maximizing sequence for $\mu_q$, this means $\mathcal{F}_q(\phi_i) \to \mu_q$ and $\| D\phi_i \|_{L^q(\Omega,g)} = 1$. By extending every $\phi_i$ on $\Omega_2 \setminus \Omega$ by zero $\phi_i \in C^\infty(\Omega_2, S)$. Now, as in the closed case the sequence $\psi_i := D\phi_i$ is bounded in $L^q(\Omega_2)$ and hence, a subsequence of $\psi_i$ converges weakly in $L^q$ to a $\psi$. Since the spinors with $L^q$-norm equal to one form a closed subset in $L^q$, $\| \psi \|_{L^q(\Omega,g)} = \| \psi \|_{L^q(\Omega_2,g)} = 1$. Because of the compactness of the embedding $L^q(\Omega_2) \hookrightarrow H^2_{-1/2}(\Omega_2)$ for $q > q_D$ (Rellich-Kondrakov, theorem 1.4.3), a subsequence converges strongly to $\psi$ in $H^2_{-1/2}(\Omega_2)$. Since $D$ can be closed, it will be assumed w.l.o.g. that $D$ is already closed. Thus, it is $\psi \in \mathrm{im} D$. Let $\phi = D^{-1}\psi$. Now, it remains to show that $\phi$ really attains the supremum, i.e. $(\phi, D\phi)_{\Omega,g} = \mu_q(\Omega)$. Using that $\phi_i, D\phi_i$ and $D\phi$ are zero on $\Omega_2 \setminus \Omega$ this follows directly from the case of a closed manifold [4] and

$$|(\phi_i, D\phi_i)_\Omega - (\phi, D\phi)_\Omega| = |(\phi_i, D\phi_i)_{\Omega_2} - (\phi, D\phi)_{\Omega_2}|.$$

Thus, due to lemma 2.2.2 $\phi$ is a solution of $D(\phi - \mu_q|D\phi|^{q-2}D\phi) = 0$. Using the regularity theorem 2.4.4 stated below for fixed $q$, $\phi \in C^{1,\alpha}$ and $\psi \in C^{0,\alpha}$. □

The next two theorems are found in [4] for $\Omega$ being closed. They also hold for $\Omega$ being an open and smoothly bounded subset of a Riemannian spin manifold. The proofs turn out to be the same when substituting $\Omega$ always by $\Omega_2$, but in $\mu_{q_D}(\Omega)$ the $\Omega$ remains unchanged.

**Theorem 2.4.4** (regularity theorem)**.** Suppose that $\phi$ is a spinor such that $\phi = D^{-1}\psi$ with $\psi \in L^q(\Omega_2)$, $q \geq q_D$ and $\phi$ fulfills (2.5) for $\mu_q = \mu_q(\Omega)$. Further suppose that there is an $r > q_D$ such that $\| \psi \|_{L^r} < \infty$. We choose $k, K > 0$ such that $\| \psi \|_{L^r} < k$ and $\mu_q \geq K$. Then, for any $\alpha \in (0,1)$ there is a constant $C$ depending only on $(\Omega, g)$, its spin structure, $r, K, k$ and $\alpha$ with

$$\| \psi \|_{C^{0,\alpha}(\Omega_2,g)} \leq C \quad \text{and} \quad \| \phi \|_{C^{1,\alpha}(\Omega_2,g)} \leq C.$$

**Theorem 2.4.5.** Let $\phi$ be a solution of (2.5) with $q \in (q_D, 2]$ such that $\phi = D^{-1}\psi$ for a spinor $\psi$ and let $\mu_q(\Omega) \geq \mu_{q_D}(S^n, g_{st}) + \epsilon$ for $\epsilon > 0$. Then there is a $q$-independent constant $C = C(\Omega, g, \sigma, \epsilon)$ such that $\| \psi \|_{L^\infty} < C$.

