Invertible Dirac operators and handle attachments

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(joint work with Mattias Dahl (Stockholm))

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Motivation

- Not every closed manifold admits a metric of positive scalar curvature.
- In contrast on every closed manifold (dim≥ 3 the space of metric with negative scalar curvature is nonempty and contractable.
- Topological obstruction for psc-metrics: (M, g) closed spin, Dirac operator D^g

Lichnerowicz formula

$$(D^g)^2 = \Delta_g + \frac{\mathrm{scal}_g}{4}$$

$\mathrm{scal}_g > 0 \Rightarrow D^g$ is invertible

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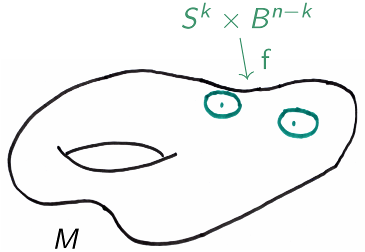
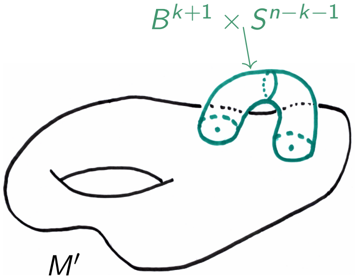
- $\mathrm{Metr}^{\mathrm{psc}}(M) \subset \mathrm{Metr}^{\mathrm{inv}}(M) \subset \mathrm{Metr}(M)$

Construction of manifolds admitting psc-metrics
- Review Surgery



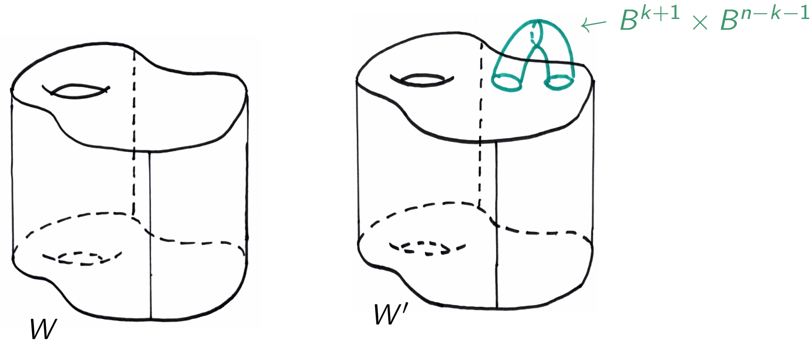


- embedding $f : S^k \times B^{n-k} \to M$
  $S := f(S^k \times \{0\})$ - surgery sphere
- $\partial(M \setminus f(S^k \times B^{n-k})) \cong S^{k-1} \times S^{n-k-1}$
- $M' = (M \setminus f(S^k \times B^{n-k})) \sqcup_{\sim} B^{k+1} \times S^{n-k-1}$

M' is obtained from M by a surgery of dim k / codim $n-k$.

Construction of manifolds admitting psc-metrics

- Review Surgery

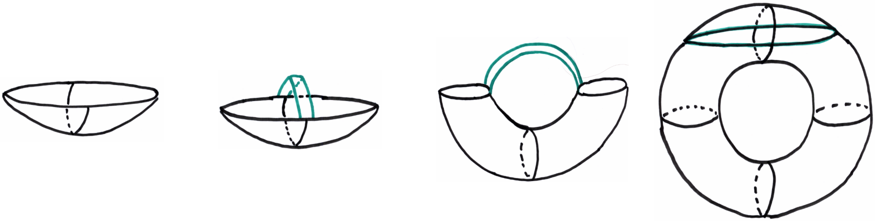


- View the cylinder $W := M \times [0,1]$ as a bordism from M to itself
- Attach $B^{k+1} \times B^{n-k-1}$ to $M \times \{1\}$
- W' is a bordism from M to M'.

W' is obtained from W by attaching a $(k+1)$-handle.

Construction of manifolds admitting psc-metrics
- Review Surgery

Each closed manifold has a handle decomposition.

