Dr. Daniel Palacin
Sommersemester 2019

Albert-Ludwigs-Universität Freiburg

Finite Simple Groups

Exercise Sheet 6

Due 25.06.2019

Exercise 1 (14 Points).

Consider the group $\mathrm{SL}(2,F)$, where F is an arbitrary field.

1. Prove that the sets of matrices

$$T = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in F \right\} \quad \text{and} \quad D = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} : b \in F^{\times} \right\}$$

   are abelian subgroups of $\mathrm{SL}(2,F)$, with matrix multiplication. Furthermore, show that T and D are isomorphic to F^+ and to $F^{\times}$ respectively.

Assume that F is the finite field of size $q = p^k$, where p is a prime.

2. Prove that D acts transitively on $T \setminus \{\mathrm{Id}\}$ by conjugation whenever $p = 2$.
   Hint: Which are the solutions of $x^2 = 1$*?*

3. Deduce that T is a p-Sylow subgroup of $\mathrm{SL}(2,q)$ which is isomorphic to $C_p \times \overset{k}{\ldots} \times C_p$ and that D is cyclic of order $q-1$.

4. Conclude that all elements of order p of $\mathrm{SL}(2,q)$ are conjugated whenever $p = 2$.

5. Prove that $\mathrm{GL}(2,q)$ has a cyclic subgroup of order $q^2 - 1$.
   Hint: $\mathbb{F}_{q^2}^{+}$ *is an* F*-vector space of dimension* 2.

6. Deduce that $\mathrm{SL}(2,q)$ has a cyclic subgroup of order $q+1$.
   Hint: The determinant is a group homomorphism from $\mathrm{GL}(2,q)$ *to* $F^{\times}$.

7. Let $r > 2$ be a prime not dividing q. Conclude that an r-Sylow subgroup of $\mathrm{SL}(2,q)$ is cyclic.

In particular, the 2-Sylow subgroups of $\mathrm{SL}(2,2^k) = \mathrm{PSL}(2,2^k)$ are isomorphic to $C_2 \times \overset{k}{\ldots} \times C_2$ and the other Sylow subgroups are all cyclic.

Exercise 2 (6 Points).

Let F be the finite field of size $q = p^k$, where $p > 2$ is a prime. Consider the group $\mathrm{SL}(2,q)$.

1. Show that $-\mathrm{Id}$ is the unique element of order 2 of $\mathrm{SL}(2,q)$.

2. Let a and b be elements of F with the property that $a^2 + b^2 = -1$, which always exist in a finite field. Consider the following elements of $\mathrm{SL}(2,q)$:

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

   Prove that $x^2 = y^2 = -\mathrm{Id}$ and that $y^x = y^{-1}$.

3. Deduce that the group $\langle x, y \rangle / \langle -\mathrm{Id} \rangle$ is isomorphic to $C_2 \times C_2$ and conclude that the 2-Sylow subgroups of $\mathrm{SL}(2,q)$ cannot be cyclic.