# Chapter 3

# Obstruction to conformal compactification

In this chapter the conformal invariant $\mu_{q_D}$ will be used to obtain a criterion when a Riemannian spin manifold is not conformally compactifiable. A similar criterion for the Yamabe invariant was given in [11].

**Definition 3.0.6.** A *conformal compactification* of a noncompact manifold $(M, g)$ is a compact manifold $(N, h)$ such that there exists a $f \in C^\infty(N)$ with $g = f^2 h$ on $M$ with $f > 0$ on $M$ and $f = 0$ else.

For that purpose, on noncompact manifolds a new figure shall be obtained from $\lambda^+_{min}$ that is "$\lambda^+_{min}$ at infinity".

**Definition 3.0.7.** Let $(M, g)$ be a noncompact complete Riemannian spin manifold. Then

$$\overline{\lambda^+_{min}(M, g)} := \lim_{r\to\infty} \lambda^+_{min}(M \setminus B_r, g),$$

where $B_r$ is a ball of radius $r$ around a fixed $p \in M$ w.r.t. the metric $g$.

The existence of the limes and $\overline{\lambda^+_{min}(M, g)} \leq \lambda^+_{min}(S^n, g_{st})$ follows with lemma 2.1.3.i and proposition 2.3.4. Further, the limes does not depend on the centre of the balls $B_r$.

Before stating the criterion, lemma 2.3.3 will be generalized to sequences of nested neighbourhoods of a fixed point where only their volumes have to converge to zero.

**Proposition 3.0.8.** Let $(M, g)$ be a Riemannian spin manifold of dimension $n > 2$. Assume that there exists a sequence $\{\Gamma_i\}$ of smoothly bounded open subsets of $(M, g)$ with $\mathrm{vol}(\Gamma_i, g) \to 0$ and $\Gamma_i \subset \Gamma_{i-1}$ for $i \in \mathbb{N}$. Then

$$\lim_{i\to\infty} \lambda^+_{min}(\Gamma_i, g) = \lambda^+_{min}(S^n, g_{st}).$$

*Proof:* This will be proved by contradiction. Assume $\lim_{i\to\infty} \lambda^+_{min}(\Gamma_i) \neq \lambda^+_{min}(S^n)$. Since $\lambda^+_{min}(\Gamma_i) \leq \lambda^+_{min}(\Gamma_{i+1}) \leq \lambda^+_{min}(S^n)$ holds for every $i$ (see lemma 2.1.3.i and proposition 2.3.4), $\lim_{i\to\infty}\lambda^+_{min}(\Gamma_i)=\lambda^+_{min}(S^n)-c$ will be assumed for a positive constant $c$. Due to theorem 2.4.2 (and since $\overline{\Gamma}_i$ is compact) there exists a spinor field $\phi_i\in C(\overline{\Gamma}_i,S)\cap C^{1,\alpha}(\Gamma_i,S)$ with

$$D_g\phi_i=\lambda_i|\phi_i|^{\frac{2}{n-1}}\phi_i \text{ on } \Gamma_i, \quad \parallel\phi_i\parallel_{L^{p_D}}=1, \tag{3.1}$$

where $\lambda_i:=\lambda^+_{min}(\Gamma_i)$, $p_D:=\frac{2n}{n-1}$. From $\parallel\phi_i\parallel_{L^{p_D}}=1$ and $\mathrm{vol}(\Gamma_i,g)\to 0$ as $i\to\infty$, $\parallel\phi_i\parallel_{L^\infty}\to\infty$ follows. Then, there is a sequence of points $s_i\in\Gamma_i$ with

$$m_i:=|\phi_i(s_i)|=\max\{|\phi_i(x)|\mid x\in\Gamma_i\}. \tag{3.2}$$

For $i\to\infty$ $m_i$ also diverges. Since $\overline{\Gamma}_i$ is compact, w.l.o.g. $s_i\to p\in\Gamma_1$ will be assumed. Let now rescaled geodesic normal coordinates on a fixed ball $\tilde{B}$ around $p$ be defined by

