Dr. Daniel Palacin
Sommersemester 2019 Albert-Ludwigs-Universität Freiburg

Finite Simple Groups
Exercise Sheet 3
Due 14.05.2019

Exercise 1 (6 Points).
Let G be a finite group. For a prime p, let n_p denote the number of p-Sylow subgroups of G.

1. Assume that $|G| = pm$ with p not dividing m. How many elements of order p has G?
2. Suppose that $|G| = p_1 \dots p_k$ where $p_1, \dots, p_k$ are distinct primes. Deduce that

$$|G| \geq 1 + \sum_{i=1}^{k} n_{p_i}(p_i - 1).$$

3. Conclude using Sylow's Theorem that if $|G| = pqr$ with $p > q > r$ primes, then G is not simple.
Hint: What are the least possible values for n_q and n_r?

Exercise 2 (8 Points).
Let $G = S_4$ and let X be the set of three partitions of $\{1, 2, 3, 4\}$ into two sets of size 2.

1. Show that any element in G gives a permutation of X. Deduce that S_4 acts on the set X.
2. Deduce that S_4 has a normal subgroup V_4 such that $S_4/V_4 \cong S_3$ and $V_4 \cong C_2 \times C_2$.

The alternating group A_n is the normal subgroup of S_n consisting of even permutations. An even permutation is a permutation whose cycle decomposition has an even number of cycles of even size.

3. Conclude that A_4 is not simple.
4. Which is the number of elements of order 5 of A_5? Do not use the fact that A_5 is simple.

Exercise 3 (6 Points).
Let G be a group acting transitively on a set X. For elements x, y in X, put

$$X(x, y) = \{g \in G : xg = y\}.$$

Fix x in X.

1. For h in G, show that $X(x, xh)$ is the right coset $\mathrm{Stab}_G(x)h$ of $\mathrm{Stab}_G(x)$.
2. Prove that the map $\phi : y \mapsto X(x, y)$ is a bijection between X and the set of right cosets of $\mathrm{Stab}_G(x)$.
3. Deduce that $\phi(xh) = (\phi(x))h$ for every h in G.