# #3. LIE ALGEBRAS (§ 1.3 - 1.4 OF THE SKRIPT)

Last time: For a matrix group $G \subset GL_n(\mathbb{R})$ we have

$$T_{Id} G = \{ A \in M_n(\mathbb{R}) \mid e^{tA} \in G \ \forall t \in \mathbb{R} \}$$

EXAMPLES

$$T_{Id} GL_n(\mathbb{R}) = M_n(\mathbb{R})$$

$$T_{Id} SL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \operatorname{tr} A = 0 \}$$

$$O_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid AA^t = Id \}$$

$$T_{Id} O_n(\mathbb{R}) \underset{\subseteq}{=} \{ A \in M_n(\mathbb{R}) \mid A = -A^t \}$$

$$\rotatebox{90}{$\in$} \qquad\qquad$$

$$X \, . \quad e^{tX} (e^{tX})^t = id \quad \text{take } \left.\frac{d}{dt}\right|_{t=0}$$

$$\| $$

$$e^{tX} e^{tX^T} \rightsquigarrow$$

$$\frac{d}{dt}\left( e^{tX} e^{tX^t} \right) = 0$$

$$\|$$

$$\frac{d}{dt}(e^{tX}) e^{0} + e^{0} \frac{d}{dt}(e^{tX^t})$$

$$\|$$

$$X + X^t \implies X + X^t = 0$$

To show $\supseteq$

$$X = -X^t \qquad \text{Want to show} \quad e^{tX} e^{tX^t} = id \quad \forall t$$

$$\| $$

$$e^{tX} e^{-tX} = id$$

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We can define a natural action of $G$ on $T_{Id} G$.

$G \subset GL_n(\mathbb{R})$

$$int(g) : M_n(\mathbb{R}) \to M_n(\mathbb{R}) \qquad \text{it's a linear map.}$$
$$X \longmapsto g X g^{-1}$$

$int(g)(G) \subset G$. because $ghg^{-1} \in G \ \forall h \in G$

$$d_{Id}\, int(g) : T_e G \longrightarrow T_e G$$

so for $X \in T_e G$ $\quad d\, \underset{\substack{\| \\ Ad(g)}}{int(g)} (X) = g X g^{-1}$

$$G \times T_e G \longrightarrow T_e G$$
$$(g, X) \longmapsto g X g^{-1}$$

is a representation of $G$ called the adjoint representation.

**Thm** The tangent space $T_{Id} G$ is closed under commutator

i.e. $A, B \in T_e G$

$$[A,B] := AB - BA \in T_{Id} G.$$

**Pf:** $A, B \in T_e G$. $e^{At} \in G \ \forall t \in \mathbb{R}$.

$$\gamma(t) = Ad(e^{At})(B) \in T_e G$$
$$\quad\quad\quad\quad \shortparallel$$
$$\quad\quad\quad\quad e^{At} B e^{-At}$$

$$\dot\gamma(0) = AB \cdot e^0 + e^0 B(-A) = AB - BA$$
$$\cap$$
$$T_e G$$

$\square$

**Def** We call Lie $G$ the vector space $T_e G$ together with the operation

$$[\ ,\ ] : T_e G \times T_e G \to T_e G$$
$$X, Y \mapsto XY - YX$$

the Lie algebra of $G$

**Def** A Lie algebra over a field $k$ is a vector space $\mathfrak{g}/_k$ together w/ a $k$-bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

s.t. 1) $[x,x] = 0$ (antisymmetry)

JACOBI IDENTITY 2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \ \forall x,y,z \in \mathfrak{g}$

Check the Jacobi identity for $T_e G$!

EXAMPLE $\mathrm{Lie}\, SL_n = sl_n$

$$o_3(\mathbb{R}) = \{A \in M_3(\mathbb{R}) \mid A = -A^t\}$$

$$\parallel$$

$$\left\{\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{R}\right\}$$

this has a basis given by

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2$$

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The Lie algebra Lie G contains a lot of information about G

We can almost recover G from Lie G

Recall $H \subset G$ subgrp that contains a nbhd of Id, + G connected $\Rightarrow H = G$

$$\exp: \begin{array}{ccc} U & \xrightarrow{\sim} & V \\ \cap & & \cap \\ T_e G & & G \end{array}$$

$\langle \exp(T_e G) \rangle$ is a subgroup of G containing $V \Rightarrow$ G is generated by $\exp(T_e G)$

$O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$ have the same Lie algebra.