$$\sigma_i(x)=\exp_p(\delta_i x)$$

with $\sigma_i\restriction_{\tilde{B}}$ being a diffeomorphism and $\delta_i:=m_i^{-\frac{2}{n-1}}\to 0$. Then the function $\tilde{\phi}_i(x):=m_i^{-1}\phi_i\circ\sigma_i(x)$ defined on $\widetilde{\Gamma}_i:=\sigma_i^{-1}(\Gamma_i\cap\sigma_i(\tilde{B}))\subseteq\mathbb{R}^n$ fulfills

$$\parallel\tilde{\phi}_i\parallel_{L_p(\tilde{\Gamma}_i,g_i)}=\parallel m_i^{-1}\phi_i\circ\sigma_i\parallel_{L_p(\tilde{\Gamma}_i,g_i)}=\parallel\phi_i\parallel_{L_p(\Gamma_i\cap\sigma_i(\tilde{B}),g)}\leq 1$$

and

$$\begin{aligned} D_i\tilde{\phi}_i &= D_i(m_i^{-1}\phi_i\circ\sigma_i)=m_i^{-\frac{n+1}{n-1}}D_g(\phi_i\circ\sigma_i)\\ &= m_i^{-\frac{n+1}{n-1}}\lambda_i|\phi_i\circ\sigma_i|^{p_D-2}\phi_i\circ\sigma_i=\lambda_i|\tilde{\phi}_i|^{p_D-2}\tilde{\phi}_i, \end{aligned}$$

where $D_i$ is the Dirac operator associated to the metric $g_i:=\delta_i^{-2}\sigma_i^*(g)$.

By stereographic projection each $\tilde{\phi}_i$ on $\tilde{\Gamma}_i\subseteq\mathbb{R}^n$ is mapped to $\hat{\phi}_i$ on $\hat{\Gamma}_i\subset S^n$. Using lemma 2.1.3.iii, lemma 2.1.6 and the conformal invariance of $\mathcal{F}_{q_D}$

$$\mu_{q_D}(S^n)=\mu_{q_D}(\widehat{\Gamma}_i)\geq\frac{(\hat{\phi}_i,D^{S^n}\hat{\phi}_i)_{\hat{\Gamma}_i}}{\parallel D\hat{\phi}_i\parallel^2_{L^{q_D}(\hat{\Gamma}_i)}}=\frac{(\tilde{\phi}_i,D^{flat}\tilde{\phi}_i)_{\tilde{\Gamma}_i}}{\parallel D^{flat}\tilde{\phi}_i\parallel^2_{L^{q_D}(\tilde{\Gamma}_i)}}$$

is obtained. It remains to show that

$$\left|\frac{(\tilde{\phi}_i,D^{flat}\tilde{\phi}_i)_{\tilde{\Gamma}_i,g_E}}{\parallel D^{flat}\tilde{\phi}_i\parallel^2_{L^{q_D}(\tilde{\Gamma}_i)}}-\frac{(\tilde{\phi}_i,D_i\tilde{\phi}_i)_{\tilde{\Gamma}_i,g_i}}{\parallel D_i\tilde{\phi}_i\parallel^2_{L^{q_D}(\tilde{\Gamma}_i)}}\right|\to 0$$

for $i\to\infty$. This is the same as (2.6) in proposition 2.3.4. Then with

$$\frac{(\tilde{\phi}_i,D_i\tilde{\phi}_i)_{\tilde{\Gamma}_i,g_i}}{\parallel D_i\tilde{\phi}_i\parallel^2_{L^{q_D}(\tilde{\Gamma}_i)}}=\frac{(\tilde{\phi}_i,\lambda_i|\tilde{\phi}_i|^{p_D-2}\tilde{\phi}_i)_{\tilde{\Gamma}_i,g_i}}{\parallel\lambda_i|\tilde{\phi}_i|^{p_D-2}\tilde{\phi}_i\parallel^2_{L^{q_D}(\tilde{\Gamma}_i)}}=\lambda_i^{-1}\frac{\parallel\tilde{\phi}_i\parallel^2_{L^{p_D}(\tilde{\Gamma}_i)}}{\parallel\tilde{\phi}_i\parallel^{2p/q}_{L^{p_D}(\tilde{\Gamma}_i)}}\geq\lambda_i^{-1}$$

