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Sommersemester 2019

Albert-Ludwigs-Universität Freiburg

Finite Simple Groups
Exercise Sheet 8
Due 09.07.2019

Exercise 1 (4 Points)**.**
Let H be a non-trivial subgroup of a group G.

1. Prove that $H \leq Z(N_G(H))$ if and only if $C_G(H) = N_G(H)$.
2. Let p be the smallest prime dividing $|G|$. Suppose that H is a cyclic subgroup of G of order p^m. Deduce that p divides $|N_G(H)/C_G(H)|$ or $H \leq Z(N_G(H))$.

Exercise 2 (4 Points)**.**
Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$. Let a and b be two elements of $C_G(P)$ which are conjugate (in G). Show that they are conjugate in $N_G(P)$, *i.e.* there is some $h \in N_G(P)$ such that $a^h = b$.
Hint: P is a p-Sylow subgroup of $C_G(a)$.

Exercise 3 (4 Points)**.**
Let G be an arbitrary group and let H be a subgroup of $Z(G)$ of index n. Let v be the transfer homomorphism from G to H. Prove that $v(g) = g^n$ for every $g \in G$.

Exercise 4 (8 Points)**.**
Let G be a finite group and let p be a prime divisor of $|Z(G) \cap G'|$. Let $P \in \mathrm{Syl}_p(G)$.

1. Show that $Z(G) \cap G' \cap P$ is non-trivial.
2. If P were abelian, then we could consider the transfer group homomorphism v from G to P. Under this assumption:
   (a) Prove that $v(z) = z^{|G:P|}$ for $z \in Z(G) \cap P$.
   (b) Show that $v(z) = 1$ for $z \in G'$.
3. Conclude that P cannot be abelian.