Lie (G) has the advantage of being a linear object
$\rightsquigarrow$ no geometry involved $\rightsquigarrow$ often simpler to study.

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**Def** A homomorphism of Lie groups $f: G \to H$
is a group hom. which is a smooth morphism.

**Def** A homomorphism of Lie algebra $\varphi: \mathfrak{g} \to \mathfrak{h}$
is a linear map s.t. $\forall X, Y \in \mathfrak{g}$ $\quad [\varphi(X), \varphi(Y)] = \varphi([X, Y])$

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Let $f: G \to H$ hom. of Lie group.

We can take
$$d_e f : \underset{\substack{\| \\ \mathrm{Lie}\, G}}{T_e G} \longrightarrow \underset{\substack{\| \\ \mathrm{Lie}\, H}}{T_e H}$$

$X \in \mathrm{Lie}\, G.$ $\quad d_e f(X) = ?$

$\gamma(t) = e^{Xt}. \qquad \dot{\gamma}(0) = X$

$$d_e f(X) = d_e f(\dot{\gamma}(0)) = \frac{d}{dt} f(e^{Xt})\Big|_{t=0}$$

Recall $\gamma : \mathbb{R} \to \mathbb{R}^n$ $\mathcal{C}^\infty$ curve.

$A \in M_n(\mathbb{R})$ then

$$\begin{cases} \dot\gamma(t) = A\gamma(t) & \text{for every } t \in \mathbb{R} \\ \gamma(0) = \gamma_0 \end{cases}$$

ORDINARY LINEAR DIFFERENTIAL EQUATION.

It has exactly one solution $\gamma(t) = e^{At}\gamma_0$

Pf of uniqueness For $v \in \mathbb{R}^n$ let $\|v\| = \sum |v_i| \in \mathbb{R}$

$A = (a_{ij}) \in M_n(\mathbb{R})$

We have $\|Av\| \leq \left(\sum_{i,j} |a_{ij}|\right) \|v\|$

Asm we have two sol$^{\underline{ns}}$ $\gamma_1, \gamma_2$ of the eq.

$$\dot\gamma(t) = A\gamma(t), \quad \gamma(0) = \gamma_0 \in \mathbb{R}^n$$

Let $\delta(t) := \gamma_1(t) - \gamma_2(t)$. $\delta(0) = 0$ $\dot\delta(t) = A\delta(t)$

$$\Rightarrow \delta(t) = \int_0^t \dot\delta(u)\,du = \int_0^t A\delta(u)\,du$$

$$\|\delta(t)\| \leq \int_0^t \|A\delta(u)\|\,du \leq \underbrace{\left(\sum_{i,j} |a_{ij}|\right)}_{C} \int_0^t \|\delta(u)\|\,du$$

$$\Rightarrow \|\delta(t)\| \leq ct \max_{u\in[1,t]} \|\delta(u)\| . \text{ Fix } t_0 = \frac{1}{2c}$$

$\forall t \in [0,t_0]$ we have

$$\|\delta(t)\| \leq c t_0 \max_{u\in[0,t_0]} \|\delta(u)\| \Rightarrow$$

$$\Rightarrow \max_{t\in[0,t_0]} \|\delta(t)\| \leq \frac{1}{2} \max_{u\in[0,t_0]} \|\delta(u)\| \Rightarrow$$

$$\Rightarrow \max_{t\in[0,t_0]} \|\delta(t)\| = 0 \Rightarrow \delta(t) = 0 \ \forall t \in [0,t_0]$$

So $\delta(x) = 0 \Rightarrow \delta(x+t) = 0 \ \forall t \in [0,t_0]$

$$\Rightarrow \delta \equiv 0 \Rightarrow \gamma_1 = \gamma_2 . \qquad \square$$

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<u>Prop</u> Let $\varphi : G \to H$ hom. of Lie groups then the following commutes

$$\begin{array}{ccc} \text{Lie } G & \xrightarrow{d_e\varphi} & \text{Lie } H \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\varphi} & H \end{array}$$

that is $\forall X \in \text{Lie } G$ we have

$$e^{d\varphi(X)} = \varphi(e^X)$$

Pf. For $X \in \mathrm{Lie}\, G$ $\quad e^{d\varphi(X)} = \varphi(e^X) \in H$

$$\gamma(t) = \varphi(e^{tX})$$

$$\dot\gamma(t) = \frac{d}{dt}\varphi(e^{tX}) = \frac{d}{ds}\varphi(e^{(t+s)X})\Big|_{s=0}$$

