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Sommersemester 2019
Albert-Ludwigs-Universität Freiburg

Finite Simple Groups
Exercise Sheet 2
Due 07.05.2019

Exercise 1 (6 Points)**.**
Let G be an abelian group of order n.

1. Suppose that m divides n. Prove that the set G_m of elements g in G such that $g^m = 1$ is a subgroup of G.
2. Let p be prime and suppose that $n = p^k r$ with p and r coprime. Prove without using Sylow's theorem that G_{p^k} has order p^k.
3. Let $p_1, \ldots, p_s$ be prime numbers and assume that $n = p_1^{k_1} \ldots p_s^{k_s}$. Show that G is isomorphic to $G_{p_1^{k_1}} \times \ldots \times G_{p_s^{k_s}}$.

Exercise 2 (8 Points)**.**
Suppose that G acts on the right on a set X.

1. Prove that the map $\phi : G \to \mathrm{Sym}(X)$ given by $g \mapsto \tau_g$ is a group homomorphism, where τ_g is a map from X to X defined by $x \mapsto xg$.
2. Prove that $\mathrm{Ker}(\phi)$ is the kernel of the action, that is to say

$$\mathrm{Ker}(\phi) = \bigcap_{x \in X} \mathrm{Stab}_G(x).$$

Assume now that X is the set of right cosets of a subgroup H in G of index $|G : H| = m$.

1. What is the stabilizer $\mathrm{Stab}_G(Hx)$ of the coset Hx?
2. Deduce that there is a subgroup N of H which is normal in G and satisfies $|G : N| \leq m!$.

Exercise 3 (2 Points)**.**
Let H be a subgroup of a group G. The normalizer of H in G is defined as the subgroup

$$N_G(H) = \left\{ g \in G : g^{-1} H g = H \right\}.$$

Show that $C_G(H)$ is a normal subgroup of $N_G(H)$ and that $N_G(H)/C_G(H)$ embedds into $\mathrm{Aut}(H)$.

Exercise 4 (4 Points)**.**

1. Show that the automorphism group of the cyclic group C_n is abelian of order $\phi(n)$, where ϕ is Euler's totient function.
2. Characterise the automorphism group of $C_p \times C_p$, where p is prime.