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Sommersemester 2019 Albert-Ludwigs-Universität Freiburg

Finite Simple Groups
Exercise Sheet 9
Due 16.07.2019

Exercise 1 (6 Points)**.**
Let G be a finite group with at least two 2-Sylow subgroups.

1. Assume that the intersection of any two distinct 2-Sylow subgroups is the identity. Let S and T be 2-Sylow subgroups and let $s \in S$ and $t \in T$ be two elements of order 2. Prove that $S = T$ whenever s and t are not conjugated.

2. Deduce that G has just one conjugacy class of elements of order 2 whenever the intersection of any two distinct 2-Sylow subgroups is the identity.

3. Suppose that the centralizer $C_G(t)$ of every element t of order 2 is an abelian 2-subgroup. Conclude then that G has just one conjugacy class of elements of order 2.

Exercise 2 (4 Points)**.**
Let G be the dihedral group of order $2n$ with $n \geq 3$ odd. Let t be an element of order 2 in G. When does the inequality

$$|G| < |C_G(t)| \left(|C_G(t)|^2 - 1 \right)$$

hold?

Exercise 3 (10 Points)**.**
An element $x \neq 1$ of a group G is said to be *strongly real* in G if there is an element t of order 2 such that $x^t = x^{-1}$. Prove the following:

1. If $x^2 = 1$, then x is strongly real.

2. If $x \neq 1$ is real (*i.e.* x and x^{-1} are conjugate) and $|C_G(x)|$ is odd, then x is strongly real.

Let $x \neq 1$ be an element of G and let $\beta(x)$ be the number of ordered pair (s, t) of elements of order 2 in G such that $st = x$.

3. Show that if $\beta(x) > 0$, then x is strongly real.

4. Prove that if x is strongly real and $\beta(x) = 0$, then x has order 2 and is the only element of order 2 in $C_G(x)$.

5. Prove that every element in D_{2n} is strongly real, for $n \geq 3$.