$$= \frac{d}{ds}\varphi(e^{tX})\varphi(e^{sX})\Big|_{s=0} = \varphi(e^{tX})\, d\varphi(X) =$$

$$= \gamma(t)\, d\varphi(X)$$

So $\gamma$ is a sol. of the diff. equation

$$\dot\gamma(t) = \underset{\substack{\cap \\ \mathrm{Lie}\, H}}{\gamma(t)\, d\varphi(X)} \qquad \gamma(0) = \mathrm{Id}$$

$$\varphi(t) = e^{t\, d\varphi(X)}, \quad \varphi(0) = \mathrm{Id}$$

$$\dot\varphi(t) = e^{t\, d\varphi(X)}\, d\varphi(X) = \varphi(t)\, d\varphi(X)$$

$$\Rightarrow \gamma(t) = \delta(t)\ \forall t \in \mathbb{R} \overset{t=1}{\Rightarrow} \varphi(e^X) = e^{d\varphi(X)}$$

Prop $d_e\varphi : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$ is a hom. of Lie algebras

$\forall X, Y \in \mathrm{Lie}\, G \quad [d\varphi(X), d\varphi(Y)] = d\varphi([X,Y])$.

Pf $X, Y \in \text{Lie}\, G$

Fix $t \in \mathbb{R}$

$$\gamma(s) = \varphi\left(e^{tX} e^{sY} e^{-tX}\right) \in H$$

$$= \varphi(e^{tX})\,\varphi(e^{sY})\,\varphi(e^{-tX})$$

$$= e^{t d_e\varphi(X)}\, e^{s d_e\varphi(Y)}\, e^{-t d_e\varphi(X)}$$

$$\dot\gamma(0) = e^{t d_e\varphi(X)}\, d_e\varphi(Y)\, e^{-t d_e\varphi(X)}$$

$$= d\varphi\left(e^{tX} Y e^{-tX}\right) \quad \forall t \in \mathbb{R}$$

Take $\frac{d}{dt}(\ \ )\big|_{t=0}$

$$d\varphi\left(XYe^0 + e^0 Y(-X)\right) = d_e\varphi(X)\, d_e\varphi(Y)\, e^0 + {} + e^0\, d_e\varphi(Y)\left(-d_e\varphi(X)\right)$$

$$d\varphi(XY - YX) = d_e\varphi(X)\, d\varphi(Y) - d\varphi(Y)\, d\varphi(X)$$

$$d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)] \quad \square$$

**Prop** $G$ is connected.

$\varphi_1 : G \to H$, $\varphi_2 : G \to H$ Lie group hom.

Then $\varphi_1 = \varphi_2 \iff d_e\varphi_1 = d_e\varphi_2$.

**Pf.** "$\Rightarrow$" Trivial

"$\Leftarrow$" $\varphi_i(e^X) = e^{d\varphi_i(X)}$ $\forall X \in \text{Lie}\, G$

$\varphi_1$ and $\varphi_2$ coincide on $e^{\text{Lie}\, G} \subset G$

$\Rightarrow$ $\varphi_1$ and $\varphi_2$ coincide on a subgroup which contains a nbhd of $1$.

$\Rightarrow$ $\varphi_1 = \varphi_2$.

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**Def** A one-parameter subgroup of $G$ is a smooth group hom $\gamma : (\mathbb{R}, +) \to G$

**Cor** Lie $G$ parametrizes one-parameter subgroups of $G$

$\forall \gamma$ one-par. subgroup $\exists! X \in \text{Lie}\, G$ s.t.

$\gamma(t) = e^{tX}$ $\forall t \in \mathbb{R}$.

**Pf** Uniqueness follows $\dot{\gamma}(0) = X$

A one par. subgroup satisfies $\gamma(s+t) = \gamma(s)\gamma(t)$ $\forall s,t$

$$\dot\gamma(t) = \frac{d}{ds}\gamma(s+t)\Big|_{s=0} = \frac{d}{ds}\gamma(t)\gamma(s)\Big|_{s=0} =$$

$$= \gamma(t)\dot\gamma(0).$$

So $\gamma$ is a sol. of a ODE which has solution $e^{t\dot\gamma(0)} = \gamma(t)$ $\square$

$$\underset{\text{Lie } G}{\overset{\cap}{\dot\gamma(0)}}$$