TODAY: § 1.1 FROM SKRIPT

PLEASE ASK QUESTIONS!

G group is a set together with two maps

$$\left.\begin{array}{l} \text{mult}: G \times G \longrightarrow G \\ \qquad (x,y) \longmapsto xy \\ \\ \text{inv}: G \longrightarrow G \\ \qquad x \longrightarrow x^{-1} \end{array}\right\} \text{ required to satisfy group axioms.}$$

EXAMPLES $(\mathbb{Z},+)$, $(\mathbb{Z}/n\mathbb{Z})$, S_n

$(\mathbb{R},+)$, $(\mathbb{R}^n,+)$, $(\mathbb{C}^*,\cdot)$ ...

Def A topological group is a group and a topological space where these two structure compatible.

Recall a top. space where we have decided what the open sets are.

EXAMPLES $\mathbb{R}^n \supset U$ is open if $\forall x \in U$ $\exists \delta > 0$ $B(x,\delta) \subset U$

<u>Recall</u> X, Y top. spaces

$f: X \to Y$ is continuous if $\forall U$ open in Y $f^{-1}(U)$ is open in X.

<u>Def</u> (more precisely) a top. group is a group and a top. space where

$$\left.\begin{array}{l} \text{mult}: G \times G \to G \\ \text{inv}: G \longrightarrow G \end{array}\right. \quad \text{are continuous maps.}$$

<u>EXAMPLES</u>

·) $(\mathbb{R}^n, +)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$

·) every group is a top. group with respect to the discrete top.

·) G top. group, H subgroup $\Rightarrow$ H top. group

·) G " ", N normal subgroup

$\Rightarrow$ G/N is a top. group.

(with the quotient top. on G/N)

$\pi: G \to G/N$, $U \subset G/N$ is open $\Leftrightarrow$ $\pi^{-1}(U)$ is open in G

<u>Def</u> A Lie group G is a group with a compatible structure of differentiable manifold

$$\left.\begin{array}{l} \text{mult}: G \times G \to G \\ \text{inv}: G \to G \end{array}\right\} \text{ are smooth morphism}$$

For the group $\mathbb{R}^n$, we are asking that

$$\left.\begin{array}{l} \text{mult} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ \text{inv} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \end{array}\right. \quad \text{are } \mathcal{C}^\infty.$$

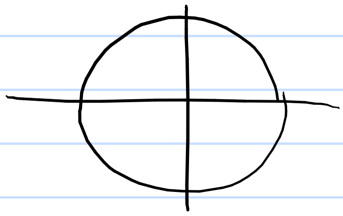
EXAMPLES OF LCB GROUPS

$(\mathbb{R}^n,+)$, $(\mathbb{C}^*,\cdot)$, $\boxed{(S^1,\cdot)}$

$(GL_n(\mathbb{R}),\cdot)$, $(GL_n(\mathbb{C}),\cdot)$

$(SL_n(\mathbb{R}),\cdot)$ ...

THE GROUP S^1



$$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

$$= \{ e^{i\theta} \mid \theta \in \mathbb{R} \} = \{ e^{i\theta} \mid \theta \in [0,2\pi) \}$$

S^1 is a compact top. group.

Recall X compact for a subset of $\mathbb{R}^n$ means it is closed and bounded ($\exists\, N>0$ s.t. $X \subset B(0,N)$).

Def A representation of a group G over a vector space V is an action of G on V via linear homomorphisms

an action is a map

$$G \times V \longrightarrow V \qquad \text{satisfying}$$
$$(g,v) \longmapsto g\cdot v \qquad (gh)\cdot v = g\cdot(h\cdot v))$$

linear means $\forall g \in G \quad \rho_g : V \longrightarrow V$ is a linear map.

$$v \longmapsto g\cdot v$$

Equivalently

A representation of G on V is a group hom.

$$\rho : G \longrightarrow GL(V)$$

Say we have linear action of G on V

define $\rho : G \to GL(V)$

$$g \longmapsto \rho_g$$

NEED TO CHECK $\quad \rho_{gh} = \rho_g \cdot \rho_h$

let $v \in V \quad \rho_{gh}(v) = gh(v) = g\cdot h(v)) = \rho_g(\rho_h(v))$

EXAMPLES

$S^1 \subset \mathbb{C}^*$, acts on $\mathbb{C}$ by mult.

$$S^1 \times \mathbb{C} \longrightarrow \mathbb{C} \qquad S^1 \hookrightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$$
$$(z,v) \longmapsto zv \qquad z \longrightarrow z$$

$S^1 \subset \mathbb{R}^2$

the group hom. $\rho: S^1 \to GL_2(\mathbb{R})$

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is a 2 dimensional real representation of S^1.