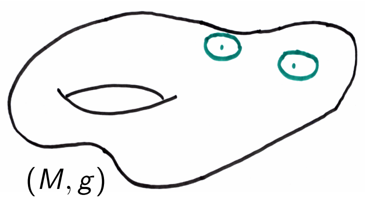


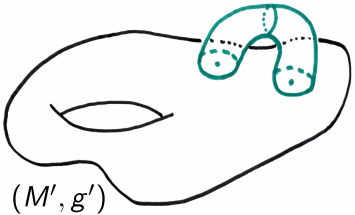
$$B^2 \ + \text{1-handle} \ + \text{1-handle} \ + B^2 \ = \ T^2$$

Construction of manifolds admitting psc-metrics

Theorem (Gromov, Lawson / Schoen, Yau; '80)

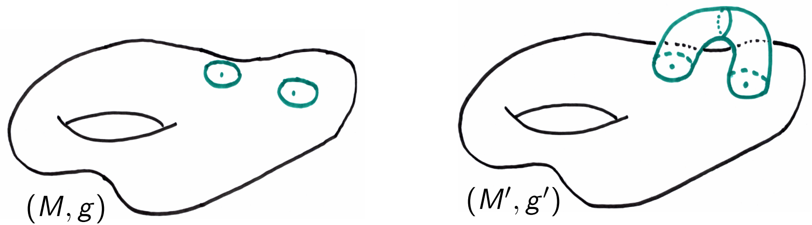
Let (M, g) be a closed Riemannian manifold with $g \in Metr^{psc}(M)$. Let M' be obtained from M by a surgery of codimension ≥ 3. Then, M' admits a psc-metric g'.





g' can be chosen such that it coincides with g outside a small neighbourhood around the surgery sphere.

Construction of manifolds admitting psc-metrics



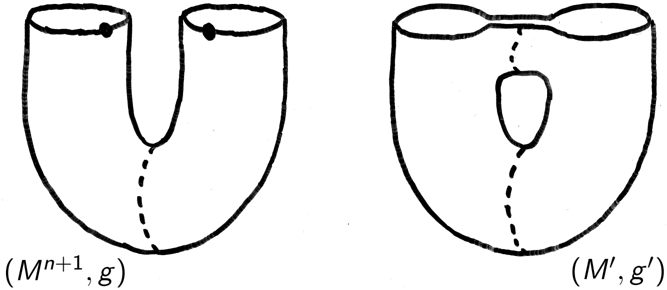
Intuition

- psc is a local property
- codim $n-k \geq 3 \;=\;$ gluing in $B^{k+1} \times S^{n-k-1 \geq 2}$
- standard product structure on $B^{k+1} \times S^{n-k-1 \geq 2}$ has psc

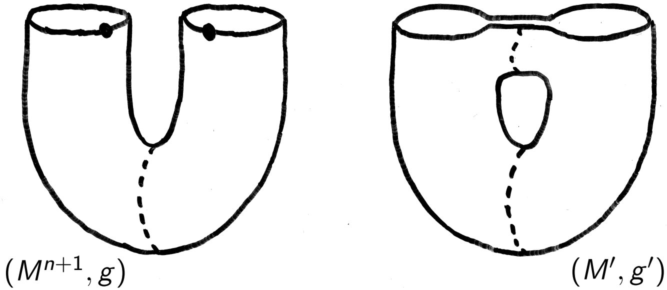
Psc-metrics and handle attachments

Theorem (Carr '88 / Gajer '87)

Let (M^{n+1}, g) be a compact Riemannian manifold with closed boundary ∂M, $g \in Metr^{psc}(M)$ and g having product structure near ∂M. Let M' be obtained from M by adding a $(k+1)$-handle of codimension $n-k \geq 3$. Then, M' admits a psc-metric g' that is again product near the (new) boundary.



