Large curvature on open manifolds

Nadine Große

Universität Leipzig

(joint with: Marc Nardmann (Hamburg))

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Motivation

Surface M - e.g. plane, sphere, torus

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Gauss curvature K:
$$\mathrm{vol}(B_\epsilon(p) \subset M) = \mathrm{vol}(B_\epsilon(0) \subset \mathbb{R}^2)\left(1 - \frac{K(p)}{12}\epsilon^2 + O(\epsilon^3)\right)$$

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Surface M - e.g. plane $K = 0$, sphere $K = const > 0$, torus

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- **Ricci curvature:** $\mathrm{Ric}_g : T_pM \otimes T_pM \to \mathbb{R}$
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- local:
- global:

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noncompact connected M:

On each M there is a metric whose scal is everywhere positive (negative).

Gromov's relative *h*-principle - Black Box

M noncompact and connected

- *h*-principle implies:
  For constants $c_1 < c_2$ each of the relations $\mathrm{scal} > c_1$, $\mathrm{scal} < c_2$, $c_1 < \mathrm{scal} < c_2$ can be fulfilled. Analog for Ric, $\sec$.

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- relative h-principle implies:
  Let $B \subset M$ be a closed subset of M such that $M \setminus B$ has exits to infinity. Let there be a metric g on M that on B fulfills a relation as above. Then there is a metric g' fulfilling the same relation on all of M and $g'|_B = g|_B$.

($M \setminus B$ has exits to infinity = each connected component of $M \setminus B$ is <u>not</u> relatively compact in M)

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Today: Can curvature grow in an arbitrary way?

Enlarging scal_g - First version

Theorem (G.-Nardmann '12)

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$$\mathrm{scal}_g > f \quad \text{on } M.$$

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Mixture out of explicit constructions on 'cylinder' and Gromov's relative h-principle

cylinders: to enlarge the curvature
h-principle: to save curvature inequalities when topology changes

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- near $N_i := \partial M_i$ cyl: $N_i \times I$

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Basic idea: $c_i := \max_{M_i' := M_i \cup (N_i \times I)} f$

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$\bar{g} = e^{2h(t)}g$:
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Enlarging scal_g - Second Version

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Let (M, g^0) be a noncompact connected manifold of dim $n \geq 2$. Let A be a closed codimension-0 submanifold-with-boundary, such that $M \setminus A$ has exits to infinity. Let $f \in C^\infty(M, \mathbb{R})$ with $\mathrm{scal}_{g^0} = f$ on A. Then there is a metric g on M with

$$g = g^0 \text{ on } A \text{ and } \quad \mathrm{scal}_g > f \text{ on } M \setminus A.$$

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$$\mathrm{ric}_g(X) := \frac{\mathrm{Ric}_g(X,X)}{|X|_g^2}, \quad X \in T_xM \setminus \{0\}$$

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