Def $(V,\rho), (W,\rho')$ are two rep. of G then a hom. of representations is a linear map $f: V \to W$ such that

$$\forall g \in G \quad f(\underbrace{g \cdot v}_{\text{action of } V}) = \underbrace{g \cdot f(v)}_{\text{action on } W}$$

EXAMPLE $V = \mathbb{R}^2$ $\quad \rho': G \to GL_2(\mathbb{R})$

$$\theta \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$f: (\mathbb{R}^2, \rho) \longrightarrow (\mathbb{R}^2, \rho')$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix} \quad \text{for } x, y \in \mathbb{R}$$

it commutes with the action of S^1

$$f\left(e^{i\theta} \underset{\rho}{\cdot} \begin{pmatrix} x \\ y \end{pmatrix}\right) = e^{i\theta} \underset{\rho'}{\cdot} f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$f\begin{pmatrix} \cos\theta\, x + \sin\theta\, y \\ -\sin\theta\, x + \cos\theta\, y \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta x + \sin\theta y \\ +\sin\theta x - \cos\theta y \end{pmatrix} = \begin{pmatrix} \cos\theta\, x + \sin\theta\, y \\ \sin\theta\, x - y\cos\theta \end{pmatrix}$$

Def An isomorphism of representation is a morphism of rep. which is an isomorphism as a linear map.

Def · Let V be a rep. of G.

A subrep. W of V is a subspace which is stable under G

(i.e. $\forall g\ \forall w \in W\quad g \cdot w \in W$)

· An irreducible representation V is a representation such that the only subrepresentations are $\{0\}$ and V.

EXAMPLE

$$\rho : S^1 \to GL_2(\mathbb{R})$$
$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is irreducible!

Otherwise $\exists\, w \in \mathbb{R}^2$ s.t. $\forall \theta$ $e^{i\theta} \cdot w = \lambda w$

but $\theta = \frac{\pi}{2}$, $\rho(e^{i\theta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\hookrightarrow t^2+1$

it has no eigenvectors

$\Rightarrow$ such a w cannot exist.

$\Rightarrow \rho$ is irreducible.

<u>Obs</u> Every rep. of dim 1 is irreducible.

$$\rho_{\mathbb{C}} : S^1 \to GL_2(\mathbb{C})$$
$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$W = \langle \begin{pmatrix} 1 \\ i \end{pmatrix} \rangle$ $\qquad e^{i\theta} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\theta + i\sin\theta \\ -\sin\theta + i\cos\theta \end{pmatrix}$

$$\parallel$$

$$e^{i\theta}\begin{pmatrix} 1 \\ i \end{pmatrix} = (\cos\theta + i\sin\theta)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\Rightarrow \rho_{\mathbb{C}}$ is not irreducible

$W' = \langle \begin{pmatrix} 1 \\ -i \end{pmatrix} \rangle$ $\qquad e^{i\theta} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{-i\theta}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

We can define $\forall n \in \mathbb{Z}$

$$\rho_n : S^1 \to GL_1(\mathbb{C}) \qquad e^{i\theta} \mapsto e^{in\theta}$$

is one-dim. complex rep. of S^1

Lemma If $n \neq m$ then $\rho_n \not\cong \rho_m$.