the contradiction $\mu_{q_D}(S^n)\geq\lim_{i\to\infty}\lambda_i^{-1}$ is obtained. $\square$

**Proposition 3.0.9** (analogon to [11] thm. 3.1.)**.** Let $(M,g)$ be a noncompact complete Riemannian spin manifold of dimension $n > 2$ with $\overline{\lambda^+_{min}(M,g)} < \lambda^+_{min}(S^n, g_{st})$. Then $(M,g)$ is not pointwise conformal to a subdomain of any compact Riemannian spin $n$-manifold.

*Proof:* The statement will be proved by contradiction in fully analogy to the proof for the Yamabe invariant in [11]. Assume that $(M,g)$ is pointwise conformal to a subdomain $(M, u^{\frac{4}{n-1}} g)$ of a compact Riemannian spin manifold $(K,h)$, where $u \in C^\infty(M,S)$. Take smooth compact domains $X_i$ in $M$ with $X_i \subset \overline{X_i} \subset X_{i+1}$ such that $\mathrm{vol}(M \setminus X_i, u^{\frac{4}{n-1}} g) \to 0$ for $i \to \infty$. Since $\overline{\lambda^+_{min}(M,g)} < \lambda^+_{min}(S^n)$ is assumed, there has to exist a spinor field $\phi_i \in C^\infty_c(M \setminus X_i, S)$ with

$$\frac{\| D_g\phi_i \|^2_{L^q}}{(D_g\phi_i, \phi_i)_g} \le \lambda^+_{min}(S^n) - c$$

for a positive constant $c$ and for all $i \in \mathbb{N}$. Take smoothly bounded open subsets $Y_i$ of $M$ with $X_i \subset \overline{X_i} \subset Y_i$ and $\mathrm{supp}\, \phi_i \subset Y_i \setminus X_i \subset M \setminus X_i$. By the conformal invariance of $\lambda^+_{min}$ it follows

$$\lambda^+_{min}(Y_i \setminus \overline{X_i}, h) = \lambda^+_{min}(Y_i \setminus \overline{X_i}, g) \le \frac{\| D_g\phi_i \|^2_{L^q}}{(D_g\phi_i, \phi_i)_g} \le \lambda^+_{min}(S^n) - c.$$

Since the volume $\mathrm{vol}(Y_i \setminus \overline{X_i}, u^{\frac{4}{n-1}} g) \to 0$, this contradicts proposition 3.0.8. □

**Example 3.0.10.** Let $M = \mathbb{R}^n$ $(n \ge 7)$ be equipped with the subsequently defined metric $g$. On each ball $B_2(p_m)$ around $p_m = (5m, 0) \in \mathbb{R}^n$ with radius 2 the metric $g$ is chosen such that it is $g_E$ on $B_2(p_m) \setminus B_1(p_m)$ but not conformally flat at the centre. Everywhere outside the balls the metric $g$ is also selected to coincide with $g_E$.
Since the balls are flat in a neighbourhood of their boundary, for every $m$ there exists a map $h : (B_2(p_m), g) \to (S^n, \tilde{g})$ such that $h$ is a conformal compactification with $h^*\tilde{g} = g$ and $h(B_2(p_m)) = S^n \setminus \{p\}$ with fixed $p \in S^n$.
Then using lemma 2.1.3.ii it is

$$\lambda^+_{min}(B_2(p_m), g) = \lambda^+_{min}(S^n \setminus \{p\}, \tilde{g}) = \lim_{\epsilon \to 0} \lambda^+_{min}(S^n \setminus B_\epsilon(p), \tilde{g}) = \lambda^+_{min}(S^n, \tilde{g}).$$