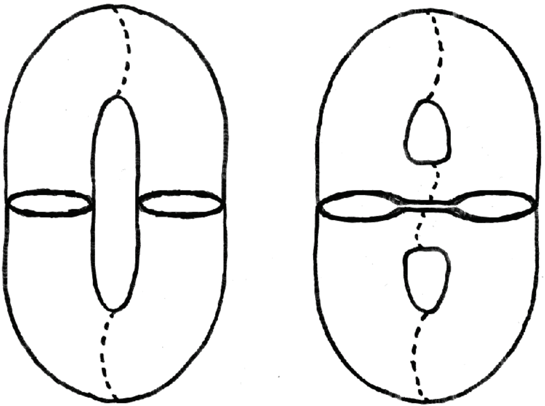
Psc-metrics and handle attachments



Intuition

- On the boundary: surgery of codim $n - k \geq 3$

Psc-metrics and handle attachments



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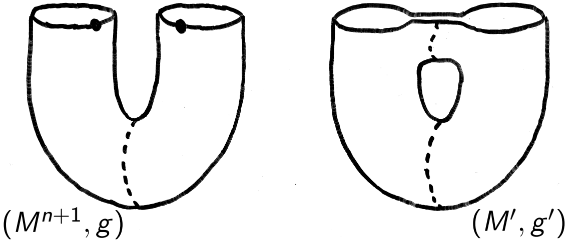

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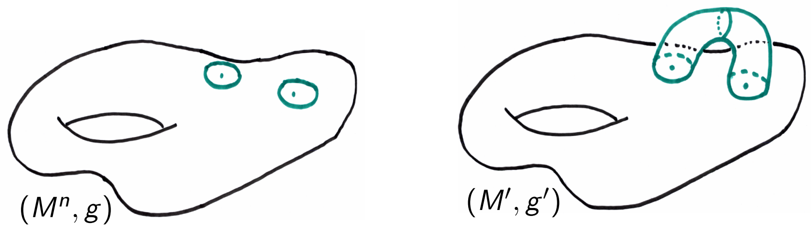
Implication

- $\mathrm{Metr}^{\mathrm{psc}}(S^{4k-1})$ has infinitely many components ($k \geq 2$) ($\mathrm{Metr}^{\mathrm{psc}}(S^3)$ is connected (Marques, 2011))

$\mathbf{Metr}^{\mathbf{inv}}(M)$

What can be done for metrics with invertible Dirac operators?

Surgery for $\mathrm{Metr}^{\mathrm{inv}}(M)$



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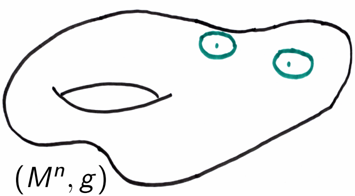

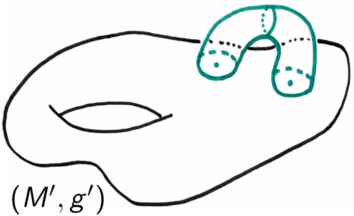

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  - $S^k \times B^{n-k}$ - induces spin structure on $S^k \times S^{n-k-1}$
  - glue in $B^{k+1} \times S^{n-k-1}$
    Its boundary should carry same spin structure.

  For $k = 1$, two spin structures on S^1 - we only allow the one that bounds the disk.

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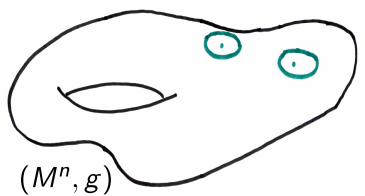

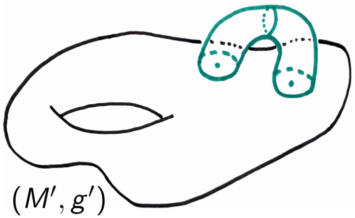

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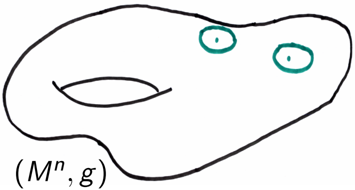
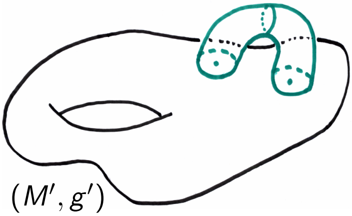
  For $k = 1$, two spin structures on S^1 - we only allow the one that bounds the disk.
- $f : S^k \times B^{n-k} \to M$ spin-preserving embedding.