Proof $\rho_m \cong \rho_n \Rightarrow m = n$

$\exists f : (\mathbb{C}, \rho_m) \xrightarrow{\sim} (\mathbb{C}, \rho_n)$ s.t. $\forall \theta \in \mathbb{R}$ we have $\forall z \in \mathbb{C}$

$$e^{i\theta} \cdot f(z) = f(e^{i\theta} \cdot z)$$

$f(z) = \lambda z$ for some $\lambda \in \mathbb{C}^*$

$$\begin{array}{ccc} e^{i\theta} \underset{\rho_n}{\cdot} f(z) & = & f(e^{i\theta} \underset{\rho_m}{\cdot} z) \\ \| & & \| \\ e^{in\theta} f(z) & & f(e^{im\theta} z) \\ \| & & \| \\ \lambda e^{in\theta} z & = & \lambda e^{im\theta} z \end{array} \Rightarrow m = n$$

Theorem The rep. ρ_n for $n \in \mathbb{Z}$ are, up to isomorphism, all the continuous irreducible complex representations of S^1.

Def A <u>continuous</u> rep. of G is a rep. $\rho : G \to GL(V)$ which is continuous.

(here $GL(V)$ seen as a top. space as a subset of $M_{n \times n}(\mathbb{R})$ or $M_{n\times n}(\mathbb{C})$) if $V \cong \mathbb{R}^n, \mathbb{C}^n$

Lemma If G is abelian, and V is an irreducible complex rep. of G, then V is of dim 1.

Pf $\rho : G \to GL(V)$.

Since G is abelian, $\rho(g), \rho(h)$ commute for every $g, h \in G$

[**Lemma** If $X \subset GL(V)$ subset s.t. its element pairwise commute, then there exist a common eigenvector in V.]

$X = \rho(G)$, $\exists v \in V$ s.t. $\forall g \in G$ $g \cdot v = \lambda v$ for some $\lambda \in \mathbb{C}$.

$\langle v \rangle = V$ because V irreducible

$\Rightarrow V$ of dim 1.

Pf of thm (Sketch) Want to classify all continuous grp. hom.

$$\rho : S^1 \to GL_1(\mathbb{C}) = \mathbb{C}^*$$

· 1st obs $\rho(S^1) \subset S^1$.

$$|\rho(e^{i\theta})| \neq 1, \quad |\rho(e^{in\theta})| \xrightarrow{\lim_{n\to\infty}} \infty$$

but $\rho(S^1)$ must be compact ↯

So $\rho : S^1 \to S^1$.

$$S^1_2 = \{ z \in S^1 \mid \operatorname{ord}(z) = 2^n \}$$
$$\cap \qquad\qquad \shortparallel$$
$$S^1 \qquad \left\{ e^{\frac{i a \pi}{2^n}} \;\middle|\; \begin{array}{l} a \in [0, 2^n] \\ n \in \mathbb{N} \end{array} \right\}$$

· S^1_2 is dense in S^1 (it insersect every open set $\neq \phi$)

It is enough to check $\rho(z) = \rho_n(z)$ for some n $\forall z \in S^1_2$.

· $\rho\left(e^{\frac{i\pi}{2^n}}\right) \xrightarrow{n \to \infty} 1$ $\implies \exists N_0$ s.t. $\forall n > N_0$

$\shortparallel$

$e^{\frac{i a_n \pi}{2^n}}$ for some $a_n \in \mathbb{N}$ $\qquad \frac{a_n}{2^n} < \frac{1}{2}$

$\implies a_n < 2^{n-1}$.

· $\rho\left(e^{\frac{i\pi}{2^{n+1}}}\right) = e^{\frac{i a_{n+1} \pi}{2^{n+1}}}$

$\shortparallel$

z

$\rho(z^2) = e^{\frac{i a_n \pi}{2^n}}$ $\implies a_{n+1} = a_n$ or $a_{n+1} = a_n + 2^n$

$\shortparallel$

$e^{\frac{i a_{n+1} \pi}{2^n}}$

$n > N_0 \implies a_{n+1} = a_n$

so we have $a_{N_0+1} = a_{N_0+2} = \dots =: a \in \mathbb{N}$

$\rho\left(e^{\frac{i\pi}{2^n}}\right) = e^{\frac{i\pi}{2^n} \cdot a}$ $\forall n > N_0$

$\rightsquigarrow \rho = \rho_a$ over S^1_2.

$\Rightarrow \rho = \rho_a$ over S^1. $\square$