Since $g$ and therewith $\tilde{g}$ is not conformally flat a, $\lambda^+_{min}(S^n, \tilde{g}) < \lambda^+_{min}(S^n, g_{st})$ (cf. remark 2.3.5.ii).
Further, for every $r$ there exists an $m \in \mathbb{N}$ such that $B_2(p_m) \subset M \setminus B_r(0)$. Due to lemma 2.1.3.i it holds $\lambda := \lambda^+_{min}(B_2(p_m), g) \ge \lambda^+_{min}(M \setminus B_r(0), g)$. Using $\lambda < \lambda^+_{min}(S^n, g_{st})$ as verified above the estimation $\overline{\lambda^+_{min}(M,g)} < \lambda^+_{min}(S^n, g_{st})$ is obtained.
Thus, $(M,g)$ is not conformally compactifiable.

# Appendix A

# Development in geodesic normal coordinates

Let the exponential map $\exp_p$ at $p \in M$ be defined on a neighbourhood $U \subset T_pM \cong \mathbb{R}^n$ and let $(x_1, \dots, x_n)$ denote the corresponding normal coordinates. Further, define the map

$$G: \; V \to S^2_+(n, \mathbb{R}); \quad m \mapsto G_m := (g_{ij}(m))_{ij},$$

where $G_m$ is the matrix of the coefficients of the metric $g$ at $m$ in the basis $\partial_i := \frac{\partial}{\partial x^i}$ and $S^2_+(n, \mathbb{R})$ is the set of all real, symmetric and positive definite $n \times n$ matrices. Thus, there exists exactly one symmetric positive-definite matrix $B_m = (b^j_i(m))_{ij}$ with $B^2_m = G^{-1}_m$.
For each $m \in M$ the matrix $B_m$ gives rise to the isometry

$$B_m : (T_{\exp_p^{-1}(m)} U \cong \mathbb{R}^n, g_E) \to (T_m V, g_m); \quad (a^1, \dots, a^n) \mapsto \sum_{i,j} b^j_i(m) a^i \partial_j(m),$$

since $g_m(\sum_i b^i_k \partial_i, \sum_j b^j_l \partial_j) = \sum_{i,j} b^i_k b^j_l g_m(\partial_i, \partial_j) = \sum_{i,j} b^i_k b^j_l g_{ij} = \delta_{kl} = g_E(\partial_k, \partial_l)$. This map is used to identify the $\mathrm{SO}(n)$-principal bundles $P_{\mathrm{SO}(n)} U_{g_E}$ and $P_{\mathrm{SO}(n)} V_g$ that lifts to an identification of the corresponding $\mathrm{Spin}(n)$-principal bundles and thus, of the spinor bundles (see [5])

$$S_{U,g_E} \to S_{V,g}; \quad \psi \mapsto \overline{\psi}.$$

Further, let $\nabla$ and $\overline{\nabla}$, respectively, denote the Levi-Civita connections on $(TU, g_E)$ and $(TM, g)$ as well as the lifted connections on the spinor bundles $S_{U,g_E}$ and $S_{V,g}$, respectively.

Firstly, the metric shall be developed in the geodesic normal coordinates $(x_1, \dots, x_n)$ in the neighbourhood $V \subset M$ around a fixed point $p \in M$. The derivation of the subsequent can be found in [13].

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{i\alpha\beta j}(p) x^\alpha x^\beta + \frac{1}{6} R_{i\alpha\beta j;\gamma}(p) x^\alpha x^\beta x^\gamma + \mathcal{O}(r^4), \tag{A.1}$$

where

$$R_{ijkl} = \langle \nabla_{e_j} \nabla_{e_i} e_k, e_l \rangle - \langle \nabla_{e_i} \nabla_{e_j} e_k, e_l \rangle - \langle \nabla_{[e_j, e_i]} e_k, e_l \rangle$$

for the orthonormal frame $(e_1, \ldots, e_n)$ of $(TV, g)$ with $e_i := b_i^j \partial_j$.