Surgery for $\mathrm{Metr}^{\mathrm{inv}}(M)$





Intuition

- Invertible Dirac operator is a global condition.
- codim $n - k \geq 3 \;=\;$ gluing in $B^{k+1} \times S^{n-k-1 \geq 2}$
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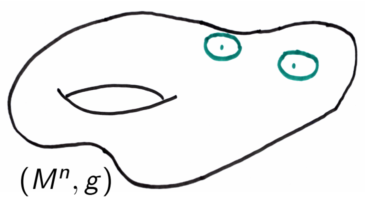

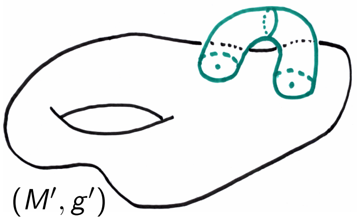

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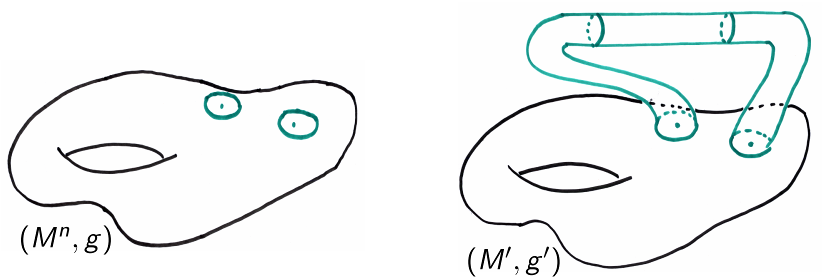

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- standard product structure on $\mathbb{R}^{k+1} \times S^{n-k-1 \geq 1}$ has invertible Dirac operator ('When taking the right S^1')
- 'If the inserted cylinder is large enough, invertibility survives.'

Construction for manifolds admitting inv-metrics

Theorem (Ammann, Dahl, Humbert; 2009)

Let (M, g) be a closed Riemannian spin manifold with $g \in Metr^{inv}(M)$. Let M' be obtained from M by a surgery of codimension ≥ 2. Then, M' admits a metric g' such that $\dim\ker D^{g'} \leq \dim\ker D^{g}$. Moreover, g' can be chosen such that it coincides with g outside a small nbh around the surgery sphere.

Consequences

$$\dim\ker D^{g} \geq \begin{cases} |\hat{A}(M)| & \text{if } n \equiv 0 \mod 4 \\ 1 & \text{if } n \equiv 1 \mod 8, \quad \alpha(M) \neq 0 \\ 2 & \text{if } n \equiv 2 \mod 8, \quad \alpha(M) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

For a generic metric, equality is attained.

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dep. on topology of M

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dep. on topology of M <u>and</u> spin str.

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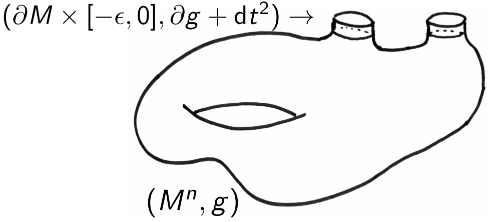
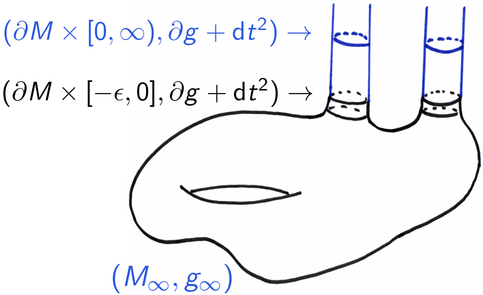
Inv-metrics on manifolds with boundary



