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Sommersemester 2019 Albert-Ludwigs-Universität Freiburg

Finite Simple Groups
Exercise Sheet 5
Due 04.06.2019

Exercise 1 (6 Points)**.**
Let G be a group. The *commutator* $[x, y]$ of two elements x, y of G is defined as $[x, y] = x^{-1}y^{-1}xy$. The *derived subgroup* G' of G is the subgroup generated by all commutators of the whole group, *i.e.*

$$G' = \langle [x, y] : x, y \in G \rangle.$$

1. Show that G' is the smallest normal subgroup of G such that G/G' is abelian.
2. Prove that $G/Z(G) = (G/Z(G))'$ whenever $G = G'$.
3. Show for $n \geq 5$ that the derived subgroup of S_n is A_n.

Exercise 2 (4 Points)**.**
Let t be an element of order 2 in A_5, say (12)(34). Show that its centralizer is isomorphic to $C_2 \times C_2$.

Exercise 3 (10 Points)**.**
Let V be an n-dimensional vector space over a field F. Define an equivalence relation $\equiv$ on $V \setminus \{0\}$ by

$$v \equiv w \text{ if there exists some } \lambda \in F^\times \text{ with } v = \lambda w.$$

Given an element v of $V \setminus \{0\}$, denote its equivalence class by $[v]$ and let $\mathrm{P}(V)$ be the collection of all equivalence classes. The set $\mathrm{P}(V)$ is *projectivization* of V.

1. Prove that the map $(g, [v]) \mapsto [g(v)]$ defines an action from $\mathrm{GL}(V)$ on $\mathrm{P}(V)$.
2. Show that $\mathrm{SL}(V)$ acts 2-transitive on $\mathrm{P}(V)$.
3. Prove that the kernel of the group action of $\mathrm{GL}(V)$ on $\mathrm{P}(V)$ consists of scalar maps, *i.e* maps of the form $v \mapsto \lambda v$ for some $\lambda \in F^\times$.
4. Show that the center of $\mathrm{GL}(V)$ is formed by the scalar matrices.