In the next step, the Dirac operators will be compared.
For this purpose, let $D$ ($\overline{D}$) denote the Dirac operators acting on $\Gamma(S_{U,g_E})$ ($\Gamma(S_{V,g})$).
It holds [5]

$$\overline{D}\,\overline{\psi} = \overline{D\psi} + \frac{1}{4} \sum_{ijk} \tilde{\Gamma}_{ij}^k e_i \cdot e_j \cdot e_k \cdot \overline{\psi}, \tag{A.2}$$

where $\tilde{\Gamma}_{ij}^k := -\langle \overline{\nabla}_{e_i} e_j, e_k \rangle$. Thus, one needs a development of $\tilde{\Gamma}_{ij}^k$:

$$\tilde{\Gamma}_{ij}^k = \partial_i b_j^k - \frac{1}{3}(R_{ik\alpha j} + R_{i\alpha kj}) x^\alpha + \mathcal{O}(r^2). \tag{A.3}$$

This calculation is implemented in [5].

Now the map $\sigma_\epsilon(x) = \exp_p(\epsilon x)$ is considered on a fixed ball $B \subset T_pM$ such that $\exp_p$ (and hence all $\sigma_\epsilon$ for $\epsilon \leq 1$) restricted to $B$ are diffeomorphisms. Define the metric $g_\epsilon = \epsilon^{-2} \sigma_\epsilon^*(g)$. With $g = g_{ij} dx^i dx^j$ it is

$$g_\epsilon = (g_{ij} \circ M_\epsilon) dx^i dx^j,$$

where $M_\epsilon$ denotes the multiplication with the scalar $\epsilon$. The corresponding Dirac operators will be denoted by $D_\epsilon$. With these preparations the following can be proved.

**Lemma A.0.11.** In the above notations and for a function $f : U \subset M \to \mathbb{R}$ it holds for $\epsilon \to 0$

$$\Big| \int_U f \mathrm{dvol}_{g_\epsilon} - \int_U f \mathrm{dvol}_{g_E} \Big| \to 0. \tag{A.4}$$

Further, for a spinor $\psi \in C^\infty(U, S)$ and $\epsilon \to 0$ it is

$$|D_\epsilon \psi - D^{flat} \psi| \to 0. \tag{A.5}$$

*Proof:* It suffices to prove the claim for a chart $\kappa : U \to \mathbb{R}^n$. Then, it holds:

$$\Big| \int_U f \mathrm{dvol}_{g_\epsilon} - \int_U f \mathrm{dvol}_{g_E} \Big| = \Big| \int_{\kappa(U)} (f \circ \kappa^{-1}) (\sqrt{|\det g_{ij} \circ M_\epsilon|} - 1) d^n x \Big|.$$

With (A.1) it follows $|\sqrt{|\det g_{ij} \circ M_\epsilon|} - 1| \to 0$ for $\epsilon \to 0$ and, thus, (A.4) is obtained.
Further, using (A.2) it is obtained that

$$|D_\epsilon \psi - D^{flat} \psi| = \frac{1}{4} \Big| \sum_{ijk} (\tilde{\Gamma}_{ij}^k)_{g_\epsilon} e_i \cdot e_j \cdot e_k \cdot \psi \Big|$$

with $(\tilde{\Gamma}_{ij}^k)_{g_\epsilon}$ denoting the Christoffel symbols for $g_\epsilon$. Since $(g_\epsilon)_{ij} = g_{ij} \circ M\epsilon$ implies that the entries of its positive definite square root are given by $b_{ij} \circ M_\epsilon$ and with (A.3) the claim is obtained. □

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