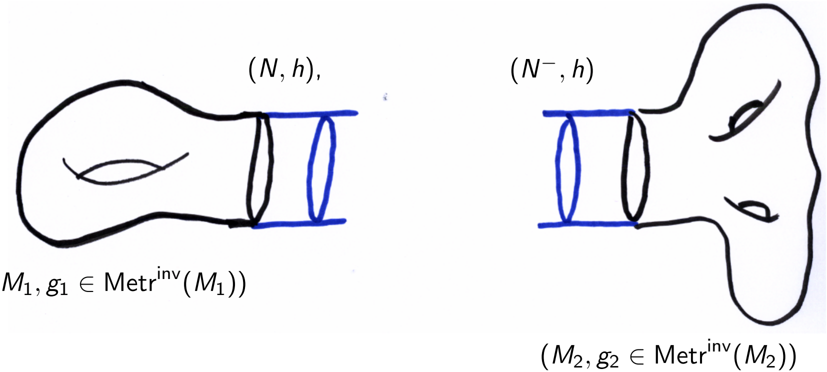
When do we call D^g invertible?

Inv-metrics on manifolds with boundary



$g \in \mathrm{Metr}^{\mathrm{inv}}(M)$ iff D^{g_∞} is invertible as operator on $L^2(M_\infty, S)$

Inv-metrics on manifolds with boundary



If M_1 and M_2 are glued together using a large enough cylinder $(N \times [-R, R], h + \mathrm{d}t^2)$, the resulting metric has again invertible Dirac operator.

Inv-metrics + handle attachments

If you have a codim ≥ 2 handle attachment result for inv-metrics,...

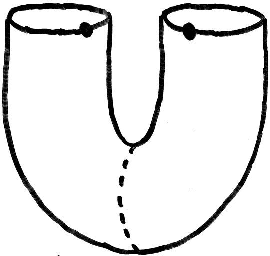
Applications

- $\mathrm{Metr}^{\mathrm{inv}}(S^{4k+3})$ has infinitely many connected components for $k \geq 0$
- genericity result
- „toy model“ for concordance theory

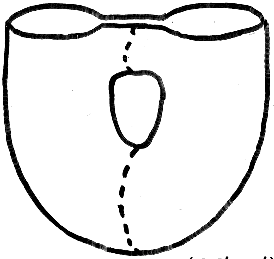
Inv-metrics and handle attachments

Theorem (Dahl, G. 2012)

Let (M^{n+1}, g) be a compact Riemannian spin manifold with closed boundary ∂M, $g \in Metr(M)^{inv}$ and g having product structure near ∂M. Let M' be obtained from M by adding a $(k+1)$-handle of codimension $n - k \geq 2$. Then, M' admits a inv-metric g' that is again product near the (new) boundary.



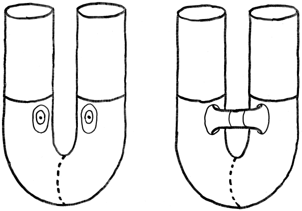
(M^{n+1}, g)



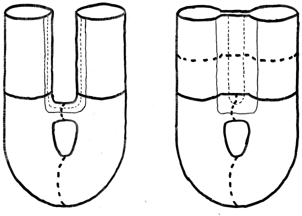
(M', g')

Strategy and Methods

- 'Topological strategy' - Decompose the handle attachment



surgery of codim $n-k$

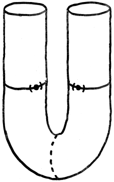


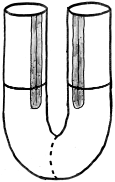
'half' surgery of codim $n-k+1$

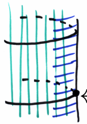
glue in '$\frac{1}{2}B^{k+1} \times S^{n-k}$'

Metric strategy

- Approx. by 'double' product metrics near the surgery sphere







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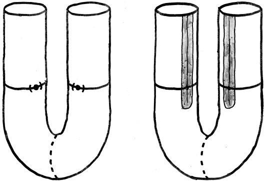
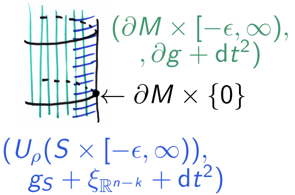
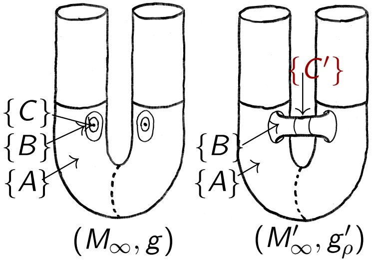


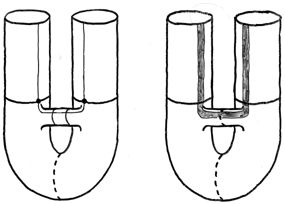


- First surgery

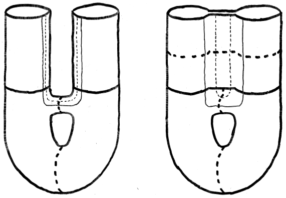


Metric strategy

- Again approx. by 'double' product metrics



- Second surgery



Proof of the single steps - Sketch

- 'Parameter for tuning': ρ
- Proof by contradiction: $\rho_i \to 0$, $g_{\rho_i} \notin \mathrm{Metr}^{\mathrm{inv}}(M')$

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- **(a priori estimate)** $\phi \neq 0$

An Application

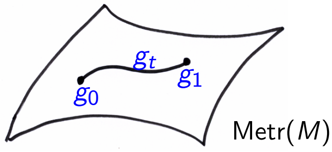
Theorem (Dahl, G.; 2012)

Let M be a closed 3-dimensional Riemannian spin manifold and $g \in Metr^{inv}(M)$. Then there are metrics $g^i \in Metr^{inv}(M)$, $i \in \mathbb{N}$, such that g^i is bordant to g but g^i is not concordant to g^j for $i \neq j$.
In particular, $Metr^{inv}(M)$ has infinitely many connected components.

Notations

Inv-metrics are indicated in blue.

- g_0, g_1 are *isotopic iff*



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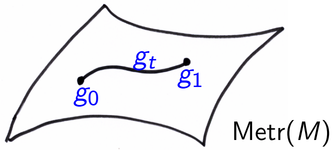

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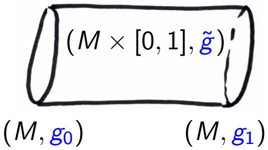

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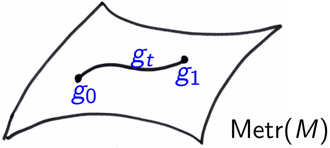

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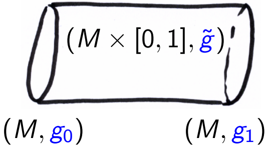

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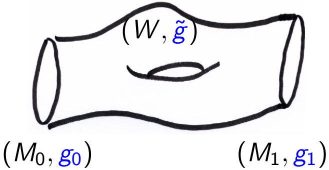

An application

Theorem (Dahl, G.; 2012)

Let M be a closed 3-dimensional Riemannian spin manifold and $g \in Metr^{inv}(M)$. Then there are metrics $g^i \in Metr^{inv}(M)$, $i \in \mathbb{N}$, such that g^i is bordant to g but g^i is not concordant to g^j for $i \neq j$.
In particular, $Metr^{inv}(M)$ has infinitely many connected components.

Lemma

There exist 4-manifolds $(Y^i, \tilde{h}^i)$ ($i \in \mathbb{N}$) with $\tilde{h}^i \in Metr^{inv}(Y^i)$, $\partial Y^i = S^3$ such that $\alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i-j)$ for a constant $c \neq 0$.

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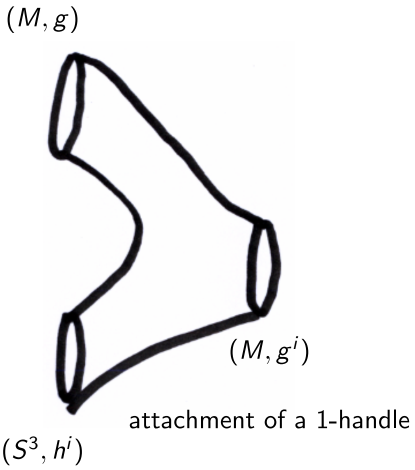
There exist 4-manifolds $(Y^i, \tilde{h}^i)$ $(i \in \mathbb{N})$ *with* $\tilde{h}^i \in Metr(Y^i)^{inv}$, $\partial Y^i = S^3$ *such that* $\alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i-j)$ *for a constant* $c \neq 0$.

Construction:

- Y^0 - B^4 with a 'torpedo metric' $\tilde{h}^0 \in \mathrm{Metr}^{\mathrm{psc}}(B^4)$ and $\tilde{h}^0|_{S^3} =$ stand. metr.
- $Y^i = \underbrace{(K3\#K3\#\cdots\#K3)}_{i \text{ times}} \setminus B^4 = Y^0 +$ several 2-handles

  $h^i := \tilde{h}^i|_{S^3}$
- $\alpha(Y^i \cup_{S^3} (Y^j)^-) = \alpha(\#_{(i-j)} K3) = (i-j)\alpha(K3) \neq 0$ for $i \neq j$

Constructions of g^i



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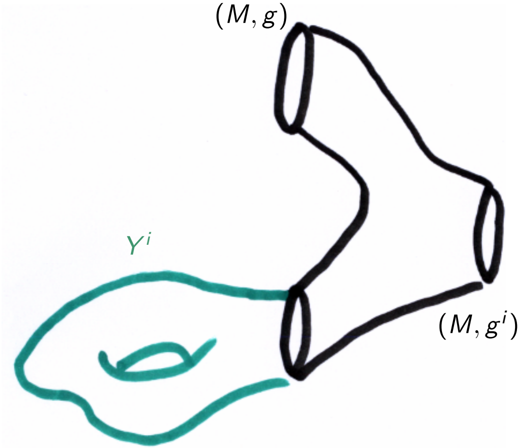
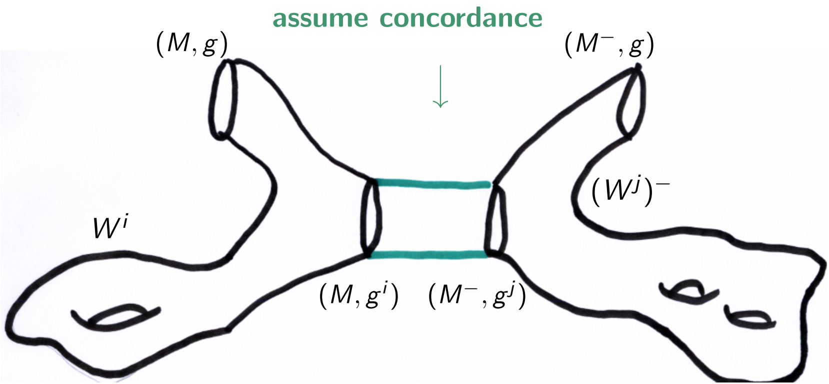


(M,g) and (M,g^i) are bordant.
Bordism $(W^i,\tilde{g}^i) \in \mathrm{Metr}(W^i)^{\mathrm{inv}}$

Constructions of g^i



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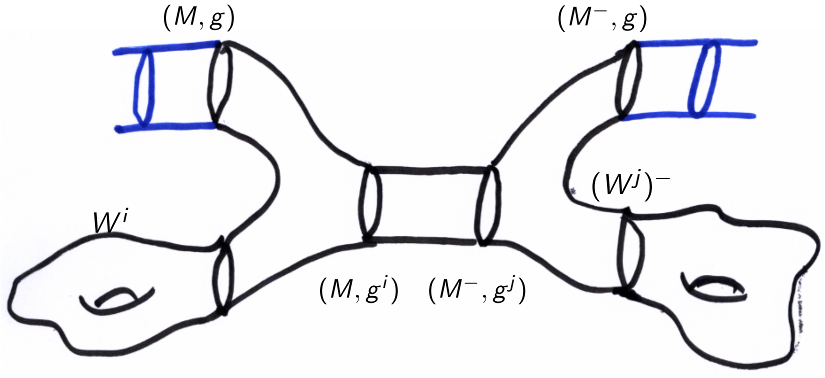


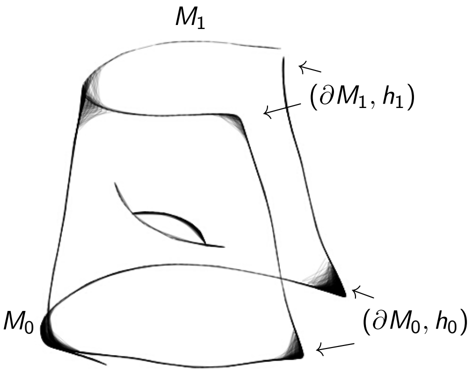
Closed manifold $(W, \tilde{g})$ with $\tilde{g} \in \mathrm{Metr}^{\mathrm{inv}}(W)$ and $\alpha(W) = (i - j)\alpha(K3)$.

Another application - Concordance theory

Definition

$$R_n^{\mathrm{inv}} = \{(M, h) \mid M \text{ is a spin } n\text{-manifold}, h \in \mathrm{Metr}^{\mathrm{inv}}(\partial M)\}/\sim$$

where $(M_0, h_0) \sim (M_1, h_1)$ if



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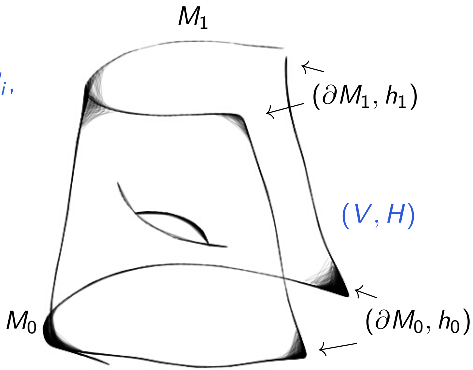
V bordism between ∂M_i,
$H \in \mathrm{Metr}^{\mathrm{inv}}(V)$



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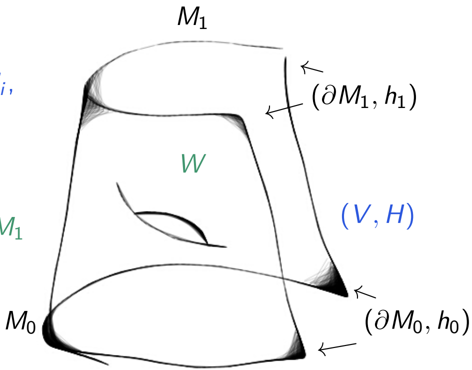
W spin
$\partial W = M_0 \sqcup_{\partial M_0} V \sqcup_{\partial M_1} M_1$



Concordance theory - exact sequence

$$\ldots \to R_{n+1}^{\mathrm{inv}} \xrightarrow{\partial} \Omega_n^{\mathrm{inv}} \xrightarrow{i} \Omega_n^{\mathrm{spin}} \xrightarrow{j} R_n^{\mathrm{inv}} \to \ldots$$

- $\partial([M,g]) := [\partial M, g]$
- $i([M,g]) := [M]$
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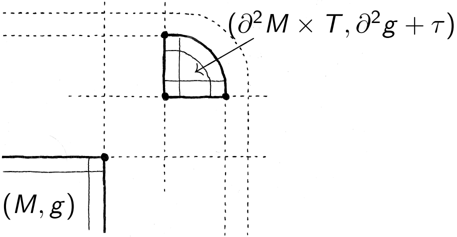
Theorem (Dahl, G; '12)

Let M be a connected spin manifold of dimension $n \geq 4$.

1. $\mathrm{Metr}^{\mathrm{inv}}(\mathrm{M\ rel\ h})$ *is nonempty if and only if* $[M, h] \in R^{\mathrm{inv}}_n$ *vanishes.*
2. *If* $\mathrm{Metr}^{\mathrm{inv}}(\mathrm{M\ rel\ h})$ *is nonempty, then* R^{inv}_{n+1} *acts freely and transitively on the set of concordance classes relative to the boundary metric* h*.*

Concordance theory

- round corners



- handle attachment result for manifolds with corners (handle are attached away from the corners)